LECTURE 20
01-MARCH-2013

The Discrete Time Linear System Response

As discussed yesterday, the hydrological observations and measurements that are of importance in surface water hydrology is:

→ Rainfall measurement
→ Stream discharge

Therefore, to interpret these hydrological phenomenon using linear system concept, we need to correlate the data to the inputs for linear system - impulse, step, pulse, etc.

Moreover, the hydrological observations are mostly in discrete time.

time	P	Q
t_0	-	Q_0
t_1	P_1	Q_1
t_2	P_2	Q_2
$\vdots$	$\vdots$	$\vdots$
t_{m-1}	P_{m-1}	Q_{m-1}
t_m	P_m	Q_m
$\vdots$	$\vdots$	$\vdots$
t_M	P_M	Q_M
t_{M+1}	-	Q_{M+1}
$\vdots$		$\vdots$
t_n	-	Q_n

Δt

★ Precipitation is pulse data

★ Stream discharge is sample data

→ The depth of rainfall at any interval 'm' is

$$P_m = \int_{(m-1)\Delta t}^{m\Delta t} I(\tau)\, d\tau$$

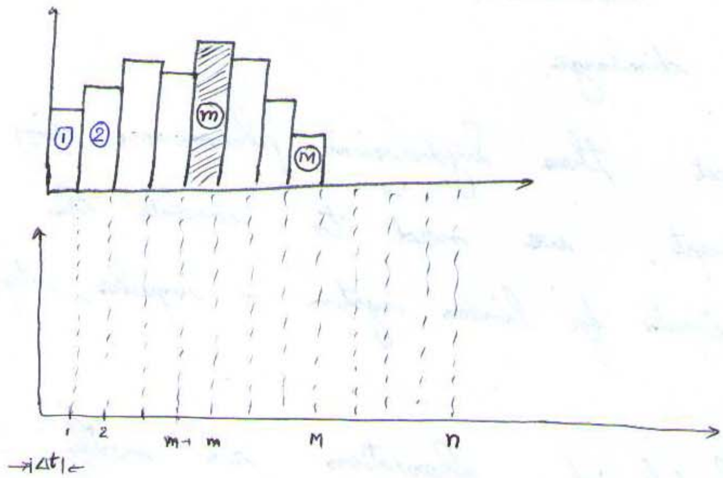
$m = 1, 2, 3, \ldots, M$

$I(\tau) \rightarrow$ rainfall intensity.

→ The stream discharge or runoff is a sample data

$Q_n \rightarrow Q(n\Delta t) \quad ; \quad n = 1, 2, 3, \ldots$

→ Now it is clear to you that input and output are measured in different dimensions.



→ We want to find the effect on runoff at n^{th} time ~~step~~ (i.e $= n\Delta t$) due to the pulse input (or rainfall) at the m^{th} time step ~~(i.e $= m\Delta t$)~~ (i.e from $(m-1)\Delta t$ to $m\Delta t$)

→ We need to find Q_n and you have P_m

→ Recall the unit pulse response function :

$$h(t) = \frac{1}{\Delta t}\left[\, g(t) - g(t-\Delta t)\right]$$

→ In this case, your pulse input began at time $(m-1)\Delta t$ and extended up to $m\Delta t$.

∴ The response from the system due to this pulse will also be after time $(m-1)\Delta t$. If your $P_m = 1$ then the response will be unit pulse response function and the response at time $n\Delta t$ will be:

$$h\left(n\Delta t - (m-1)\Delta t\right) = h\left((n-m+1)\Delta t\right)$$

$$= \frac{1}{\Delta t}\int_{(n-m)\Delta t}^{(n-m+1)\Delta t} u(l)\, dl$$

where $u(l) \rightarrow$ unit impulse response function.

