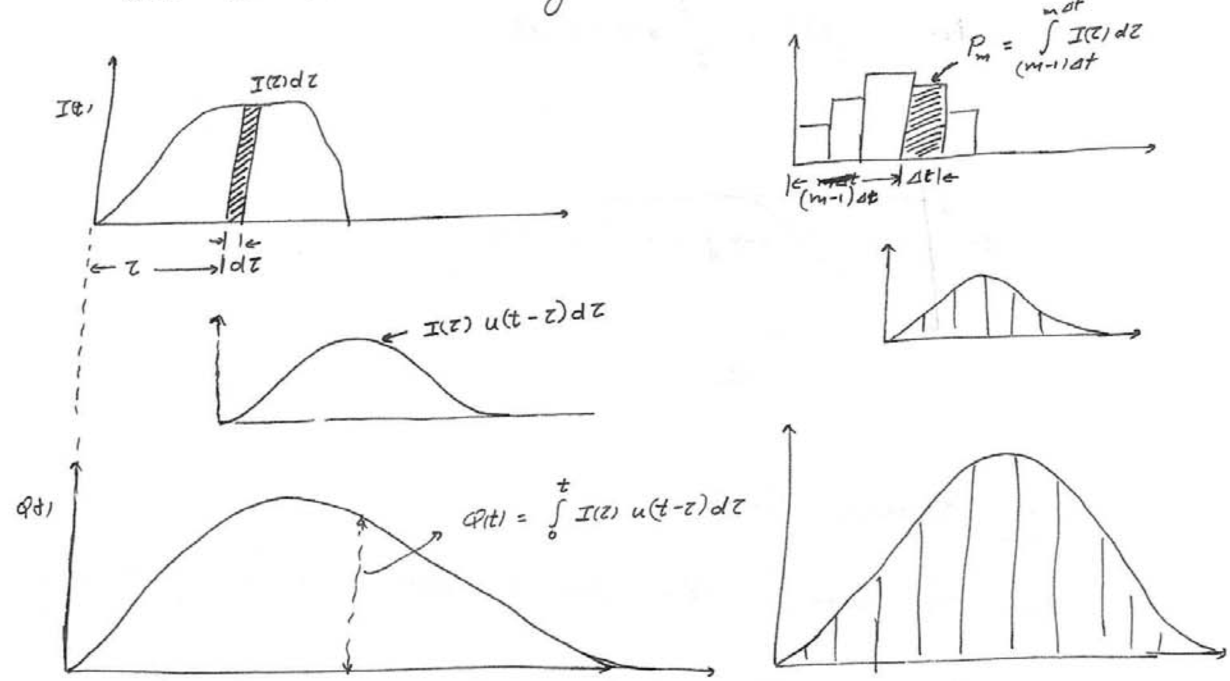
Then your input is $= \int_0^t I(\tau)\, d\tau$

This will given you same as rainfall depth, $I(\tau)$ is rainfall ~~system~~ intensity.

The response from the system will be

$$Q = \int_0^t I(\tau)\, u(t-\tau)\, d\tau \qquad \rightarrow \text{the surface runoff}$$

This is convolution integral used in continuous time scale.



In hydrology, we mostly have discrete measurements (or observations)

Step Response Function

Q: What is step input? ~~and~~

~~Th~~ At any time τ the input goes from 0 to some magnitude and continues.

$\Rightarrow$ A unit step input $\rightarrow$ the input goes from 0 to 1 at time 0 and continues indefinitely.

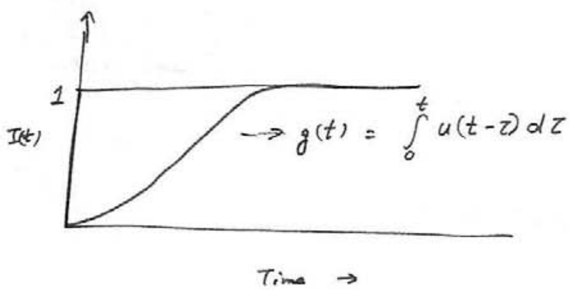
For such an unit input, what will be the response from linear system.

This will be unit step response function $g(t)$

Note $I(\tau) = 1$ for $\tau \geq 0$

$$\therefore \; Q(t) = g(t) = \int_0^t u(t-\tau)\, d\tau$$

$$\text{i.e.} \quad g(t) = \int_0^t u(t-\tau)\, d\tau$$



Pulse Response Function

How the linear system respond when a pulse input is given?

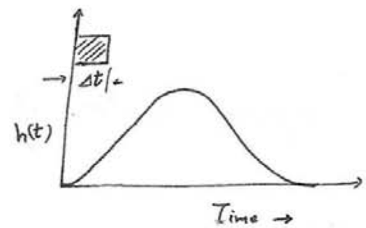
A unit pulse input → input of unit amount occurring in duration Δt.

