22/02/2017

LECTURE – 19

Reynold's Transport Theorem

Yesterday, we discussed about

- ✓ Material derivative of any arbitrary property $P_{ij}(\vec{x},t)$,where the property is described in Eulerian coordinate.
- ✓ Material derivative of any enclosed volume of fluid (or continuum) , i.e.,
  $$\frac{D(dU)}{Dt} = \frac{\partial v_p}{\partial x_p} dU$$
- ✓ Subsequently, for an arbitrary control volume (say U), any property $B_{ij....}(t) = \iiint_U \gamma_{ij....}(\vec{x},t)dU$,which is the volumetric integral of $\gamma_{ij....}(\vec{x},t)$ in that volume, the material derivative was formulated:

$$\frac{D}{Dt}[B_{ij....}(t)] = \frac{D}{Dt}[\iiint_U \gamma_{ij....}(\vec{x},t)dU]$$

$$= \iiint_U \frac{D\gamma_{ij....}(\vec{\mathrm{x}},\mathrm{t})\,\mathrm{dU}}{Dt} + \gamma_{ij....}(\vec{x},t)\frac{D(dU)}{Dt}]$$

$$= \iiint_U [\frac{\partial \gamma_{ij....}(\vec{x},t)}{\partial t} + v_p \frac{\partial \gamma_{ij....}(\vec{x},t)}{\partial x_p}]dU + \iiint_U \gamma_{ij....}(\vec{x},t)\frac{\partial v_p}{\partial x_p}dU$$

$$= \iiint_U \frac{\partial \gamma_{ij....}(\vec{x},t)}{\partial t} + \frac{\partial(\gamma_{ij....}(\vec{x},t)}{\partial x_p} v_p]dU$$

Recall the Gauss theorem & applying that on the second volumetric integral

i.e, $$\iiint_V \frac{\partial}{\partial t}[\gamma_{ij....}(\vec{x},t)]dU = \iint_A \gamma_{ij....}(\vec{\mathrm{x}},\mathrm{t})\,\mathrm{v}_p\, n_p dA$$

Where, $\hat{n}$ is unit outward normal vector.

$$\frac{D}{Dt}[B_{ij....}(t)] = \iiint_U \frac{\partial}{\partial t}[\gamma_{ij....}(\vec{x},t)]dU + \iint_A \gamma_{ij....}(\vec{x},t)v_p\,\mathrm{n}_p\, dA \quad \text{......................}(1)$$

This is same as the Reynold's Transport Theorem, we studied earlier connecting Lagrangian system with Eulerian control volume.

$$\left.\frac{dB}{dt}\right|_{system} = \frac{d}{dt}\iiint_{CV} \beta\rho dU + \iint_{CS} \beta\rho(\vec{v}.\hat{n})dA$$

Please note that the first integral of R.H.S i.e., $\frac{d}{dt}\iiint_{CV} \beta\rho dU$ for the control volume that remain as it is $= \iiint_{CV} \frac{\partial}{\partial t}(\beta\rho)dU$

$$\left.\frac{dB}{dt}\right|_{system} = \iiint_{CV} \frac{\partial}{\partial t}(\beta\rho)dU + \iint_{CS} \beta\rho(\vec{v}.\hat{n})dA \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

You can also correlate now $B_{ij\ldots}(t)$ same as B in (2)

$\gamma_{ij\ldots.}(\vec{x},t) \rightarrow$ same as $\beta\rho$ in (2)

<u>Application of Reynold's Transport Theorem to various Conservation principles:</u>

1. **<u>Conservation of Mass</u>:**

The Reynold's Transport theorem

$$\frac{D}{Dt}[B_{ij\ldots.}(t)] = \iiint_{U} \frac{\partial}{\partial t}[\gamma_{ij\ldots.}(\vec{x},t)]dU + \iint_{A} \gamma_{ij\ldots.}(\vec{x},t)v_p \, \mathrm{n}_p \, dA$$

or

$$\left.\frac{dB}{dt}\right|_{system} = \iiint_{CV} \frac{\partial}{\partial t}(\beta\rho)dU + \iint_{CS} \beta\rho(\vec{v}.\hat{n})dA$$

Let B = extensive property = mass of fluid in the control volume

$B = \iiint_{CV} \rho dU = m$ so, $\beta = 1$.

$$\left.\frac{dB}{dt}\right|_{system} = \frac{Dm}{Dt} = 0 = \iiint_{CV} \frac{\partial\rho}{\partial t}dU + \iint_{CS} \rho(\vec{v}.\hat{n})\,\mathrm{dA}$$

For steady flow, $\frac{\partial\rho}{\partial t} = 0$

$$\iint_{CS} \rho(\vec{v}.\hat{n})\, dA = 0$$

i.e., the net outflow through the control volume is zero.

