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LECTURE – 39

BUCKINHAM PI THEORY:

- In the last class, you have seen the explanation for dimensionally homogeneous equation.
- Also, you were introduced the concept of dimensionless variables (or numbers) that can be formulated from a given expression.
- There are several methods to reduce the number of dimensional variables into a smaller number of dimensionless groups.

"A physical process satisfying the principle of dimensional homogeneity & having n-dimensional variables can be reduced to a relation between k dimensionless variables or π's, where k < n."

Here k = n-j

where j is the no of repeating variables that can be determined based on the number of basic dimensions in the expression.

In the force example, F = f(L,U,ρ,μ). You know, n = 5.

In the π- theorem, after identifying 'j', then you need to select j – scaling variables that do not form a π-group among themselves.

Then each π-group will be a power product of these j- variables plus one additional variable.

There are six steps involved in Buckinham Pi Theorem. We will explain it through our example:

F = f(L,U,ρ,μ)

I. List and count the n-variables involved in the problem: In this example it is F, L,U,ρ,μ, i.e, n = 5.

II. List the basic dimensions of each variable

F	L	U	ρ	μ
$[MLT^{-2}]$	$[L]$	$[LT^{-1}]$	$[ML^{-3}]$	$[ML^{-1}T^{-1}]$

So, you can see the basic dimensions involved in this example is 3.

III. Find 'j' such that they are variables that do not form a π – product (dimensionless variable) among themselves. (How??)

The basic dimension in the case is 3. Therefore, you may see three variables (out of n = 5) that may be independent or the combination of them will not form a π – product.

In this case here,

L is an independent variable & have dimension [L]

U is independent, because it is the one having basic dimension [T]

ρ is independent, because it is the one having the basic dimension [M]

The remaining two variables, i.e, F & μ can be expressed in term of L, U & ρ

Therefore, here we have j = 3 (the number of repeating variables)

Here, k = n – j = 5 – 3 = 2

These will be two dimensionless π's that can be formed.

IV. The scaling parameters that do not form dimensionless π's are L, U, ρ.

There j – scaling parameters will appear now in each of the π – groups.

V. Add one additional variable to the j – repeating variables to form a power product π.

As there are two π's possible in the example:

$$\pi_1 = L^a U^b \rho^c F \rightarrow [M^0 L^0 T^0]$$

& $$\pi_2 = L^a U^b \rho^c \mu = [\mathrm{M}^0\, L^0 T^0]$$

So, if you evaluate:

$$\pi_1 = L^a U^b \rho^c F$$
$$[L]^a [LT^{-1}]^b [\mathrm{ML}^{-3}]^c [\mathrm{M}\, LT^{-2}] = [\mathrm{M}^0\, L^0 T^0]$$

On solving, for [L], a+b-3c+1=0

For [T], -b-2=0 or, b = -2

For [M], c+1 = 0 or, c=-1

So, a-2+3+1=0 or, a =-2

Hence, $\pi_1 = \dfrac{F}{\rho U^2 L^2}$

We called this dimensionless variable as coefficient of force.

$$\pi_2 = L^a U^b \rho^c \mu$$
$$[L]^a [LT^{-1}]^b [\mathrm{ML}^{-3}]^c [\mathrm{M} L^{-1} T^{-1}] = [\mathrm{M}^0 L^0 T^0]$$

i.e, a+b-3c-1=0

-b-1=0 or, b= -1

c+1=0 or, c= -1

So, a = -1.

So, $\pi_2 = \dfrac{\mu}{\rho UL} = \dfrac{1}{\mathrm{Re}}$

VI. The dimensionless functions need to be formed from the developed π's.

Here it can be $\dfrac{F}{\rho U^2 L^2} = g(\mathrm{Re})$

Example (As adopted from FM White's Fluid Mechanics)

At low velocities (laminar flow), the volume flow Q through a small-bore tube is a function only of the tube radius R, the fluid viscosity μ, and the pressure drop per unit tube length dp/dx. Using the Pi Theorem, find an appropriate dimensionless relationship.

<u>**Solution**</u>:
As stated in the problem description, you can express the volume flow Q as:
Q =f(R, μ, $\frac{dp}{dx}$)
So, using the six steps of Buckinham Pi theory:

I. The number of variables in the problem → Q, R, μ, $\frac{dp}{dx}$. So, n = 4

II. The basic dimensions in the problem,

Q	R	μ	dp/dx
$[L^3 T^{-1}]$	$[L]$	$[ML^{-1}T^{-1}]$	$[ML^{-2}T^{-2}]$

There are three basic dimensions M,L,T for the given problem.

III. To find the repeating variables or the variables that cannot form dimensionless quantities among themselves.

As the basic dimensions are M, L, T

You can see R is independent & having dimension [L]

Since Q = f(R, μ, $\frac{dp}{dx}$) , we are not interested to take Q as a repeating variables

Let us check, whether you need to take $\frac{dp}{dx}$ as a repeating variable or μ or both.

You have j≤3 for this case.

Say, if you take R & μ as repeating variables.

$$\pi_1 = R^a \mu^b (\frac{dp}{dx}) \to [M^0 L^0 T^0]$$
$$\pi_2 = R^a \mu^b Q \to [M^0 L^0 T^0]$$

You will see that, you will not be able to solve for a & b for both the cases.

So, j = 3 has to be taken.

IV. If j =3, the number of scaling parameters = 3

They are R, μ, $\frac{dp}{dx}$

V. The number of π – group = k = n – j = 4 – 3 = 1

Therefore, there will be only one π – group.

$$\pi_1 = R^a \mu^b (\frac{dp}{dx})^c Q \to M^0 L^0 T^0$$
$$i.e, [L]^a [ML^{-1}T^{-1}]^b [ML^{-2}T^{-2}]^c [M^0 L^3 T^{-1}] = [M^0 L^0 T^0]$$

i.e, a-b-2c+3=0

-b-2c=1

b+c=0

Solving we get, a= -4, b=1, c=-1

$$\pi_1 = R^{-4} \mu (\frac{dp}{dx})^{-1} Q$$

$$\pi_1 = \frac{Q\mu}{R^4 (\frac{dp}{dx})} = cons\tan t$$

This is a dimensionless constant for the given laminar flow.