08/02/2017

LECTURE – 13

Control Volume Theory (Contd……)

&

Brief Introduction to Tensors

Yesterday, we started discussing the control volume theory that can be adopted in fluid mechanics.

- We considered an arbitrary shaped control volume of fluid & then dealt with the transport of extensive property, B.
- We come up with the Reynold's Transport Theorem that relates the conservation principles which are readily applicable in Lagrangian form, to the Eulerian form.

$$\left.\frac{dB}{dt}\right|_{system} = \frac{d}{dt}[\int_{CV} \beta\rho dU] + \int_{CS} \beta\rho(\vec{\mathrm{v}}.\hat{\mathrm{n}})\,\mathrm{dA}$$

The left side of the equation is in Lagrangian form & right side of the equation is in Eularian form.

The Lagrangian time derivative is also called Material Derivative & we will therefore express it as $\frac{DB}{Dt}$.

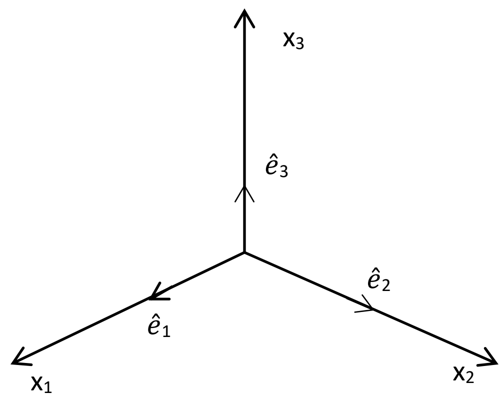
$$\frac{DB}{Dt} = \frac{d}{dt}[\int_{CV} \beta\rho dU] + \int_{CS} \beta\rho(\vec{\mathrm{v}}.\hat{\mathrm{n}})\,\mathrm{dA}$$

Before dealing with some applications of the integral or control volume theorem in fluid flow analysis, it is advisable to go through the basis of tensor.

Introduction to Tensors

In physics & other mechanics courses you have already studied what is a scalar & what is a vector.

- In a 3-D Cartesian coordinate system



You know that you can assign unit vectors in each of the coordinate axis direction.

Here $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ are unit vectors for the coordinate system.

- For a vector , you require three components e.g. $\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3$
- There are many properties that require more than three components to describe that property at a mathematical point in space.

For example, you have studied about stress in last semester (solid mechanics). To describe stress at a mathematical point in space, you require more than three components.

In Cartesian coordinates, you require nine components to describe stress at a point.

$$\bar{\bar{\tau}} = \begin{matrix} \tau_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \tau_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{matrix}$$

Recall, $\vec{v} = \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix}$

The components τ_{11}, τ_{22}, τ_{33} are normal stresses & others are shear stresses.

- ✓ Stress is a tensor.
- ✓ Velocity is a first rank tensor called <u>vector</u>. Stress is a <u>second ranked tensor</u>.
- ✓ Scalar is a <u>zero ranked tensor</u>.
- ✓ Similarly, there are higher ranked tensors present to describe various properties.

In fluid mechanics, mostly we will deal with up to <u>second ranked tensor</u>.

- The number of components for a property is based on the rank of a tensor. If a tensor is ranked 'r', then in 3-D Cartesian coordinates, you require 3^r components.
- Symbolically we represent vectors with a top arrow e.g $\vec{v}$, $\vec{F}$, $\vec{a}$,etc.

When we write, $\vec{v}$, you can automatically understand that,

$$\vec{v} = \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix}$$

Tensor is given by as $\overline{\overline{\tau}}, \overline{\overline{\mathrm{K}}}, \mathrm{etc}.$

$$\overline{\overline{\tau}} = \begin{matrix} \tau_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \tau_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{matrix}$$

- However, to ease the representation, we can use index notations to deal vectors & tensors.

In the index notation,

$\vec{v} \to v_i$

$\vec{F} \to F_i$

$\vec{a} \to a_i$

Where the subscript 'i' is a free index. If a free index is present, you have to interpret the representation as

$v_i \to \vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3$

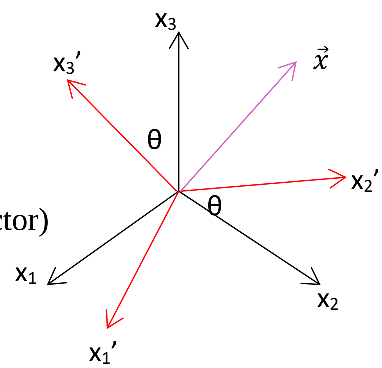
The stress tensor is given as

$$\tau_{ij} = \begin{matrix} \tau_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \tau_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{matrix}$$

For a second ranked tensor, there will be two free indices in the subscript.

