10 – March 2017

Lecture – 23

Conservation of Angular Momentum

In the last class, we discussed about the conservation of angular momentum principle.

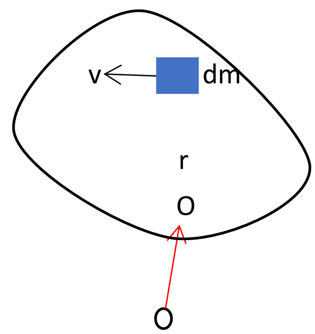
Using RTT, the angular momentum principle was given as

$$\frac{D\overrightarrow{Ho}}{Dt} = \frac{d}{dt}\iiint_{cv} (\vec{r}\times\vec{v})\rho dU + \iint_{cs} (\vec{r}\times\vec{v})\rho(\vec{v}_r.\hat{n})dA$$

Where $\vec{H}o$ is the

$$\overrightarrow{Ho} = \iiint_{system} (\vec{r}\times\vec{v})dm$$

the total angular momentum at any point O in the system

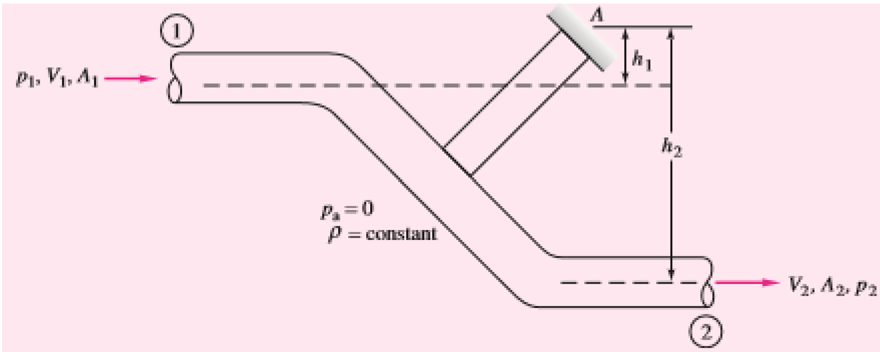


Also recall, $\frac{D\vec{H}0}{Dt} = \sum Mo$, the net moment about the point O.

$$\sum Mo = \frac{D\overrightarrow{Ho}}{Dt} = \frac{d}{dt}\iiint_{cv} (\vec{r}\times\vec{v})\rho du + \iint_{cs} (\vec{r}\times\vec{v})\rho(\vec{v}.\hat{n})dA$$

Example: - (As adopted from FM White's Fluid Mechanics)

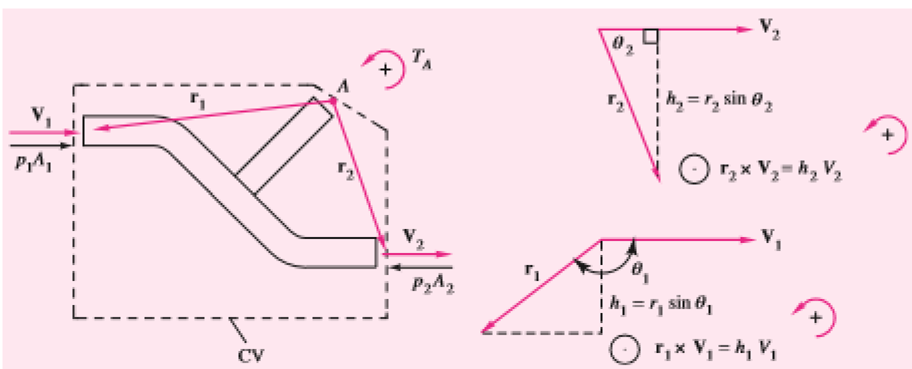
A pipe bend is supported at point A and connected to a flow system by flexible couplings at sections 1 and 2. The fluid is incompressible, and ambient pressure p_a is zero. (a) Find an expression for the torque T that must be resisted by the support at A, in terms of the flow properties at sections 1 and 2 and the distances h1 and h2. (b) Compute this torque if $D_1 = D_2 = 8$ cm, $p_1 = 0.69\times10^6$ Pa gage, $p_2 = 0.55\times10^6$ Pa gage, $V_1 = 15$ m/s, $h_1 = 5$ cm, $h_2 = 25$ cm, and $\rho = 1000$ kg/m^3.