→ However your rainfall depth in the m^{th} time interval i.e $P_m \neq 1$, then you can consider that the rainfall occurs at uniform intensity from $(m-1)\Delta t$ to $m\Delta t$ and this intensity is: $I(\tau) = \frac{P_m}{\Delta t}$

where $(m-1)\Delta t \leq \tau \leq m\Delta t$

$$\therefore \quad P_m = \int_{(m-1)\Delta t}^{m\Delta t} I(\tau)\, d\tau$$

$$I(\tau) = \begin{cases} P_m/\Delta t & ; \quad (m-1)\Delta t \leq \tau \leq m\Delta t \\ 0 & ; \quad \tau > m\Delta t \end{cases}$$

~~Again, in more general form we ca~~

Therefore, the output from the system due to this input is: $Q_{n,m} = \int_{(m-1)\Delta t}^{m\Delta t} I(\tau)\, u(t-\tau)\, d\tau$

Your $t = n\Delta t$, $I(\tau) = \frac{P_m}{\Delta t}$

$$\therefore \; Q_{n,m} = \int_{(m-1)\Delta t}^{m\Delta t} \frac{P_m}{\Delta t} \, u(n\Delta t - \tau) \, d\tau$$

$\Rightarrow$ As you can see there are a total of M rainfall pulses to the system. i.e $m = 1, 2, 3, \ldots, M$

We can define now the rainfall intensities for each of the pulses

$$I(\tau) = \begin{cases} P_m/\Delta t \; ; & (m-1)\Delta t \le \tau \le m\Delta t \; ; \; m = 1, 2, \ldots, M \\ 0 \; ; & \tau > M\Delta t \; ; \; \tau < (m-1)\Delta t \; ; \\ & \tau > m\Delta t \end{cases}$$

$\Rightarrow$ The output from the system will be convolution of all the outputs due to individual pulses

i.e. At any time $t = n\Delta t$

$$Q_n = \frac{P_1}{\Delta t} \int_0^{\Delta t} u(n\Delta t - \tau) \, d\tau + \frac{P_2}{\Delta t} \int_{\Delta t}^{2\Delta t} u(n\Delta t - \tau) \, d\tau$$
$$+ \cdots + \frac{P_m}{\Delta t} \int_{(m-1)\Delta t}^{m\Delta t} u(n\Delta t - \tau) \, d\tau + \cdots + \frac{P_M}{\Delta t} \int_{(M-1)\Delta t}^{M\Delta t} u(n\Delta t - \tau) \, d\tau$$

Assign $l = n\Delta t - \tau$

$\therefore \; d\tau = -dl$

For any $\frac{P_m}{\Delta t} \int_{(m-1)\Delta t}^{m\Delta t} u(n\Delta t - \tau) \, d\tau = \frac{P_m}{\Delta t} \int_{(n-m+1)\Delta t}^{(n-m)\Delta t} - u(l) \, dl$

i.e.

$$= \frac{P_m}{\Delta t} \int_{(n-m)\Delta t}^{(n-m+1)\Delta t} u(l)\, dl$$

$$= P_m \, h\left((n-m+1)\Delta t\right)$$

$$\therefore \boxed{\begin{aligned} Q_n = {} & P_1 h(n\Delta t) + P_2 h((n-1)\Delta t) + \cdots \\ & \cdots + P_m h((n-m+1)\Delta t) + \cdots + P_M h((n-M+1)\Delta t) \end{aligned}}$$

This is convolution equation with input in pulses and output in sample or instantaneous form.

Again note: If $\underset{= J\Delta t}{n\Delta t} < M\Delta t$

then,

$$Q_n = P_1 h(n\Delta t) + P_2 h((n-1)\Delta t) + \cdots + P_m h((n-m+1)\Delta t) + \cdots + P_J h((n-J+1)\Delta t)$$

Pulse Response Function in Discrete Form

As seen earlier, the response for a pulse P_m in the m^{th} interval was correlated with unit pulse response function $h((n-m+1)\Delta t)$

$h((n-m+1)\Delta t) \rightarrow$ is a continuous response function

We can represent this continuous pulse response in a discrete form:

