08/03/2017

Lecture 22

Linear Momentum Principle Through RTT

Yesterday, we discussed about the conservation of linear momentum principle through RTT.

$$\left.\frac{DB}{Dt}\right|_{system} = \frac{d}{dt}\iiint_{CV} \beta\rho dU + \iint_{CS} \beta\rho(\vec{v}_r.\hat{n})dA$$

For linear momentum,

$$\left.\frac{D(m\vec{v})}{Dt}\right|_{system} = \sum\vec{F} = \frac{d}{dt}\iiint_{CV} \vec{v}\rho dU + \iint_{CS} \vec{v}\rho(\vec{v}_r.\hat{n})dA$$

- We worked on a couple of examples.
  Today, we will see one more example on the topic.

Example:(Adopted from F.M. White's Fluid Mechanics-Text Book)
A sluice gate controls the flow in open channels. The flow is uniform at sections 1 and 2. Pressure of liquid is assumed as hydrostatic. Neglect bottom friction and atmospheric pressure. Derive a formula for the horizontal force F required to hold the gate.
Soln: Given the figure of sluice gate problem:

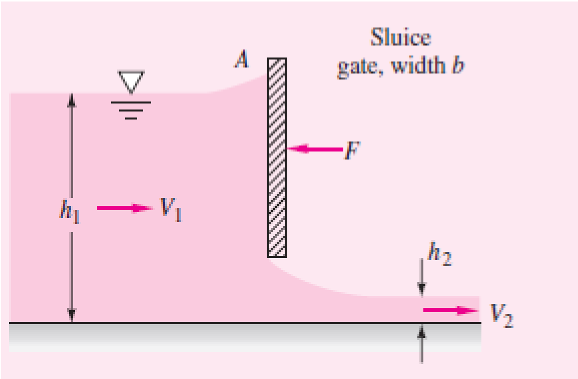

Fig.1: Problem Figure
(Source: Fluid Mechanics by Frank White)

- We need to appropriately choose the control volume.
- Sections 1 and 2 have uniform flow. So we include those sections as part of control surfaces.

- As the flow is uniform, it automatically implies the flow upstream and downstream are steady.
- Assuming the coordinate system and $\vec{v} = u\hat{e}_1 + v\hat{e}_2$

  Conservation of mass equation suggests :

  $$\iint_{CS} \rho(\vec{v}.\hat{n})\,\mathrm{dA} = 0 \quad \text{for steady state flow}$$

  i.e., $\rho u_2 b h_2 - \rho u_1 b h_1 = 0$ (b is width of section into the paper)

  $$\dot{m}_2 - \dot{m}_1 = 0$$

  $$u_2 = u_1 \frac{h_1}{h_2}$$

- Pressure is considered hydrostatic at section 1 and 2.
  - ⇨ $P_{1b} = p_{atm} + \rho g h_1$

    Neglecting atmospheric pressure, $p_{1b} = \rho g h_1$

    Similarly, at section 2, $p_{2b} = \rho g h_2$

- The force on the sluice gate wall should be horizontal, so as to balance the hydrostatic and the other forces:

  $$\sum F_x = \frac{d}{dt}\iiint_{CV} u\rho dU + \iint_{CS} u\rho(\vec{v}_r.\hat{n})dA$$

  Since, $$\frac{d}{dt}\iiint_{CV} u\rho dU = 0$$

  $$\sum F_x = u_2\rho(\vec{v}_2.\hat{n})bh_2 - u_1\rho(\vec{v}_1.\hat{n})bh_1$$

  Recall, $$\vec{v}_2 = u_2\hat{e}_1 + 0\hat{e}_2$$
  $$\vec{v}_1 = u_1\hat{e}_1 + 0\hat{e}_2$$

  $$\sum F_x = \rho u_2^2 b h_2 - \rho u_1^2 b h_1$$

  $$\sum F_x = \rho g \frac{h_1}{2} b h_1 - \rho g \frac{h_2}{2} b h_2 - F_{gate}$$

  $$i.e. \rho g b(\frac{h_1^2}{2} - \frac{h_2^2}{2}) - \mathrm{F}_{gate} = \rho b(u_2^2 h_2 - u_1^2 h_1)$$

  $$\frac{\rho g b}{2}(\mathrm{h}_1 - \mathrm{h}_2)(\mathrm{h}_1 + \mathrm{h}_2) - \mathrm{F}_{gate} = \rho b(h_2 u_2^2 - h_1 u_1^2)$$

  $$\mathrm{F}_{gate} =$$

  Substitute the relation $u_2 = u_1 \frac{h_1}{h_2}$ to get the final expression for F_{gate} .

Example:

A liquid jet of velocity V_j and area A_j strikes a single 180^0 bucket on a turbine wheel rotating at angular velocity ω . Derive an expression for the power P delivered to this wheel at this instant as a function of the system parameters. At what angular velocity is the maximum power obtained?

Soln: As like previous analyses, we need to choose appropriate control volume as shown below.

