21/02/2017

LECTURE 18

Material Derivative (Contd…..)

Recall, in the last class, we started discussing about material derivative.

For any scalar, vectorial or tensorial quantity P_{ij} ….., the material derivative was given by:

$$\frac{DP_{ij....}(\vec{x},t)}{Dt} = \frac{\partial P_{ij....}(\vec{x},t)}{\partial t} + v_k \frac{\partial P_{ij....}(\vec{x},t)}{\partial x_k}$$

The first term indicates **local rate change** & second term indicates **convective rate change**.

- ✓ Also recall, the Lagrangian strain tensor & Eulerian Strain tensor.
- ✓ In a similar way, if the spatial gradient of velocity $\vec{v}(\vec{x},t)$ is taken, it will provide you velocity gradient tensor $\frac{\partial vi}{\partial xj}$
- ✓ Using tensor property :

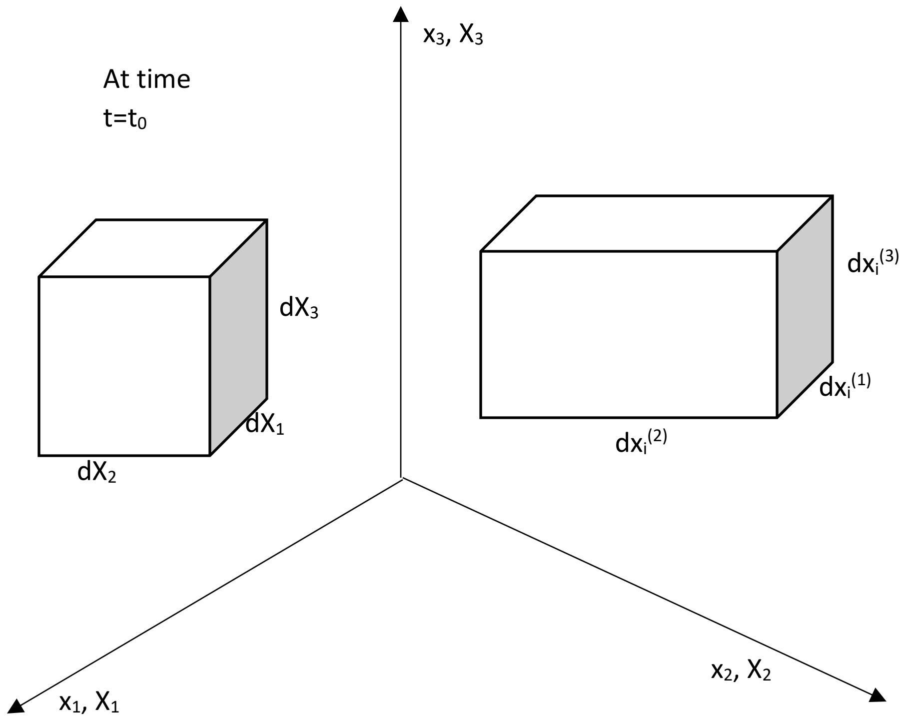
$$\frac{\partial v_i}{\partial x_j} = \frac{1}{2}\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right] + \frac{1}{2}\left[\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right]$$

1st part is symmetric tensor & 2nd part is anti-symmetric or skew symmetric tensor.

- ✓ The symmetric tensor is the rate of deformation tensor or it is the strain rate tensor.
- ✓ The skew symmetric tensor is called Vorticity tensor or spin tensor.
- ✓ So, the velocity gradient term consist of strain rate & rotational rate.
- ✓ Later on you will see that, if the vorticity tensor is zero, the corresponding velocity flow field is irrotational.

Material Derivative of Volume:

Consider a fluid continuum to be in motion. In that fluid continuum, consider an elementary volume having initial configuration as such:



The elementary configuration's sides are $dX_1\hat{e}_1$, $dX\hat{e}_2$, $dX_3\hat{e}_3$. The box product is given the volume:

$$dU_0 = dX_1\hat{e}_1 \times dX_2\hat{e}_2 . dX_3\hat{e}_3$$

Volume is parallelepiped is $dU_0 = dX_1 dX_2 dX_3$

- ✓ As the fluid is moving, this parallelepiped is displaced & deformed. Let the deformed volume at time t = t be as given in the figure (for our convenience, we are right now assuming it to be parallelepiped).
- ✓ Unlike in the initial configuration, in the later configuration, the sides will be vectors having components in all three directions. The sides are: $d\vec{x}^{(1)}, d\vec{x}^{(2)}, d\vec{x}^{(3)}$

$$dx_i = \frac{\partial x_i}{\partial X_j} dX_j$$

$$dx_i^{(1)} = \frac{\partial x_i^{(1)}}{\partial X_j} dX_j$$

$$dx_i^{(2)} = \frac{\partial x_i^{(2)}}{\partial X_j} dX_j$$

$$dx_i^{(3)} = \frac{\partial x_i^{(3)}}{\partial X_j} dX_j$$

- ✓ The volume of the deformed parallelepiped will be

$$dU = d\vec{x}^{(1)} \times d\vec{x}^{(2)} . d\vec{x}^{(3)}$$

$$= \varepsilon_{ijk} d\vec{x}_j^{(1)} d\vec{x}_k^{(2)} d\vec{x}_i^{(3)}$$ [Box product]

As the original initially configured volume had edges dX_1, dX_2, dX_3 & there are particles associated with these edges, we assume in the distorted configuration the same particles occupy the edges $dx^{(1)}$, $dx^{(2)}$, $dx_i^{(3)}$ respectively.