(Source: Fluid Mechanics by F.M. White)

Solution :-

a) The control volume chosen in Fig. above given cuts through sections 1 and 2 and through the support at A, where the torque T_A is desired.
The flexible couplings description specifies that there is no torque at either section 1 or 2, and so the cuts there expose no moments.
For the angular momentum terms $\vec{r}\times\vec{v}$, $\vec{r}$ should be taken from point A to sections 1 and 2.
Note that the gage pressure forces p_1A_1 and p_2A_2 both have moments about A.



(Source: Fluid Mechanics by Frank M. White)

$\vec{r}_1$ =position vector to section 1.

$\vec{r}_2$ =position vector to section 2.

$$\sum M_A = T_A + \vec{r}_1 \times (-p_1 A_1 \hat{n}_1) + \vec{r}_2 \times (-p_2 A_2 \hat{n}_2)$$

In RTT :

$T_A + \vec{r}_1 \times (p_1 A_1 n_1) + \vec{r}_2 \times (- p_2 A_2 n_2) = (\vec{r}_1 \times \vec{V}_1)(+ m°_{out}) – (\vec{r}_2 \times \vec{V}_2)(- m°_{in})$

Figure 2 shows that all the cross products are associated with either $r_1 \sin\theta_1 = h_1$ or $r_2 \sin\theta_2 = h_2$, the perpendicular distances from point A to the pipe axes at 1 and 2.

Therefore, $\sum M_A = T_A + p_1 A_1 h_1 - p_2 A_2 h_2$

Remember that from the steady flow continuity relation. In terms of counterclockwise moments, Eq. (1) then becomes

$$T_A + p_1 A_1 h_1 - p_2 A_2 h_2 = \dot{m}\left[h_2 V_2 - h_1 V_1\right]$$

or $\quad T_A = -p_1 A_1 h_1 + p_2 A_2 h_2 + \dot{m}\left[h_2 V_2 - h_1 V_1\right]$

i.e. $\quad T_A = h_2\left(p_2 A_2 + \dot{m} V_2\right) - h_1\left(p_1 A_1 + \dot{m} V_1\right)$

The counterclockwise momentum or torque about A is expressed above.

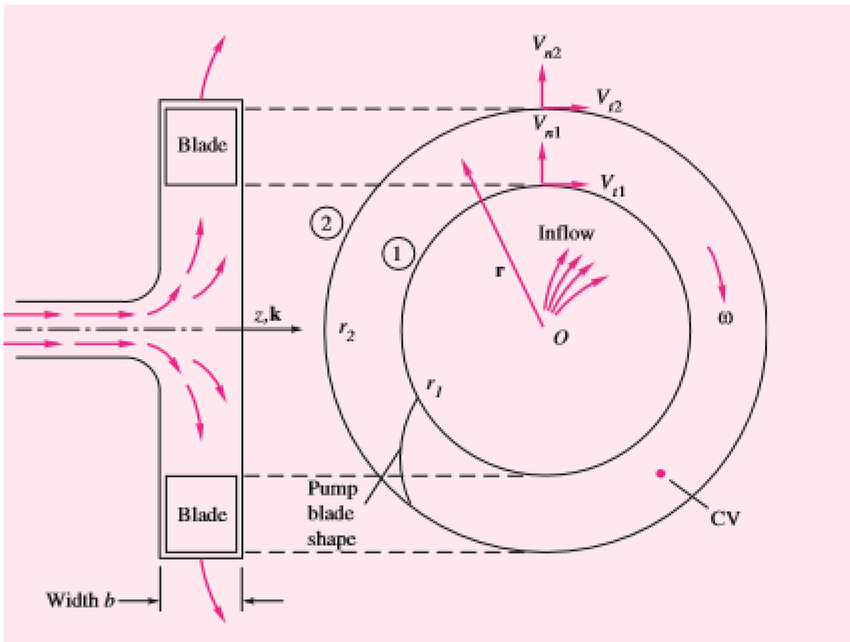
The quantities p1 and p2 are gage pressures. Note that this result is independent of the shape of the pipe bend and varies only with the properties at sections 1 and 2 and the distances h_1 and h_2.

