Lecture 37

12-APRIL-2017

Chapter

<u>Differential Analyses (Contd…)</u>

<u>Flow due to pressure gradient between two fixed plates</u>

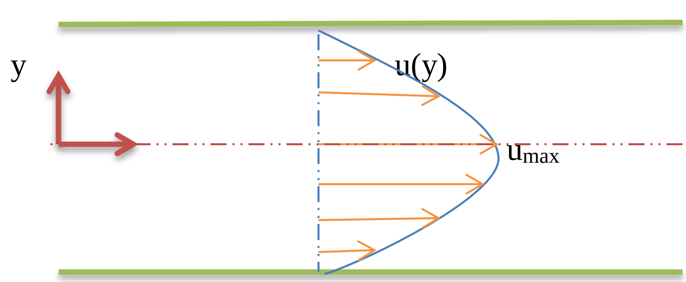
Last class we solved the case of Couette flow between two infinitely parallel plates. It was considered at that time that:

- Flow is steady
- Liquid is incompressible
- Effects of gravity neglected
- Effects of pressure neglected

As the upper plate was moved with a velocity, V, the flow was essentially only in one-dimension.

Today, we will quickly go through another case where flow of liquid occurs <u>due to the pressure gradient</u> between two fixed plates.

Consider the two fixed plates and a liquid flows between the plates as shown below:



- ➔ Let us consider the liquid is flowing in the x-direction.
- ➔ The flow is steady and the liquid is incompressible.

➔ The pressure of fluid varies in the x-direction.

This is a two-dimensional problem with the flow occurring only in the x-direction.

⇨ The components of velocity are u = u,

v = 0,

w = 0

From continuity equation, you can see that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

As $v = 0, \frac{\partial u}{\partial x} = \frac{du}{dx} = 0$

⇨ $u = u(y)$

The x-momentum equation:

$$\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

⇨ $\rho[0] = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial y^2}\right]$

⇨ $\frac{\partial p}{\partial x} = \mu\frac{\partial^2 u}{\partial y^2}$ …..1

The y-momentum equation:

$$\rho\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

⇨ $\rho[0 + 0] = -\frac{\partial p}{\partial y} + \mu[0 + 0]$

⇨ $\frac{\partial p}{\partial y} = 0$ …..2

The z-momentum equation:

$$\rho\left[u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right] = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right]$$

(Note: We have neglected the effects of gravity)

⇨ $\rho[0 + 0 + 0] = -\frac{\partial p}{\partial z} + \mu[0 + 0]$

⇨ $\frac{\partial p}{\partial z} = 0$ ……3

From equations $\boxed{1}$, $\boxed{2}$ and $\boxed{3}$, it is clear that:

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} , \frac{\partial p}{\partial y} = 0 \; and \; \frac{\partial p}{\partial z} = 0$$

Means: $p = p(x)$ only. Moreover, $u = u(y)$

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} \qquad \ldots \boxed{4}$$

i.e., you can equate their differentials only if the relation is a constant.

(It is the principle from calculus).

⇨ $\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} = constant$

This constant should be negative, because pressure must decrease in the flow direction to overcome the resisting wall shear stress.

The velocity profile will be $u(y)$.

$$u = \frac{1}{\mu}\left(\frac{\partial p}{\partial x}\right)\frac{y^2}{2} + C_1 y + C_2$$

At $y = \pm h \, ; u = 0$

⇨ $C_1 = 0 \; and \; C_2 = -\left(\frac{\partial p}{\partial x}\right)\frac{h^2}{2\mu}$

⇨ $u = -\left(\frac{\partial p}{\partial x}\right)\frac{h^2}{2\mu}\left[1 - \frac{y^2}{h^2}\right]$

This is Poiseulle's flow between two parallel plates.

***Example** (As adopted from FM White's Fluid Mechanics)*

For flow between two parallel fixed plates due to the pressure gradient, compute:

(a) The wall shear stress,
(b) The stream function,
(c) The vorticity,
(d) The velocity potential,
(e) The average velocity.

Answer:

We have seen that flow between parallel plates (fixed) for incompressible, steady state flow is:

$$u = -\left(\frac{\partial p}{\partial x}\right)\frac{h^2}{2\mu}\left[1 - \frac{y^2}{h^2}\right]$$

For Newtonian fluid, the shear stress was given as say:

$\tau_{xy} = 2\mu\frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]$

(Recall in index notation we had earlier given expression for viscous stress as $\tau_{ij} = 2\mu\frac{1}{2}\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right]$)

Therefore, the shear stress on the walls :

$$\tau_{xy_{wall}} = \mu\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]_{y=\pm h}$$

$$=\mu\left[\frac{\partial}{\partial y}\left\{-\left(\frac{\partial p}{\partial x}\right)\frac{h^2}{2\mu}\left[1 - \frac{y^2}{h^2}\right]\right\} + 0\right]_{y=\pm h}$$

$$=\mu\left(-\frac{\partial p}{\partial x}\right)\frac{h^2}{2\mu}\left[0 - \frac{1}{h^2}2y\right]_{y=\pm h}$$

i.e., At $y = +h,\ \tau_{xy_{wall}} = \mu\left(-\frac{\partial p}{\partial x}\right)\frac{h^2}{2\mu}\left(0 - \frac{2}{h}\right)$

At $y = -h,\ \ \tau_{xy_{wall}} = \mu\left(-\frac{\partial p}{\partial x}\right)\frac{h^2}{2\mu}\left(0 + \frac{2}{h}\right)$

➔ As the flow is steady, incompressible and plane flow, stream function can be defined.

$$u = \frac{\partial \psi}{\partial y} = u_{max}\left(1 - \frac{y^2}{h^2}\right)$$

$$v = -\frac{\partial \psi}{\partial x} = 0$$

Solving by integrating,

$$\psi = u_{max}\left(y - \frac{y^3}{3h^2}\right)$$

You can check at $y = +h$ and $y = -h$

- For plane flow:

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & 0 & 0 \end{vmatrix}$$

$$= \hat{k}\left(0 - \frac{\partial u}{\partial y}\right)$$

$$= \frac{2u_{max}}{h^2} y$$

Vorticity wil be highest at the walls.

- As the vorticity exists, the flow is not irrotational and we cannot develop a potential function.

- Average velocity, $V_{av} = \frac{Q}{A}$

$$Q = \int u \, dA$$

⇨ $V_{av} = \frac{1}{A}\int u \, dA = \frac{1}{2h \times b}\int_{-h}^{+h} u_{max}\left(1 - \frac{y^2}{h^2}\right) b \, dy$

$= \frac{2}{3} u_{max}$