If so, then,

$$dx_i^{(1)} = \frac{\partial x_i}{\partial X_1} dX_1$$

$$dx_i^{(2)} = \frac{\partial x_i}{\partial X_2} dX_2$$

$$dx_i^{(3)} = \frac{\partial x_i}{\partial X_3} dX_3$$

Therefore the distorted configured volume

$$dU = \varepsilon_{ijk} dx_i^{(1)} dx_j^{(2)} dx_k^{(3)}$$

$$dU = \varepsilon_{ijk} \frac{\partial x_i}{\partial X_1} \frac{\partial x_j}{\partial X_2} \frac{\partial x_k}{\partial X_3} dX_1 dX_2 dX_3$$

$$\varepsilon_{ijk} \frac{\partial x_i}{\partial X_1} \frac{\partial x_j}{\partial X_2} \frac{\partial x_k}{\partial X_3} = J = \left| \frac{\partial x_i}{\partial X_j} \right| \qquad (\text{The Jacobian})$$

$$\mathrm{d}U = J dU_0$$

Hence, the material derivative of the volume

$$\frac{D(dU)}{Dt} = \frac{D(JdU_0)}{Dt} = \frac{DJ}{Dt} dU_0$$

(You can easily guess that $\frac{D(dU_0)}{Dt} = 0$)

From further operations, it is proved that

$\frac{DJ}{Dt}$ = divergence of velocity vector × Jacobian,

i.e. $\frac{DJ}{Dt} = \frac{\partial v_p}{\partial x_p}$

Therefore,
$$\frac{D(dU)}{Dt} = J\frac{\partial v_p}{\partial x_p} dU_0$$
$$\frac{D(dU)}{Dt} = \frac{\partial v_p}{\partial x_p} dU$$

<u>Material Derivative of property in a Volume</u>:

You know that mass is a property associated with volume. If the density of the fluid is ρ then mass is $M = \int_U \rho dU$

i.e. You need to volumetrically integrate density to get mass.

Similarly, there may be several properties in a closed continuum (fluid), where these properties are obtained by volumetric integration.

Let us say that any property (scalar or Vector or Tensor) B_{ij} … is obtained by volumetric integration.

i.e, $B_{ij....}(t) = \iiint_U \beta_{ij....}(\vec{x},t) dU$

(**Note**: here we are not saying function of $\vec{x}$ & t. We say only function of t. this is because volumetric integration done.

The material derivative of the property B_{ij}….. will be

$$\frac{DB_{ij....}(t)}{Dt} = \frac{D}{Dt}[\iiint_U \beta_{ij....}(\vec{x},t)dU]$$
$$= \iiint_U \frac{D}{Dt}(\beta_{ij....}(\vec{x},t)dU)$$
$$= \iiint_U [\frac{D}{Dt}\beta_{ij....}(\vec{x},t)dU + \beta_{ij....}(\vec{x},t)\frac{D}{Dt}(dU)]$$
$$= \iiint_U [\frac{\partial}{\partial t}\beta_{ij....}(\vec{x},t) + \mathrm{v}_p \frac{\partial}{\partial x_p}\beta_{ij....}(\vec{x},t)]dU + \iiint_U \beta_{ij....}(\vec{x},t)\frac{\partial v_p}{\partial x_p}dU$$
$$= \iiint_U [\frac{\partial}{\partial t}\beta_{ij....}(\vec{x},t) + \mathrm{v}_p \frac{\partial}{\partial x_p}\beta_{ij....}(\vec{x},t) + \beta_{ij....}(\vec{x},t)\frac{\partial v_p}{\partial x_p}]dU$$
$$= \iiint_U [\frac{\partial}{\partial t}\beta_{ij....}(\vec{x},t) + \frac{\partial}{\partial x_p}(v_p\beta_{ij....}(\vec{x},t))]dU$$

Applying Gauss Divergence theorem on the second portion of the volumetric integral, you get,

$$\frac{DB_{ij....}(t)}{Dt} = \iiint_U [\frac{\partial}{\partial t}\beta_{ij....}(\vec{x},t)]\mathrm{dU} + \iint_A v_p n_p \beta_{ij....}(\vec{x},t)dA$$

This equation is very similar to the Reynold's Transport theorem expression you had studied.

Acknowledgement:

The concept of deriving **material derivative of volume** is referred from the following text:

George E. Mase (1970). "Continuum Mechanics". Schaum's Outline Series.