11/04/2017

LECTURE – 36

Frictionless Flow – BERNOULLI'S EQUATION

Recall, in the last class, we described about potential function $\phi(x, y, t)$ for two dimensional irrotational, incompressible flow.

i.e, $u = \frac{\partial \phi}{\partial x}$ & $v = \frac{\partial \phi}{\partial y}$

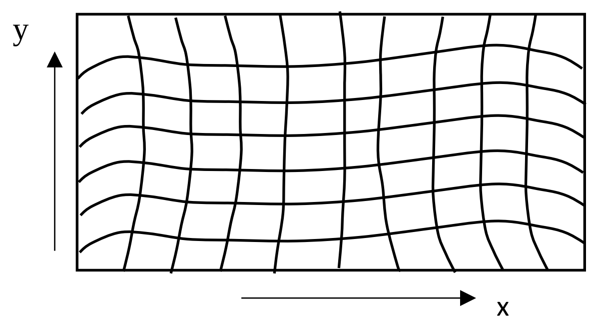
So, in two dimensions, if the fluid flow is incompressible, frictionless (inviscid) & irrotational, then both stream function ($\psi(x, y, t)$) & potential function ($\phi(x, y, t)$) exist.

You can draw streamlines & potential lines as solutions for the flow.

i.e, you can solve either

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

or, $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$



Another interesting aspect is that:

Streamlines and potential lines are orthogonal to each other. **<u>How??</u>**

For a line of constant potential (i.e. ϕ), $d\phi = 0$

You can define,

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

or, $$d\phi = udx + vdy$$

For $d\phi = 0$,

$$0 = udx + vdy$$

or, $$udx = -vdy$$

or, $$\left(\frac{dy}{dx}\right)_{\phi = \text{constant}} = -\frac{u}{v} \text{.....................(1)}$$

Similarly, along a streamline ψ = constant

The equation of a streamline

$$vdx - udy = 0 \quad \text{(Described earlier)}$$

$$\text{i.e,} \qquad \left(\frac{dy}{dx}\right)_{\psi = constant} = \frac{v}{u} \text{......................(2)}$$

On a mathematical point, where a streamline & potential line intersect,

$$\left(\frac{dy}{dx}\right)_{\phi = constant} = -\frac{u}{v} = -\frac{1}{v/u} = -\frac{1}{\left(\frac{dy}{dx}\right)_{\psi = constant}}$$

This property shows the streamlines & potential lines are orthogonal to each other.

Again, recall in last lecture we suggested that for a <u>frictionless flow</u>, if the fluid is incompressible & steady you have:

$$\frac{p}{\rho} + \frac{v^2}{2} + gz = \quad \text{constant} \quad \text{(Along a streamline)}$$

Also, recall for a <u>frictionless flow</u>, the dot product of linear momentum & position vector was given by:

$$\left[\frac{\partial \vec{v}}{\partial t} + \nabla\left(\frac{|\vec{v}|^2}{2}\right) + (\vec{q} \times \vec{v}) + \frac{1}{\rho}\nabla p - \vec{g}\right].\overrightarrow{dr} = 0 \text{...............(3)}$$

This was possible only if:

$$[\vec{q} \times \vec{v}].\overrightarrow{dr} = 0 \text{....................................(4)}$$

As mentioned at that time, equation (4) is possible only if

i) $\vec{v} = 0$

ii) $(\vec{q} \times \vec{v})$ is perpendicular to $\overrightarrow{dr}$

iii) $\vec{v}$ is parallel to $\overrightarrow{dr}$ (i.e, along a streamline)

iv) $\vec{q} = 0$ (i.e, flow is irrotational)

Now note for irrotational flow, which is incompressible as well as flow is steady, this frictionless flow will be given by:

$$d\left(\frac{1}{2}\left|\vec{v}\right|^2\right) + \frac{dp}{\rho} + gdz = 0 \text{..................................} (5)$$

For frictionless, incompressible, irrotational & steady flow, integrating equation (5) within any two points (1) & (2) in the domain, you get

$$\int_{(1)}^{(2)} \frac{dp}{\rho} + \frac{1}{2}(v_2^2 - v_1^2) + g(z_2 - z_1) = 0$$

i.e, $\frac{1}{\rho}(p_2 - p_1) + \frac{1}{2}(v_2^2 - v_1^2) + g(z_2 - z_1) = 0$

i.e, $\frac{p_2}{\rho} + \frac{v_2^2}{2} + gz_2 = \frac{p_1}{\rho} + \frac{v_1^2}{2} + gz_1 =$ constant

throughout the flow domain

This is the world famous Bernoulli's equation you have studied in schools.

