LECTURE 18

GENERALISED LINEAR SYSTEM

27-FEB-2013

If you recall, in the last class we discussed on generalised linear system. Some of the hydrological processes can be linearised, etc.

The storage for a system can be represented using a linear equation

$$S = a_1 Q + a_2 \frac{dQ}{dt} + \cdots + b_1 I + b_2 \frac{dI}{dt} + \cdots$$

∴ In this linear form, one can represent output

$$Q(t) = [\quad] \, I(t)$$

↳ Some transfer function or Response function

Response Functions of Linear Systems

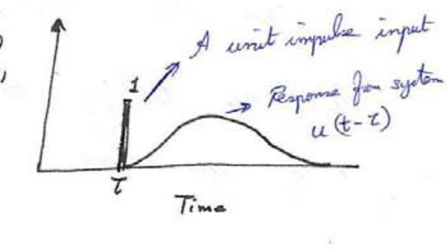
We also suggested earlier that this transfer function (or response function) follows basic principles for linear system operations.

→ Principle of proportionality
→ Principle of superposition

Then we started discussing on the response function. The first one was

Impulse Response Function

An input of unit magnitude is impulsively provided to the system. The response is shown.

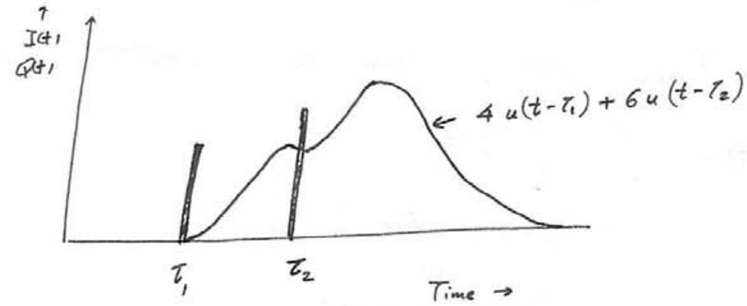


~~If~~ For this unit impulse the response is given as $u(t - \tau)$

You $(t - \tau)$ is time lag as impulse is applied only at time τ.

For unit impulse $\Rightarrow$ Unit Response Function $u(t - \tau)$

If 4 units at τ_1 and 6 units at τ_2, then the system response will be $= 4\,u(t - \tau_1) + 6\,u(t - \tau_2)$



→ We can also consider continuous input as sum of infinitesimal impulses.

Let us denote impulse input as $I(\tau)$

∴ The continuous input between time τ and $\tau + d\tau$ can be given as: $I(\tau)\,d\tau$

(Recall your rainfall depth measurement → Depth = Intensity × Time)

∴ For the system if the input is $I(\tau)\,d\tau$ then the response $= I(\tau)\,d\tau\; u(t - \tau)$

We are talking about input from time τ to $\tau + d\tau$.

→ If you have continuous input from $t = 0$ to time t.

