Lecture 2 **05-January-2013**

CONTROL VOLUME APPROACH

Yesterday, we discussed that

→ Hydrologic cycle (and phenomena) are highly complex

→ For simplicity - we adopt systems concept to analyse hydrologic processes.

→ One method to use systems concept is control volume approach.

→ We have also seen various classification of hydrological models.

Today we will see how we can use the control volume method to analyse hydrological processes.

As discussed we require a three dimensional frame of reference in space with boundaries that enclose to form a volume - called control volume (CV).

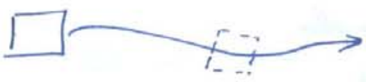
Reynolds Transport Theorem

* So in any arbitrary CV the physical laws like

→ conservation of mass, momentum, energy, etc. are valid.

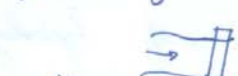
* If you recall fluid mechanics - the Lagrangian and Euler approach to view motion (of objects or particles, etc.).

* In Lagrangian ~~analys~~ approach, the focus is on the motion of a body (when you apply physical laws) and the analysis follows the body wherever it moves.



Used in solid mechanics

* In fluids, we are not interested in individual fluid particle motion.
  Fluid is continuum – the motion is analysed in fixed frame in space (cv) through which fluid passes.



  This is Eulerian view of motion

* As studied in fluid mechanics to relate the Lagrangian and Eulerian analysis we need to incorporate substantial or material derivative.

* RTT gives the equation for material derivative of any extensive fluid property.

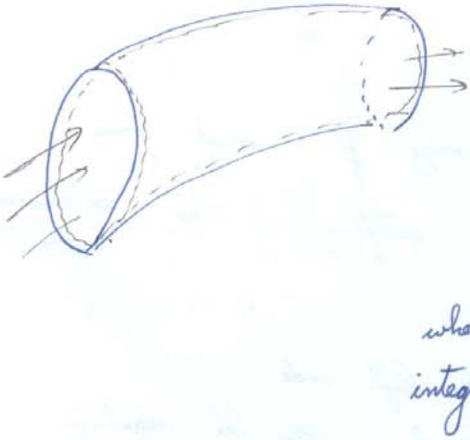
* Therefore, RTT uses the physical laws and applies them on the fluid flowing continuously through the cv.

  → There are two types of fluid properties
    ↳ Extensive
    ↳ Intensive

→ Extensive properties are those that are related to the mass of fluid present in the CV. You may say B.
→ Intensive properties are independent of mass.
Symbolically given as β.
β can also be defined as the quantity of that property B per unit mass of fluid.

Some of the extensive properties are
→ Mass, Momentum, etc.

Some intensive properties are
→ Velocity, specific energy, etc.



For any arbitrary CV (the dotted black line)
the extensive property

$$B = \iiint \beta \rho \, dU$$

where dU is elementary volume for integration

Without going into the derivation of RTT, the equation is given as:

$$\frac{DB}{Dt} = \frac{d}{dt} \iiint_{cv} \beta \rho \, dU + \iint_{cs} \beta \rho \, \vec{v} \cdot \hat{n} \, dA \quad \rightarrow (1)$$

The explanation to this equation is:

→ The material derivative ~~of~~ or time rate of any extensive property in the CV is equal to sum of the time rate of change of the extensive property stored inside the control volume and the net outflux of the extensive property through the surfaces of this control volume.

→ If your CV is fixed in space and time (i.e. non deformable) you will get

$$\frac{DB}{Dt} = \frac{\partial}{\partial t}\iiint_{cv} \beta \rho \, dV + \iint_{cs} \beta \rho \, \vec{v}\cdot\hat{n} \, dA$$

See the physical meaning of second term
→ It is the net outflux of the extensive property.

On the left side
$\vec{v}\cdot\hat{n}$ can be negative

On the right side
$\vec{v}\cdot\hat{n}$ can be positive

On the curved surfaces around there is no flow.
Then $\vec{v}\cdot\hat{n} = 0$.

Continuity Equation

Let us take the continuity principle or principle of conservation of mass for a fluid control volume.