Please note that the above Bernoulli's equation can be applied only if

i) The flow is frictionless

ii) Flow is incompressible

iii) Flow is irrotational

iv) Flow is steady

In reality, most of flows you see in nature or in industrial applications etc may not be frictionless.

That is effect of viscosity plays significant role in the fluid flow.

The no slip conditions may prevails in the flow.

The irrotational flow assumptions may not be applicable.

What will you do to solve flow problem in such situations??

You have to use Navier-Stoke's equations to solve such fluid flow.

For viscous flows, one can solve Navier-Stoke's equations appropriately.

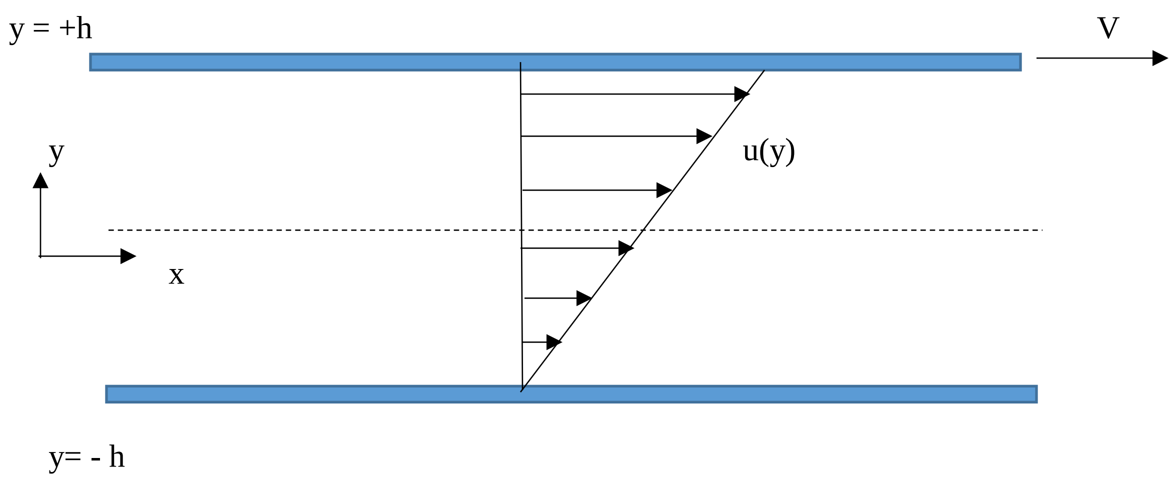
Recall,   (for incompressible flow)

You can visualize from equation (7) that

i) If the flow is frictionless, neglect the corresponding terms.
ii) Similarly, if the effect of gravity are negligible, neglect the gravitational term
iii) If the effect of pressure force are negligible, neglect the corresponding term.
iv) If the flow is steady, take $\frac{\partial \vec{v}}{dt} = 0$

<u>Couette Flow between fixed & moving plate</u>:

Consider, fluid flows between two parallel plates.



Upper plate is moving with a velocity V & bottom plate is fixed.

It is assumed that plates are very wide & the movement of plate causes fluid movement only in x direction. That is, flow is axial.

Therefore, the fluid velocity contains only u. other velocity components v and w are zero.

You can neglect gravity effects & also assume pressure gradient as zero.

Liquid is incompressible.

So, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$, v = 0 & w = 0

$i.e, \frac{\partial u}{\partial x} = 0$

So, u = u(y) only

$$\rho[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}] = \rho g_x - \frac{\partial p}{\partial x} + \mu[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}]$$

i.e, $\rho[u\frac{\partial u}{\partial x}] = \mu[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}]$

(as, $\frac{\partial u}{\partial t} = 0, v\frac{\partial u}{\partial y} = 0, w\frac{\partial u}{\partial z} = 0, \frac{\partial p}{\partial x} = 0, \rho g_x = 0$)

As, u=u(y), so, $\frac{\partial u}{\partial x} = 0, \frac{\partial^2 u}{\partial x^2} = 0$

So, $0 = \mu\frac{d^2 u}{dy^2}$

So, $u = C_1 y + C_2$

At y=+h, u = V = C_1h +C_2

At y=-h, u = 0 = -C_1h +C_2

So, $C_1 = \frac{V}{2h}$ & $C_2 = \frac{V}{2}$

So, $u = \frac{V}{2h}y + \frac{V}{2}$ for $-h \le y \le +h$

Is the required solution for Couette flow.