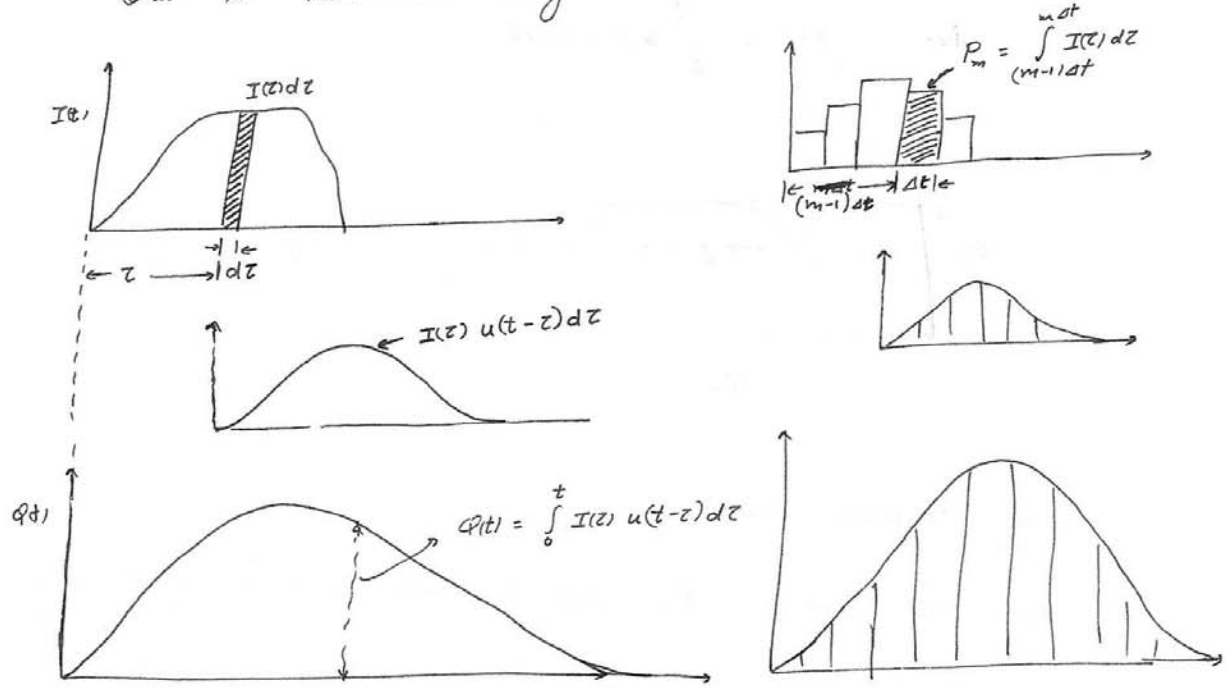
Then your input is $\cancel{P} = \int_0^t I(\tau)\, d\tau$

This will give you same as rainfall depth, $I(\tau)$ is rainfall ~~system~~ intensity.

The response from the system will be

$$Q = \int_0^t I(\tau)\, u(t-\tau)\, d\tau \quad \rightarrow \text{the surface runoff}$$

This is convolution integral used in continuous time scale.



In hydrology, we mostly have discrete measurements (or observations)

Step Response Function

Q: What is step input? ~~and~~ the input goes from 0 to some magnitude
~~Ah~~ At any time τ
and continues.

$\Rightarrow$ A unit step input $\rightarrow$ the input goes from 0 to 1 at time 0 and continues indefinitely.

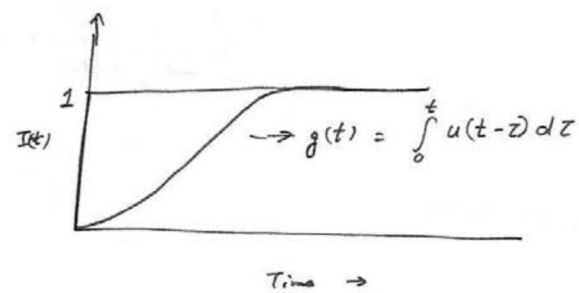
For such an unit input, what will be the response from linear system.

This will be unit step response function $g(t)$

Note $I(\tau) = 1$ for $\tau \geq 0$

$$\therefore \quad Q(t) = g(t) = \int_0^t u(t-\tau)\, d\tau$$

$$\text{i.e.} \quad g(t) = \int_0^t u(t-\tau)\, d\tau$$



Pulse Response Function

How the linear system respond when a pulse input is given?

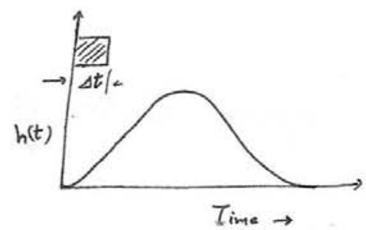
A unit pulse input → input of unit amount occurring in duration Δt.

