Green Gauss Theorem

Yesterday, we continued discussions on index notations that can be used to represent tensors.

In index notations we saw the

- Gradient of a scalar
- Divergence of a vector
- Cross product of vector etc.

We also saw that a second rank tensor can be written as a sum of a symmetric & asymmetric tensor.

\[ B_{ij} = \frac{1}{2} [B_{ij} + B_{ji}] + \frac{1}{2} [B_{ij} - B_{ji}] \]

First part is symmetric & second part is asymmetric

Today we will briefly discuss about Green Gauss Theorem etc.

Green Gauss Theorem relates the volume & surface integrals.

- The most common form of Gauss’s theorem is the Gauss divergence theorem which you have studied in previous courses.
If you have a volume $U$ formed by surface $S$, the Gauss divergence theorem for a vector $\vec{v}$ suggest that

$$\oiint_{\bar{S}} \vec{v} \hat{n} dS = \iiint_{U} \nabla \vec{v} dU$$

where, $S =$ bounding surface on closed surface

$U =$ volumetric domain

$\hat{n} =$ the unit outward normal vector of the elementary surface area $ds$

- This Gauss Divergence theorem can also be represented using index notations.

$$\oiint_{\bar{S}} v_i n_i dS = \iiint_{U} \frac{\partial v_i}{\partial x_i} dU$$

For any scalar multiple or factor for vector $\vec{v}$, say $\alpha \vec{v}$, the Green Gauss can be represented as

$$\oiint_{\bar{S}} (\alpha \vec{v}) \hat{n} dS = \iiint_{U} \nabla (\alpha \vec{v}) dU$$

i.e. $\oiint_{\bar{S}} (\alpha \vec{v}) \hat{n} dS = \iiint_{U} \alpha (\nabla \vec{v}) dU + \iiint_{U} (\vec{v} \nabla) \alpha dU$

or $\oiint_{\bar{S}} \alpha v_i n_i dS = \iiint_{U} \alpha \frac{\partial v_i}{\partial x_i} dU + \iiint_{U} v_i \frac{\partial \alpha}{\partial x_i} dU$

Gauss’s theorem is applicable not only to a vector $\vec{v}$. It can be applied to any tensor $\bar{B}$ (say)

$$\oiint_{\bar{S}} \alpha B_{ij} n_i dS = \iiint_{U} \alpha \frac{\partial B_{ij}}{\partial x_i} dU + \iiint_{U} B_{ij} \frac{\partial \alpha}{\partial x_i} dU$$
**Stoke’s Theorem**

Stoke’s theorem relates the integral over an open surface \( S \), to line integral around the surface’s bounding curve (say \( C \)).

You need to appropriately choose the unit outward normal vector to the surface.

\[
\iiint_S (\nabla \times \vec{v}) \cdot \hat{n} dS = \oint_C \vec{v} \cdot d\vec{r}
\]

\[
i.e. \iiint_S \varepsilon_{ijk} \frac{\partial v_k}{\partial x_j} \hat{n} dS = \oint_C v_i t_i dr
\]
**Kinematics of Fluid Flow**

- You know kinematics is the study of motion without reference to the forces or stresses that produce that motion.
- You have also seen the concept of fluid particles & that the collection of particles form the continuum of the fluid.
- You have also seen Lagrangian & Eularian description of the flow field.
- Lagrangian description is based on the motion of fluid particles.

The path of the particle is given in the $X_1, X_2, X_3$ coordinate system (as drawn).

Initially the position vector of the particle was $\vec{r}_0(t_0)$

Therefore $\vec{r}_0$ is a known as referred value.

Based on the particle at $\vec{r}_0$ at time $t_0$, that particle will be having positions at different times.

$$\vec{r} = \vec{r}(\vec{r}_0, t)$$

- The velocity of the field particle will be $\bar{v} = \frac{d\vec{r}(\vec{r}_0, t)}{dt}$
- The acceleration of the particle will be $\bar{a} = \frac{d^2\vec{r}(\vec{r}_0, t)}{dt^2}$
- This descriptions of velocity & acceleration (i.e, the Lagrangian approach) is same as that you has studied in mechanics
Any property $F = F(\vec{r}(\vec{r}_0, t), t)$

- As was suggested earlier, it may be cumbersome to describe fluid motion through Lagrangian approach.
- Moreover, the medium is getting completely deformed during the motion.
- In Eularian approach, we need to interpret velocity & acceleration vectors.
- In Eularian description, we will involve the Eularian vectors $x$ & time $t$.
- Any fluid property (say $F$), depends directly on $\vec{x}$ & $t$.
  
  $$F = F(\vec{x}, t)$$

- As we are not following individual fluid particles, we have to relate Lagrangian & Eularian descriptions.
- At any snapshot time ‘$t$’, if we merge the Lagrangian & Eularian coordinate, then
  
  $$F(\vec{r}(t, \vec{r}_0, t_0), t) = F(\vec{x}, t)$$

  Where $x = \vec{r}(t, \vec{r}_0(t_0))$