14-02-2017

LECTURE 14

Index Notations (Contd….)

In the last class, we introduced you the index notations for vectors, tensors, etc.

- Say e.g. a vector $\vec{v}$ was given as

$$\vec{v} \rightarrow v_i \,, \quad \vec{F} \rightarrow F_i \,, \quad \text{etc.}$$

Tensor $\quad \overline{\overline{\tau}} \rightarrow \tau_{ij}$

- We described about free index and dummy index.
- Also described about index notations for various operations.

e.g. $\quad \vec{\nabla} p \rightarrow \dfrac{\partial p}{\partial x_i}$,

$$\vec{\nabla}.\vec{v} \rightarrow \frac{\delta v_i}{\delta x_i}, \quad \text{etc.}$$

- Recall, the Kronecher delta

$$\delta_{ij} = \begin{cases} 1 & for \ \ i = j \\ 0 & for \ \ i \neq j \end{cases}$$

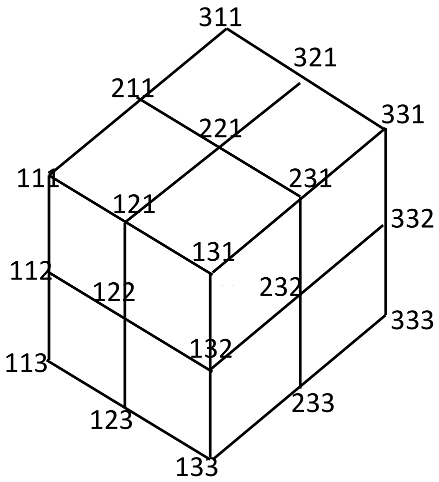
i.e.

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Like the kronecher delta, there is another isotropic tensor called Permutation symbol

$$\epsilon_{ijk} = \begin{cases} 1 & \text{for even permutations } (123, 231, etc.) \\ -1 & \text{for odd permutations } (132, 213, etc.) \\ 0 & \text{for two or more equal indices} \end{cases}$$

i.e. $\in_{ijk}$ =



- We can represent the cross product of two vectors

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \hat{v}_1 & \hat{v}_2 & \hat{v}_3 \\ \widehat{w}_1 & \widehat{w}_2 & \widehat{w}_3 \end{vmatrix}$$

$$= \hat{e}_1[v_2 w_3 - v_1 w_2] - \hat{e}_2[v_1 w_3 - v_3 w_1] + \hat{e}_3[v_1 w_2 - v_2 w_1]$$

$$= \in_{ijk} v_i w_j$$

Similarly, $\nabla \times \vec{v} = \in_{ijk} \frac{\partial v_j}{\partial x_i}$

You can also write $\nabla \times \vec{v} = \in_{ijk} \frac{\partial v_k}{\partial x_j}$

Also if you want, you can check

$$\nabla \times (\nabla \times \vec{v}) \rightarrow \frac{\partial}{\partial x_k} \times (\in_{ijk} \frac{\partial v_j}{\partial x_i})$$

$$\rightarrow \in_{klm} \in_{ijk} \frac{\partial}{\partial x_l} \frac{\partial}{\partial x_i} v_j$$

- Divergence operator increases the order of a tensor by one.

$$\nabla . \overline{\overline{\tau}} = \frac{\partial \tau_{ij}}{\partial x_j}$$

- Gradient operation increases the order of a tensor by one.

Symmeteric and Antisymmeteric Tensors

- A tensor $\bar{\bar{\tau}}$ is symmetric, if the components do not change when i and j are interchanged

$$\bar{\bar{\tau}} \quad \rightarrow \quad \tau_{ij} = \tau_{ji} \qquad \text{(symmetric tensor)}$$

- A tensor is anti symmetric , if

$$\tau_{ij} = -\tau_{ji}$$

- We can write any tensor as sum of symmetric part and anti symmetric part.

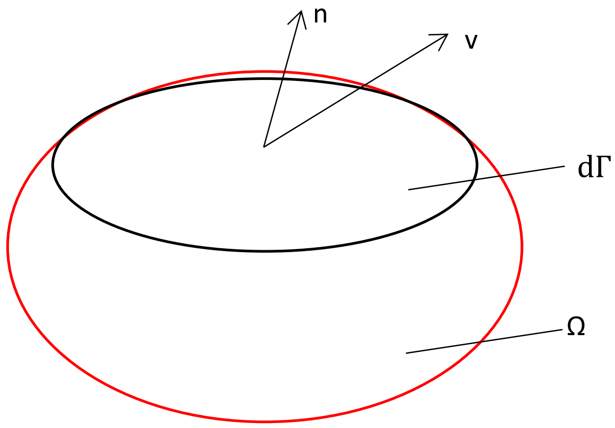
i.e. $$\tau_{ij} = \underbrace{\tfrac{1}{2}[\tau_{ij} + \tau_{ji}]}_{\text{Symmetric Tensor}} + \underbrace{\tfrac{1}{2}[\tau_{ij} - \tau_{ji}]}_{\text{Anti symmetric tensor}}$$

Green Gauss Theorem and Bounding surfaces

Green gauss theorem → relation that connects the surface integral for a vector $\vec{v}$ in the direction normal to an inclined surface with the volume integral of the divergence of $\vec{v}$.

i.e. $$\int_{\Gamma} \vec{v}.\hat{n} d\Gamma = \int_{\Omega} \nabla.\vec{v} d\Omega$$

where Γ is the bounding surfaces
Ω is volumetric domain



In index notation, we write green guass theorem

$$\int_{\Gamma} v_i n_i d\Gamma = \int_{\Omega} \frac{\partial v_i}{\partial x_i} d\Omega$$

If $\propto$ is any index, then green guass theorem in a general form

$$\int_{\Gamma} \alpha\vec{v}.\hat{n}d\Gamma = \int_{\Omega} \nabla.(\alpha\vec{v})d\Omega$$

$$= \int_{\Omega} \alpha\nabla.\vec{v}d\Omega + \int_{\Omega} (\vec{v}.\nabla)\alpha d\Omega$$

i.e. $$\int_{\Gamma} \alpha v_i n_i d\Gamma = \int_{\Omega} \alpha \frac{\partial v_i}{\partial x_i} d\Omega + \int_{\Omega} v_i \frac{\partial \alpha}{\partial x_i} d\Omega$$

- ❖ The index notation for same equation

$$\rightarrow \quad \nabla^2\Psi = 0$$

I.e. $$\frac{\partial^2\Psi}{\partial x_i \partial x_i} = 0$$

$$\frac{\partial\Psi}{\partial t} = -c\left[\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} +\right]$$

i.e. $$\frac{\partial\Psi}{\partial t} = -c \frac{\partial^2\Psi}{\partial x_i \partial x_i}$$

Q. Write in index notation of following equations :

1. $$\delta\vec{g} - \vec{\nabla}p + \nabla.\bar{\bar{\tau}} = \delta\frac{\partial\vec{v}}{\partial t}$$

2. $$\left.\frac{d\bar{\bar{B}}}{dt}\right|_{system} = \frac{d}{dt}\left[\iiint \beta\delta \; du\right] + \iint \beta\delta(\vec{v}.\hat{n})\, dA$$

where $\bar{\bar{B}}$ is a second order tensor.