The total magnitude of unit pulse input = 1

$\therefore$ Pulse input rate $I(\tau) = \frac{1}{\Delta t}$

that ranges from $0 \leq \tau \leq \Delta t$

$$\text{i.e.} \quad I(\tau) = \begin{cases} \frac{1}{\Delta t} & ; \quad 0 \leq \tau \leq \Delta t \\ 0 & ; \quad \tau > \Delta t \end{cases}$$



The pulse response for unit input is given as $h(t)$

Recall $g(t) = \int_0^t u(t-\tau)\, d\tau$

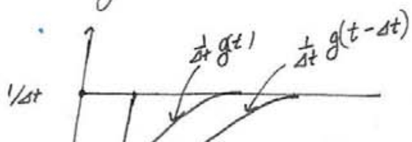
This was for unit step input

$\therefore$ the step response for an input rate of $\frac{1}{\Delta t}$ is

$$= \int_0^t \frac{1}{\Delta t}\, u(t-\tau)\, d\tau,$$

$$\text{ie} \quad = \frac{1}{\Delta t}\, g(t)$$

→ Consider another step input of $I(\tau) = \frac{1}{\Delta t}$ that begins at time $t = 0 + \Delta t$



Now from the figure the response of the system for unit pulse input

$$h(t) = \frac{1}{\Delta t}\, g(t) - \frac{1}{\Delta t}\, g(t-\Delta t)$$

$$\text{i.e.} \quad h(t) = \frac{1}{\Delta t}\Big[\int_0^t u(t-\tau)\, d\tau - \int_{\Delta t}^t u(t-\Delta t-\tau)\, d\tau$$

For integration simplicity, let us assign $l = t - \tau$

$\therefore dl = -d\tau$

When $\tau = t, \quad l = 0$

$\tau = 0, \quad l = t$

$$\therefore g(t) = -\int_t^0 u(l)\, dl$$

$$\text{or} \quad g(t) = \int_0^t u(l)\, dl$$

$$\therefore h(t) = \frac{1}{\Delta t}\Big[\int_0^t u(l)\, dl - \int_0^{t-\Delta t} u(l)\, dl \Big] = \frac{1}{\Delta t} \int_{t-\Delta t}^t u(l)\, dl$$

A quick example:

Find unit impulse, step, and pulse response functions of a linear reservoir.

$\Rightarrow$ A linear reservoir $\quad S = k\,\varphi$

and $\quad \dfrac{dS}{dt} = I - \varphi$

or $\quad k\dfrac{d\varphi}{dt} + \varphi = I$

i.e. $\quad \dfrac{d\varphi}{dt} + \dfrac{1}{k}\varphi = \dfrac{I}{k}$

To solve such equations:

The integrating factor $\quad e^{t/k}$

The solution is:

$$\varphi(t) = \varphi_0\, e^{-t/k} + \int_0^t \frac{1}{k} e^{-(t-\tau)/k}\, I(\tau)\, d\tau$$

where $\varphi_0 \rightarrow$ output at time $t = 0$.

Compare with convolution integral

$$\varphi(t) = \int_0^t I(\tau)\, u(t-\tau)\, d\tau$$

If you had $\varphi_0 = 0$, then

$$\int_0^t I(\tau)\, u(t-\tau)\, d\tau = \int_0^t \frac{1}{k} e^{-(t-\tau)/k}\, I(\tau)\, d\tau$$

$$\therefore \quad u(t-\tau) = \frac{1}{k} e^{-(t-\tau)/k}$$

Unit step response is: $g(t) = \int_0^t u(l)\,dl$

i.e. $g(t) = \int_0^t \frac{1}{k} e^{-l/k}\,dl$

$$= \left[-e^{-l/k}\right]_0^t = 1 - e^{-t/k}$$

The unit pulse response is:

$$h(t) = \frac{1}{\Delta t}\left[g(t) - g(t-\Delta t)\right]$$

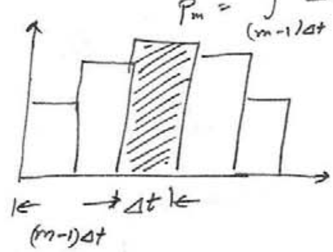
You check what will happen for $g(t-\Delta t)$ for $0 \le t \le \Delta t$

In Discrete Time The Linear System Response

We already suggested that most of hydrological observations or measurements are in discrete time. Therefore, we need to interpret the unit impulse, step, and pulse response functions in discrete time.

Q: Where do you see pulse input data?