Your RTT is :

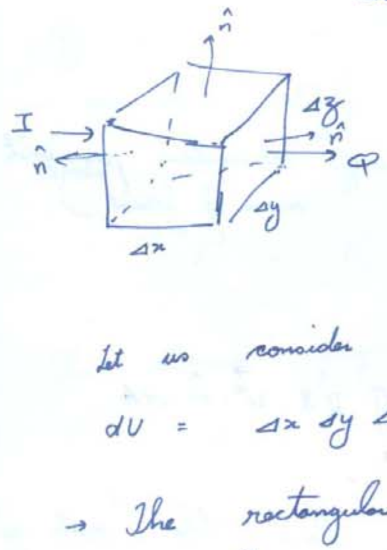
$$\frac{DB}{Dt} = \frac{d}{dt}\iiint_{cv} \beta \rho \, dU + \iint_{cs} \beta \rho \, \vec{v}\cdot\hat{n} \, dA$$

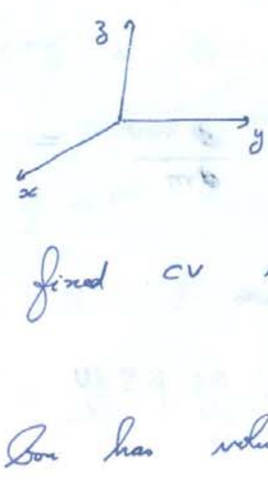
Now $B = m$

$\beta = 1$

As per conservation of mass principle $\frac{Dm}{Dt} = 0$.

$$\therefore \quad 0 = \frac{d}{dt}\iiint_{cv} \rho \, dU + \iint_{cs} \rho \, \vec{v}\cdot\hat{n} \, dA .$$





Let us consider a fixed cv in the space

$dU = \Delta x \, \Delta y \, \Delta z$:

→ The rectangular box has volume = $\Delta x \Delta y \Delta z$

→ It has six faces.

→ Let us assume the flow is present only in y-direction.

The term $\iint_{cs} \rho \vec{v}\cdot\hat{n}\, dA = \rho\left(\frac{v}{y}A\right)_{outlet} - \rho\left(\frac{v}{y}A\right)_{inlet}$

$$= \rho\varphi - \rho I$$

$$\frac{d}{dt}\iiint_{cv} \rho\, dV = \rho\frac{dS}{dt} \qquad \text{where } S = \text{Storage in CV.}$$

(Assuming the fluid is incompressible).

$$\therefore \quad 0 = \rho\frac{dS}{dt} + \rho\varphi - \rho I$$

$$\text{i.e.} \quad \frac{dS}{dt} = I - \varphi$$

This is the continuity equation that are used for unsteady flow in hydrology.

Momentum Equation

Momentum $= B = m\vec{v}$

$$\therefore \beta = \frac{m\vec{v}}{m} = \vec{v} \quad \text{(Momentum per unit mass of fluid).}$$

$\therefore$ RTT gives

$$\frac{DB}{Dt} = \frac{d}{dt}\iiint_{cv} \beta\rho\, dV + \iint_{cs} \beta\rho\, \vec{v}\cdot\hat{n}\, dA$$

Now as per Newton's second law the rate of change of momentum is equal to the net force acting on the body.

$$\therefore \frac{DB}{Dt} = \frac{D}{Dt}(m\vec{v}) = \Sigma\vec{F}$$

$$\therefore \Sigma\vec{F} = \frac{d}{dt}\iiint_{cv} \vec{v}\,\rho\,dU + \iint_{cs} \vec{v}\,\rho\,\vec{v}.\hat{n}\,dA$$

The meaning here is :

The net force acting on the fluid cv = Rate of change of momentum stored inside the cv + The Net Outflow of Momentum through the control surface

RTT in Energy Balance Equation

We need to Balance all inputs and outputs of energy to and from a system (cv).
Say, if E is the amount of energy in a fluid cv, it can be actually sum of internal energy E_u, kinetic energy $\frac{1}{2}mv^2$, potential energy mgz

$$\therefore E = E_u + \frac{1}{2}mv^2 + mgz$$

The extensive property $B = E$

$$\therefore \beta = \underbrace{\frac{E_u}{m}}_{e_u} + \frac{v^2}{2} + gz$$

e_u (internal energy per unit mass)

$$\therefore \frac{DE}{Dt} = \frac{d}{dt}\iiint_{cv}\left(e_u + \frac{v^2}{2} + gz\right)\rho\,dU + \iint_{cs}\left(e_u + \frac{v^2}{2} + gz\right)\rho\,\vec{v}.\hat{n}\,dA$$

As per first law of thermodynamics:

" Net rate of energy transfer into the fluid

= Rate at which heat is transferred into the fluid

− Rate at which fluid does work on its surrounding

i.e. $\frac{DE}{Dt} = \frac{dH}{dt} - \frac{dW}{dt}$

$$\therefore \frac{dH}{dt} - \frac{dW}{dt} = \frac{d}{dt}\iiint_{cv}\left(e_u + \frac{v^2}{2} + gz\right)\rho\, dV + \iint_{cs}\left(e_u + \frac{v^2}{2} + gz\right)\rho\, \vec{v}\cdot\hat{n}\, dA$$