Assign

$$U_{n-m+1} = h\left((n-m+1)\Delta t\right)$$

$\therefore$ For $m = 1$, $\quad U_n = h(n\Delta t)$

$$\text{llly} \quad U_{n-1} = h\left((n-1)\Delta t\right)$$

$$\vdots$$

$$U_{n-M+1} = h\left((n-M+1)\Delta t\right)$$

$$\therefore \quad Q_n = P_1 U_n + P_2 U_{n-1} + \cdots + P_m U_{n-m+1} + \cdots + P_M U_{n-M+1} \; ; \quad n \geq M$$

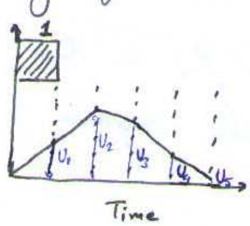
i.e. $$Q_n = \sum_{m=1}^{M} P_m U_{n-m+1} \quad ; \quad n \geq M$$

If $n < M$, then we can write in general form

$$\boxed{Q_n = \sum_{m=1}^{n \leq M} P_m U_{n-m+1}} \rightarrow \text{Discrete Convolution Equation}$$

Illustration :

Say for a linear system, the PRF in discrete form is



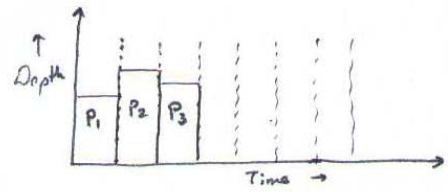
The ordinates of PRF give the discrete values U_1, U_2, U_3, U_4. $U_5 = 0$ as shown in figure.

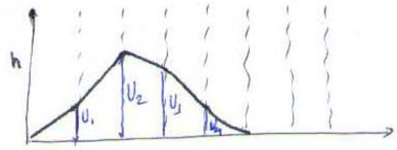
(See also next page for complete illustration).

In the figure, In I^{st} time interval, $P_1 \rightarrow$ pulse

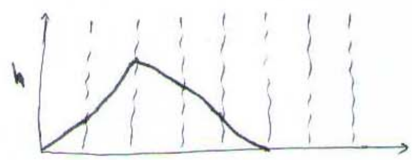
II^{nd} " " $P_2 \rightarrow$ pulse.

III^{rd} time interval $P_3 \rightarrow$ pulse

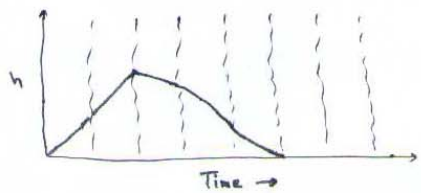




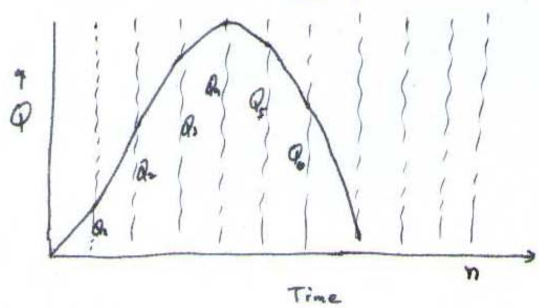
→ Unit response for First Pulse



→ Unit response for second pulse



→ Unit response for third pulse.



Here $M = 3$

Now $$Q_n = \sum_{m=1}^{n \leq M} P_m U_{n-m+1}$$

$\therefore$ For $n = 1$, that is first interval

$$Q_1 = \sum_{m=1}^{1} P_m U_{n-m+1} = P_1 U_1$$

$n=2$, $\quad Q_2 = P_1 U_2 + P_2 U_1$

$n = 3$, $\quad Q_3 = P_1 U_3 + P_2 U_2 + P_3 U_1$

$Q_4 = P_1 U_4 + P_2 U_3 + P_3 U_2 \qquad ; n > M$

$Q_5 = P_1 U_5 + P_2 U_4 + P_3 U_3 = P_2 U_4 + P_3 U_3$

$Q_6 = P_3 U_4 \quad ; \quad Q_7 = 0$