18-FEB-2017

LECTURE 17

Material Derivative

Yesterday, we were discussing about the

- Lagrangian form of expressing position vector
  $x_i = x_i (X_1, X_2, X_3,t)$
- Eulerian form of expressing equation
  $X_i = X_i (x_1, x_2, x_3, t)$
- Also recall O X_1 X_2 X_3 –Material derivative
  o x_1 x_2 x_3- Spatial derivative
- Subsequently we briefly mentioned (although not essential for this course)
- We were discussing about deformation tensor:
  - We considered two neighbouring particles P_0 and Q_0 in material coordinates (i.e. position at time t= t_0)
  - The square of the length between P_0 and Q_0 was:

$$(dX)^2 = dX_i dX_i$$
$$= \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j} dx_i dx_j$$
$$= C_{ij} dx_i dx_j$$

Where C_{ij} is Cauchy's deformation tensor.

- In the deformed configuration, where the same particles are at the position P and Q at time t, the square of the distance between them is:

$$(dx)^2 = dx_i dx_i$$
$$dx_i = \frac{\partial x_i}{\partial X_j} dX_j$$
$$(dx)^2 = dx_i dx_i = \left[\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j}\right] dX_i dX_j$$
$$(dx)^2 = G_{ij} dX_i dX_j$$

Where G_{ij} is the Green's deformation tensor

- As mentioned yesterday, it is the difference between dx and dX that describes the deformation in the continuum.
- Therefore the measure of deformation is:

$$(dx)^2 - (dX)^2 = \left[\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j}\right] dX_i dX_j - dX_i dX_i$$

$$= \left[\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right] dX_i dX_j$$

$$= 2L_{ij} dX_i dX_j$$

(Please remember $\delta_{ij}\, dX_i\, dX_j = dX_i dX_j$)

Where L_{ij} is the Lagrangian finite strain tensor

$$L_{ij} = \frac{1}{2}\left[\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right]$$

You know, what is meant by strain, etc.

- In a similar way, Eulerian strain tensor can also be defined as:

  $$E_{ij} = \frac{1}{2}\left[\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right]$$

  And $(dx)^2 - (dX)^2 = 2\, G_{ij}\, dx_i\, dx_j$
- If $(dx)^2 - (dX)^2 = 0$, it represents rigid body motion.

Fluid Kinematics

Any fluid property can be described in Lagrangian or Eulerian form.

- For example, density of fluid in material description will be:

  $\rho = \rho\,(X_1, X_2, X_3, t) = \rho\,(X_i, t)$
- It will be the density of the fluid particle at the material position (X_1, X_2, X_3, t).
- If someone wants to describe density of the fluid in Eulerian form:

  $\rho = \rho\,(X_i\,(x_1, x_2, x_3, t)\,, t) = \rho\,(Xi(x_j, t), t) = \rho^*\,(x_i\,, t)$

  The star superscript is to show that the material is not same.

The Material Derivative

You know that the fluid continuum encompasses several particles.

- The properties of the particles can change with respect to time.
- The time rate of change of any property of the fluid (continuum) with respect to the particles is called the material derivative of the property. (Also called substantial derivative)
- For example, if we consider the instantaneous position of a particle as a property

$x_i = x_i (X_1, X_2, X_3, t) = x_i (X_j, t)$

The material derivative is given by $\frac{D}{Dt}$

$\frac{Dx_i}{Dt} = v_i$ - The velocity vector.

Note: $\frac{Dx_i}{Dt} = \frac{D\ (s_i + X_i)}{Dt} = \frac{Ds_i}{Dt} = v_i$

$\frac{DX_i}{Dt} = 0$ Always (Why??)

- For any general property – scalar or vector, or tensor (say $P_{ij....}$)
- We can represent it in Lagrangian form $P_{ij....} = P_{ij....} (X_1, X_2, X_3, t)$

$$\frac{DP_{ij....}}{Dt} = \frac{\partial P_{ij....}(X_i,t)}{\partial t}$$

Because by this time you know $\frac{DX_i}{Dt} = 0$

- If the property $P_{ij....}$ Is represented in Eulerian form, i.e. $P_{ij....} = P_{ij....} (x_i, t)$

$$\frac{DP_{ij....}(x_i,t)}{Dt} = \frac{\partial P_{ij....}(x_i,t)}{\partial t} + \frac{\partial P_{ij....}}{\partial x_k} \frac{Dx_k}{Dt}$$

$$\frac{DP_{ij....}}{Dt} = \frac{\partial P_{ij....}}{\partial t} + v_k \frac{\partial P_{ij....}}{\partial x_k}$$

$\frac{\partial P_{ij....}}{\partial t}$ = Local rate of change and $v_k \frac{\partial P_{ij....}}{\partial x_k} =$ Convective rate of change

- For defining acceleration, if $v_i = v_i(X_j, t)$ (Lagrangian)

Then:

$\frac{Dv_i}{Dt} = a_i = \frac{\partial v_i}{\partial t}$

If velocity is expressed in Eulerian form

i.e. $v_i = v_i\,(x_j, t)$

Then, $\frac{Dv_i}{Dt} = a_i = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}$

$\frac{\partial v_i}{\partial t}$ =Local acceleration and $v_j \frac{\partial v_i}{\partial x_j}$ =Convective acceleration

- We can also form partial derivatives of instantaneous velocity with respect to spatial coordinates.

This again gives a second rank tensor i.e. $\frac{\partial v_i}{\partial x_j}$

- Recall earlier, we told that any tensor can be described in symmetric and anti-symmetric form:

i.e. $\frac{\partial v_i}{\partial x_j} = \frac{1}{2}\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right] + \frac{1}{2}\left[\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right]$

$= D_{ij} + V_{ij}$

Where D_{ij} – the symmetric tensor is called rate of deformation tensor or strain rate tensor

$= \frac{1}{2}\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right]$

V_{ij} – The skew- symmetric tensor is called vorticity tensor or spin tensor

$= \frac{1}{2}\left[\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right]$

- You can derive on your own that rate of deformation tensor is material derivative of Eulerian linear strain tensor.