The total magnitude of unit pulse input = 1

$\therefore$ Pulse input rate $I(\tau) = \frac{1}{\Delta t}$

that ranges from $0 \leq \tau \leq \Delta t$

$$\text{i.e.} \quad I(\tau) = \begin{cases} \frac{1}{\Delta t} & ; \; 0 \leq \tau \leq \Delta t \\ 0 & ; \; \tau > \Delta t \end{cases}$$



The pulse response for unit input is given as $h(t)$

Recall $\quad g(t) = \int_0^t u(t-\tau)\,d\tau$

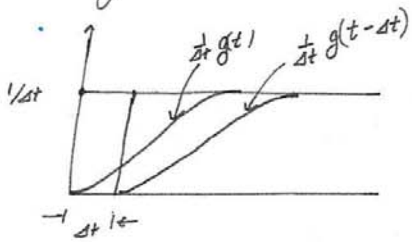
This was for unit step input

$\therefore$ the step response for an input rate of $\frac{1}{\Delta t}$ is

$$= \int_0^t \frac{1}{\Delta t}\, u(t-\tau)\,d\tau,$$

$$\text{ie} \quad = \frac{1}{\Delta t}\, g(t)$$

→ Consider another step input of $I(\tau) = \frac{1}{\Delta t}$ that begins at time $t = 0 + \Delta t$



Now from the figure the response of the system for unit pulse input

$$h(t) = \frac{1}{\Delta t}\, g(t) - \frac{1}{\Delta t}\, g(t-\Delta t)$$

$$\text{i.e.} \quad h(t) = \frac{1}{\Delta t}\left[\int_0^t u(t-\tau)\,d\tau - \int_{\Delta t}^t u(t-\Delta t-\tau)\,d\tau \right.$$

For integration simplicity, let us assign $\quad l = t - \tau$

$$\therefore dl = -d\tau$$

When $\tau = t, \quad l = 0$

$\qquad \tau = 0, \quad l = t$

$$\therefore g(t) = -\int_t^0 u(l)\,dl$$

$$\text{or} \quad g(t) = \int_0^t u(l)\,dl$$

$$\therefore h(t) = \frac{1}{\Delta t}\left[\int_0^t u(l)\,dl - \int_0^{t-\Delta t} u(l)\,dl \right] = \frac{1}{\Delta t} \int_{t-\Delta t}^{t} u(l)\,dl$$

A quick example :

Find unit impulse, step, and pulse response functions of a linear reservoir.

$\Rightarrow$ A linear reservoir $\quad S = kQ$

$$\text{and} \quad \frac{dS}{dt} = I - Q$$

$$\text{or} \quad k\frac{dQ}{dt} + Q = I$$

$$\text{i.e.} \quad \frac{dQ}{dt} + \frac{1}{k}Q = \frac{I}{k}$$

To solve such equations:

The integrating factor $\quad e^{t/k}$

The solution is:

$$Q(t) = Q_0 e^{-t/k} + \int_0^t \frac{1}{k} e^{-(t-\tau)/k} I(\tau)\, d\tau$$

where $Q_0 \rightarrow$ output at time $t = 0$.

Compare with convolution integral

$$Q(t) = \int_0^t I(\tau)\, u(t-\tau)\, d\tau$$

If you had $Q_0 = 0$, then

$$\int_0^t I(\tau)\, u(t-\tau)\, d\tau = \int_0^t \frac{1}{k} e^{-(t-\tau)/k} I(\tau)\, d\tau$$

$$\therefore \quad u(t-\tau) = \frac{1}{k} e^{-(t-\tau)/k}$$

Unit step response is: $g(t) = \int_0^t u(l)\, dl$

i.e. $g(t) = \int_0^t \frac{1}{k} e^{-l/k} dl$

$$= \left[- e^{-l/k} \right]_0^t = 1 - e^{-t/k}$$

The unit pulse response is :

$$h(t) = \frac{1}{\Delta t} \left[g(t) - g(t - \Delta t) \right]$$

You check what will happen for $g(t - \Delta t)$ for $0 \leq t \leq \Delta t$

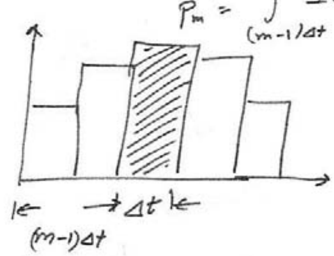
In Discrete Time The Linear System Response

We already suggested that most of hydrological observations or measurements are in discrete time. Therefore, we need to interpret the unit impulse, step, and pulse response functions in discrete time.

Q: Where do you see pulse input data ?

→ The rainfall or precipitation

P_m → depth of precipitation from time $(m-1)\Delta t$ to $m\Delta t$.



$$P_m = \int_{(m-1)\Delta t}^{m\Delta t} I(\tau)\, d\tau \quad ; \quad m = 1, 2, 3 \ldots$$

where $I(\tau)$ → rainfall intensity.

→ The streamflow at any gauging station is measured instantaneously Q_n ; $n = 1, 2, 3, \ldots$

Maybe at every Δt time, streamflow measured.

→ Inflow to the system - precipitation is measured in pulse form

→ Outflow from the system - Q_n is measured instantaneously.

Q: How will you correlate the data now?

Q_n means $Q(n\Delta t)$

$$P_m = \int_{(m-1)\Delta t}^{m\Delta t} I(\tau)\, dt \qquad ; \quad (m-1)\Delta t \le t \le m\Delta t$$

Now unit pulse response function

$$h(t) = \frac{1}{\Delta t}\left[g(t) - g(t-\Delta t) \right]$$

The output time $t = n\Delta t$

We want the response at $t = n\Delta t$ when the precipitation of intensity $I(\tau)$ occurred between $(m-1)\Delta t$ and $m\Delta t$ time.

i.e. $$h(t-(m-1)\Delta t) = h(n\Delta t - m\Delta t + \Delta t)$$

$$= h((n-m+1)\Delta t)$$

$$= \frac{1}{\Delta t}\int_{(n-m)\Delta t}^{(n-m+1)\Delta t} u(l)\, dl .$$

⇒ The m^{th} input pulse (or precipitation) $= P_m = \int_{(m-1)\Delta t}^{m\Delta t} I(\tau)\, d\tau$