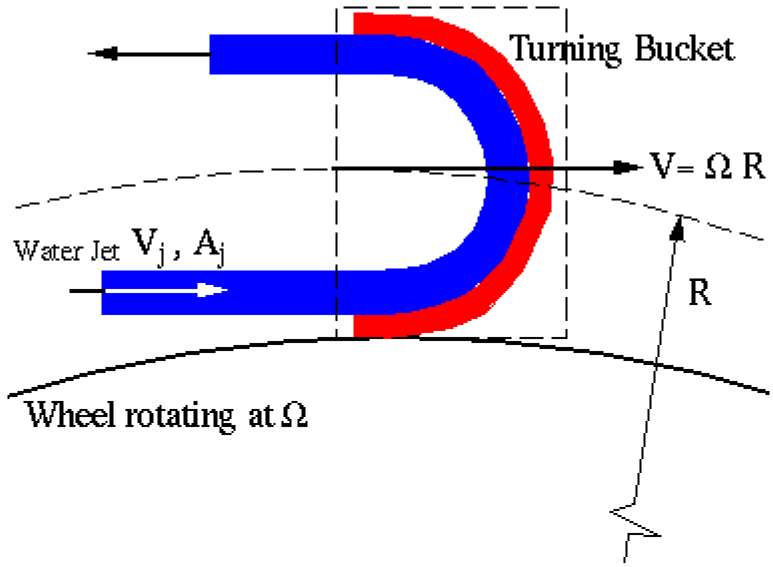

Fig. 2: A 180^0 jet bucket. The control volume is shown dotted.

(Image Source: http://www.mne.psu.edu/cimbala/Learning/Fluid/CV_Momentum/home.htm)

You know velocity v = Radius x Angular velocity

= R x ω

Also, inlet : $\vec{v}_j = v_j\hat{e}_1 + 0\hat{e}_2$

Outlet: $\vec{v}_j = v_j\hat{e}_1 + 0\hat{e}_2$

Liquid jet enters control volume at speed v_j

The bucket moves right at a speed v = R ω

⇨ The relative speed of jet entering the control volume = $v_j - v = v_j - R\omega$

Apply mass conservation first, we get, $\dot{m}_1 = \dot{m}_2 = \dot{m} = \rho A_j(v_j - R\omega)$

Applying momentum principle in steady state conditions:

$$\sum F_x = -F_{bucket} = \dot{m}u_{out} - \dot{m}u_{in}$$
$$= \dot{m}[-(v_j - R\omega)] - \dot{m}[v_j - R\omega]$$
$$= 2\dot{m}[v_j - R\omega] = 2\rho A_j [v_j - R\omega]^2$$
$$P = F_{bucket} * Velocity$$
$$= 2\rho A_j [v_j - R\omega]^2 * R\omega$$

Maximum power will be felt for the angular velocity ω, as such

$\frac{dP}{d\omega} = 0$ which gives finally $\frac{dP}{d\omega} = 2\rho A_j [R(v_j - R\omega)^2 + R\omega.2(v_j - R\omega)(-R)] = 0$

For $\frac{dP}{d\omega} = 0$

$$R(v_j - R\omega)(v_j - R\omega) = 2R^2\omega(v_j - R\omega)$$
$$2R\omega = v_j - R\omega$$
$$3R\omega = v_j$$
$$R\omega = \frac{v_j}{3}$$

Conservation of Angular Momentum

If in the Reynolds Transport Theorem, the extensive property B is taken as angular momentum, then B = $\vec{H}$, the angular momentum.

Recall, angular momentum at any point O can be written as :

$$\vec{H}_0 = \int_{system} (\vec{r} \times \vec{v}) dm$$

Where $\vec{r}$ is the position vector from O to elemental mass dm, $\vec{v}$ is the velocity of that element.

- The angular momentum per unit mass that is β is given by:

  $$\beta = \frac{d\vec{H}_0}{dm} = \vec{r} \times \vec{v}$$

- The RTT will be :

$$\left.\frac{D\vec{H}_0}{Dt}\right|_{system} = \frac{d}{dt}[\iiint_{CV}(\vec{\mathrm{r}}\times\vec{\mathrm{v}})\rho\,\mathrm{dU}] + \iint_{CS}(\vec{\mathrm{r}}\times\vec{\mathrm{v}})\rho(\vec{v}_r.\hat{\mathrm{n}})\,\mathrm{dA}$$

- The rate of change of angular momentum of a system should be equal to the sum of all ***moments*** about the point O in that system.

$$\frac{D\vec{H}_0}{Dt} = \sum M_0 = \frac{d}{dt}[\iiint_{CV}(\vec{\mathrm{r}}\times\vec{\mathrm{v}})\rho\,\mathrm{dU}] + \iint_{CS}(\vec{\mathrm{r}}\times\vec{\mathrm{v}})\rho(\vec{v}_r.\hat{\mathrm{n}})\,\mathrm{dA}$$