$P_m = \int_{(m-1)\Delta t}^{m\Delta t} I(\tau)\,d\tau$



→ The rainfall or precipitation

P_m → depth of precipitation from time $(m-1)\Delta t$ to $m\,\Delta t$.

$$P_m = \int_{(m-1)\Delta t}^{m\Delta t} I(\tau)\,d\tau \quad ; \quad m = 1, 2, 3, \ldots$$

where $I(\tau)$ → rainfall intensity.

→ The streamflow at any gauging station is measured instantaneously Q_n ; $n = 1, 2, 3, \ldots$

Maybe at every Δt time, streamflow measured.

→ Inflow to the system - precipitation is measured in pulse form

→ Outflow from the system - Q_n is measured instantaneously.

Q: How will you correlate the data now?

Q_n means $Q(n\Delta t)$

$$P_m = \int_{(m-1)\Delta t}^{m\Delta t} I(\tau)\, d t \quad ; \quad (m-1)\Delta t \le t \le m\Delta t$$

Now unit pulse response function

$$h(t) = \frac{1}{\Delta t}\left[g(t) - g(t-\Delta t)\right]$$

The output time $t = n\Delta t$

We want the response at $t = n\Delta t$ when the precipitation of intensity $I(\tau)$ occurred ~~at~~ between $(m-1)\Delta t$ and $m\Delta t$ time.

$$\text{i.e.} \quad h(t-(m-1)\Delta t) = h(n\Delta t - m\Delta t + \Delta t)$$

$$= h((n-m+1)\Delta t)$$

$$= \frac{1}{\Delta t}\int_{(n-m)\Delta t}^{(n-m+1)\Delta t} u(l)\, dl \,.$$

⇒ The m^{th} input pulse (or precipitation) $= P_m = \int_{(m-1)\Delta t}^{m\Delta t} I(\tau)\, d\tau$

i.e. $$I(\tau) = \begin{cases} \dfrac{P_m}{\Delta t} & \text{from} \quad (m-1)\Delta t \le \tau \le m\Delta t \\ 0 & ; \quad \tau > m\Delta t \end{cases}$$

Say if there are a total of $m = 1, 2, 3, \ldots, M$ input pulses, then we have

$$I(\tau) = \begin{cases} \dfrac{P_m}{\Delta t} & ; \quad (m-1)\Delta t \le \tau \le m\Delta t \quad ; \quad m = 1, 2, 3, \ldots, M \\ 0 & ; \quad \tau > M\Delta t \end{cases}$$

Now $$Q_n = \int_0^{n\Delta t} I(\tau)\, u(n\Delta t - \tau)\, d\tau$$

$$= \frac{P_1}{\Delta t}\int_0^{\Delta t} u(n\Delta t - \tau)\,d\tau + \frac{P_2}{\Delta t}\int_{\Delta t}^{2\Delta t} u(n\Delta t - \tau)\,d\tau$$

$$+ \cdots + \frac{P_m}{\Delta t}\int_{(m-1)\Delta t}^{m\Delta t} u(n\Delta t - \tau)\,d\tau + \cdots + \frac{P_M}{\Delta t}\int_{(M-1)\Delta t}^{M\Delta t} u(n\Delta t - \tau)\,d\tau$$

Again for integration benefit, $l = n\Delta t - \tau$

$\therefore d\tau = -dl$

~~At~~ for any

$$\frac{P_m}{\Delta t}\int_{(m-1)\Delta t}^{m\Delta t} u(n\Delta t - \tau)\,d\tau = \frac{P_m}{\Delta t}\int_{(n-m+1)\Delta t}^{(n-m)\Delta t} -u(l)\,dl$$

$$= \frac{P_m}{\Delta t}\int_{(n-m)\Delta t}^{(n-m+1)\Delta t} u(l)\,dl$$

$$= \underline{P_m\, h((n-m+1)\Delta t)}$$

$$\therefore \; Q_n = P_1 \, h(n\Delta t) + P_2 \, h((n-1)\Delta t) + \cdots$$

$$\cdots + P_m \, h((n-m+1)\Delta t) + \cdots + P_M \, h((n-M+1)\Delta t)$$

~~which~~ This is a convolution equation

$P_m \rightarrow$ input in pulses.

$Q_n \rightarrow$ instantaneous streamflow data.