If in the C.V, the fluid in consideration is same & incompressible, as well as the inlets/outlets on the volume are one dimensional.

$$\iint_{CS} \rho(\vec{v}.\hat{n})\, dA = 0$$

$$\iint_{CS} (\vec{v}.\hat{n})\, dA = 0 = \iint_{Outlet} (\vec{v}.\hat{n})\, dA - \iint_{inlet} (\vec{v}.\hat{n})\, dA$$

Or, $\sum_{inlet} A_i V_i = \sum_{outlet} A_j V_j$

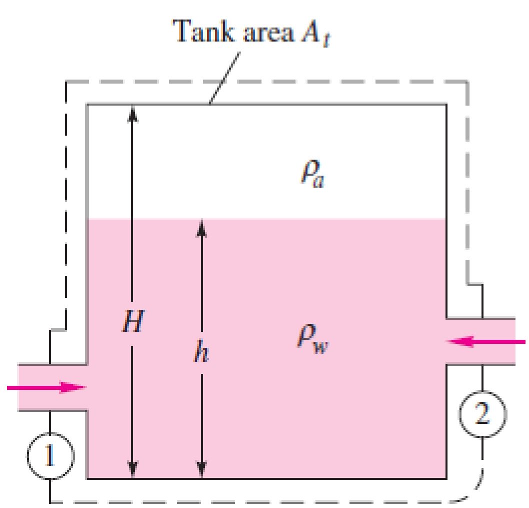
Or, in index notation: $A_i V_i = A_j V_j$

Example: (As adopted from FM White's Fluid Mechanics)

A tank closed at the top is filled with water by two one dimensional inlets. The water height is 'h' & the air is trapped at top.

a) Find the expression for $\frac{dh}{dt}$

b) Compute $\frac{dh}{dt}$ if the diameter at section I is D_I = 3 cm, V_I = 1.5 m/sec & at section 2, D_2 = 8 cm , V_2 = 1.0 m/sec . The tank top area is A_t = 0.25 m^2. Water is at 20°C.



(Source: Fluid Mechanics by F.M. White)

Solution:

We are taking the blue dotted line as part of control surface of the control volume.

As the blue dotted lines cover the tank in entirely, you can see that the tank volume is not going to change with respect to time.

However, the height of water 'h' increases as water enters through the two inlets 1 & 2.

Let us apply conservation of mass principle to the control volume. The control volume consist of mass of water & mass of air. i.e., B = mass in control volume.

$$\left.\frac{DB_{ij....}}{Dt}\right|_{system} = 0 = \iiint_U \frac{\partial}{\partial t}(\gamma_{ij....})dU + \iint_A \gamma_{ij....} v_p \, \mathrm{n}_p \, dA$$

In this case, the time rate of change in mass of air trapped in the control volume will be

$$\frac{\partial}{\partial t}\iiint_{CV} \gamma_{ij...} dU \approx \frac{d}{dt}[\rho_a A_t (H-h)] = 0$$

However, for water, $\frac{\partial}{\partial t}\iiint_{CV} \gamma_{ij...} dU \approx \frac{d}{dt}[\rho_w A_t h] = \rho_w A_t \frac{dh}{dt}$

The RTT gives, $0 = \rho_w A_t \frac{dh}{dt} - \rho_w A_I v_I - \rho_w A_{II} v_{II}$

Recall, water is incompressible.

$$A_t \frac{dh}{dt} - A_I v_I - A_{II} v_{II} = 0$$

$$\frac{dh}{dt} = \frac{1}{A_t}[A_I v_I + A_{II} v_{II}]$$

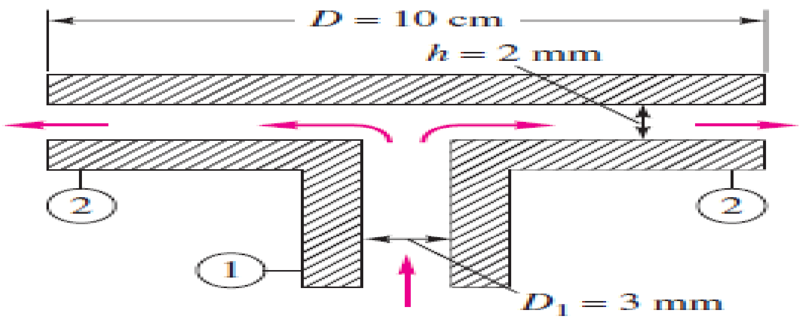
Given, $A_t = 0.25 \text{ m}^2 = 2500 \text{ cm}^2$

$A_1 = \frac{\pi}{4} * 3^2 = 7cm^2 \quad A_2 = \frac{\pi}{4} * 8^2 = 50.3cm^2$

$$\frac{dh}{dt} = \frac{1}{2500}[7*150 + 50.3*100] = 2.432 cm/s$$

<u>Quiz</u>:

Oil of specific gravity 0.89 enters at section 1 with inflow of 240 kg/sec. The steady oil flow exits at two cylindrical outlets. Determine the outlet volume flux in mL/sec.



(Source: Fluid Mechanics by F.M. White)