b) $D_1 = D_2 = 8$ cm,
$p_1 = 0.69\times10^6$ Pa gage,
$p_2 = 0.55\times10^6$ Pa gage,
$V_1 = 15$ m/s,
$h_1 = 5$ cm,
$h_2 = 25$ cm and
$\rho = 1000$ kg/m^3
$T_A = 0.25 \times(.055\times10^6 \times (3.14/4) \times (0.08)^2 + m° V_2) – (0.05) \times (0.69\times10^6\times3.14/4\times (0.08)^2 + \dot{m} \times15)$

Work it out and complete yourself.

Example: - (As adopted from FM White's Fluid Mechanics)

Figure below shows a schematic of a centrifugal pump. The fluid enters axially and passes through the pump blades, which rotate at angular velocity ω; the velocity of the fluid is changed from V1 to V2 and its pressure from p1 to p2. (a) Find an expression for the torque T_o that must be applied to these blades to maintain this flow. (b) The power supplied to the pump would be P = ω*T_o. To illustrate numerically, suppose r_1 = 0.2 m, r_2 = 0.5 m, and b = 0.15 m. Let the pump rotate at 600 r/min and deliver water at 2.5 m3/s with a density of 1000 kg/m3. Compute the torque and power supplied.



(Source: Fluid Mechanics by Frank M. White)

Solution:-

a) The control volume is chosen to be the annular region between sections 1 and 2 where the flow passes through the pump blades . The flow is steady and assumed incompressible. The contribution of pressure to the torque about axis O is zero since the pressure forces at 1 and 2 act radially through O.

$$\sum Mo = To = (\vec{r}_2 \times \vec{v}_2)(+\dot{m}_{out}) - (\vec{r}_1 \times \vec{v}_1)(\dot{m}_{in})$$

where steady flow continuity tells us that

$\dot{m}_{in} = \rho v_{n1} 2\pi r_1 b = \dot{m}_{out} = \rho v_{n2} 2\pi r_2 b = \rho Q$

The cross product $\vec{r} \times \vec{v}$ is found to be clockwise about O at both sections:

$r_2 \times v_2 = r_2 v_{12} \sin 90° k = r_2 V_{t2} k$ clockwise

$r_1 \times v_1 = r_1 v_{t1} k$ clockwise

Above Equation thus becomes the desired formula for torque:

$To = \rho Q(r_2 v_{t2} - r_1 v_{t1})k$ clockwise

This relation is called Eulers' turbine formula . In an idealized pump, the inlet and outlet tangential velocities would match the blade rotational speeds $V_{t1} = \omega r_1$ and $V_{t2} = \omega r_2$. Then the formula for torque supplied becomes

$To = \rho Q\omega(r_2^2 - r_1^2)k$ clockwise

b) Convert ω to $600(2\pi/60) = 62.8$ rad/s. The normal velocities are not needed here but follow from the flow rate

$$V_{n1} = \frac{Q}{2\pi r_1 b}$$

$$V_{n2} = \frac{Q}{2\pi r_2 b}$$

For the idealized inlet and outlet, tangential velocity equals tip speed:
$V_{t1} = \omega r_1 = (62.8 \text{ rad/s})(0.2 \text{ m}) = 12.6 \text{ m/s}$
$V_{t2} = \omega r_2 = 62.8(0.5) = 31.4 \text{ m/s}$

Above Equation predicts the required torque to be
$T_o = (1000 \text{ kg/m3})(2.5 \text{ m3/s})[(0.5 \text{ m})(31.4 \text{ m/s}) - (0.2 \text{ m})(12.6 \text{ m/s})]$
$= 33000(\text{kg-m}^2)/\text{s}^2 = 33000 \text{ Nm}$

The power required is
$P = \omega To = (62.8 \text{ rad/s})(33{,}000 \text{ N m}) = 2{,}070{,}000 \text{ (N m)/s}$

$= 2.07 \text{ MW } (2780 \text{ hp})$

In actual practice the tangential velocities are considerably less than the impeller-tip speeds, and the design power requirements for this pump may be only 1 MW or less.