- ✓ **Coordinate Transformation**

$\vec{x} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$ (Position vector)



If you want to transform coordinates in the traditional form you did

$$\vec{x}' = [A]\vec{x}$$

That is you use, some matrix transformation.

In index notation, the transformation can be given as

$$x_j^{'} = A_{ij}x_i$$

- You can see that on right side, $A_{ij}x_i$, the subscript "i" is appearing for A as well as X in the product. This means that the subscript "i" is a <u>dummy</u> index. Therefore, in the expression, there is only one <u>free index</u> j.
- The number of free index suggest about the rank of tensor. (example- the position vector is the rank one tensor & therefore, only one free index)
- A tensor that has same components in all rotated orthogonal coordinate system is referred to as an isotropic tensor. (example- Kronecker Delta)

$$\delta_{ij} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

This tensor remains as it is, in any referral axis.

Recall, the divergence (or grad)

$$\vec{\nabla} = \frac{\partial}{\partial x_1}()\,\hat{\mathrm{e}}_1 + \frac{\partial}{\partial x_2}()\,\hat{\mathrm{e}}_2 + \frac{\partial}{\partial x_3}()\,\hat{\mathrm{e}}_3$$

For a scalar field Ψ,

$$\vec{\nabla}\psi = \hat{\mathrm{e}}_1\frac{\partial\psi}{\partial x_1} + \hat{\mathrm{e}}_2\frac{\partial\psi}{\partial x_2} + \hat{\mathrm{e}}_3\frac{\partial\psi}{\partial x_3}$$

For a vector field, $\vec{v}$

$$\vec{\nabla}.\vec{v} = [\hat{\mathrm{e}}_1\frac{\partial}{\partial x_1} + \hat{\mathrm{e}}_2\frac{\partial}{\partial x_2} + \hat{\mathrm{e}}_3\frac{\partial}{\partial x_3}].[\mathrm{v}_1\,\hat{e}_1 + \mathrm{v}_2\,\hat{e}_2 + \mathrm{v}_3\,\hat{e}_3]$$

$$= \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

These things can be represented easily in index notation as such:

$\vec{\nabla}\psi \rightarrow \frac{\partial\psi}{\partial x_i}$ (as "i" is free index, you need to take i = 1,2,3)

$\vec{\nabla}.\vec{v} \rightarrow \frac{\partial v_i}{\partial x_i}$ In the expression "i" is dummy, therefore the expression has 0 rank (as scalar).

✓ Curl of a vector

$$\vec{\nabla}\times\vec{v} = \begin{matrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{matrix}$$

This is given in index notation as, $\frac{\partial v_j}{\partial x_i}\varepsilon_{ijk}$ where ε_{ijk} = permutation symbol

You can see that there is only one free index

$$\epsilon_{ijk} = \begin{cases} 1 & \text{for even permutations } (123, 231, 312, \text{etc.}) \\ -1 & \text{for odd permutations } (132, 213, \text{etc.}) \\ 0 & \text{for two or more equal indices} (112, 113, 221, \text{etc.}) \end{cases}$$

$$\nabla \psi - \nabla p = \vec{m}, \frac{\partial \psi}{\partial x_i} - \frac{\partial p}{\partial x_i} = m_i$$

The double derivative,

$$\nabla^2 \psi = [\hat{e}_1 \frac{\partial}{\partial x_i}() + \hat{e}_2 \frac{\partial}{\partial x_2}() + \hat{e}_3 \frac{\partial}{\partial x_3}()].[\hat{e}_1 \frac{\partial}{\partial x_i}() + \hat{e}_2 \frac{\partial}{\partial x_2}() + \hat{e}_3 \frac{\partial}{\partial x_3}()]\psi$$

$$= \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_3^2}$$

In index notation, this becomes $\frac{\partial^2 \psi}{\partial x_i \partial x_i}$ [Here “i” is a dummy index]

Quiz:

Write the index notation form for following equations

1. $\frac{\partial g}{\partial t} + \nabla.(\rho \vec{v}) = 0$
2. $\rho \frac{dv}{dt} = \rho g - \nabla p$