Frictionless Irrotational Flows:

Recall, earlier we had described stream function for 2-D flows.

e.g. for a flow in x-y plane, you had defined

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

where, $\psi = \psi (x, y)$ is a stream function in x-y plane.

Also, for incompressible, inviscid, irrotational flows the continuity equation will be

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Again recalling, the linear momentum equation,

$$\rho \left[ \frac{\partial \vec{v}}{\partial t} + u \frac{\partial \vec{v}}{\partial x} + v \frac{\partial \vec{v}}{\partial y} + w \frac{\partial \vec{v}}{\partial z} \right] = \rho g - \vec{v} \cdot \nabla p + \mu \nabla^2 \vec{v}$$

i.e. $\rho \left[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \rho g - \vec{v} \cdot \nabla p + \mu \nabla^2 \vec{v}$

For steady flow

$$\rho \left[ (\vec{v} \cdot \nabla) \vec{v} \right] = \rho g - \vec{v} \cdot \nabla p + \mu \nabla^2 \vec{v}$$

(2)

If you take the curl of equation (2)

$$\nabla \times \left[ u \frac{\partial \vec{v}}{\partial x} + v \frac{\partial \vec{v}}{\partial y} \right] = \nabla \times \left[ \vec{g} - \frac{1}{\rho} \vec{v} \cdot \nabla p + \frac{\mu}{\rho} \left( \frac{\partial^2 \vec{v}}{\partial x^2} + \frac{\partial^2 \vec{v}}{\partial y^2} \right) \right]$$

For inviscid flow the above equation become

$$\nabla \times \left[ (\vec{v} \cdot \nabla) \vec{v} \right] = \nabla \times \left[ \vec{g} - \frac{1}{\rho} \vec{v} \cdot \nabla p \right]$$

There is a vector identity that suggest
\(( \vec{v} \cdot \nabla ) \vec{v} = \vec{\nabla} \times \left( \frac{1}{2} |\vec{v}|^2 \right) + \vec{q} \times \vec{v} \)

Where \( \vec{q} = \vec{\nabla} \times \vec{v} \) = curl of a velocity vector or vorticity vector

Therefore, for frictionless flows, or inviscid flows:

The momentum equation:

\[
\vec{\nabla} \times \left[ \vec{\nabla} \times \left( \frac{1}{2} |\vec{v}|^2 \right) \right] + \vec{\nabla} \times (\vec{q} \times \vec{v}) = \vec{\nabla} \times \vec{g} - \vec{\nabla} \times \left( \frac{1}{\rho} \vec{\nabla} p \right)
\] (3)

At this stage, we are not further proceeding with the expansion as well as explanation of equation (3)

So coming back to Euler’s inviscid equation (frictionless)

\[
\rho \left[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \rho \vec{g} - \vec{\nabla} p
\] (4)

Using the above vector identity in this equation (4)

\[
\frac{\partial \vec{v}}{\partial t} + \vec{\nabla} \times \left( \frac{1}{2} |\vec{v}|^2 \right) + \vec{q} \times \vec{v} - \vec{g} + \frac{1}{\rho} \vec{\nabla} p
\]

Or else, if we dot product this equation with position vector \( d\vec{r} = (dx)\hat{i} + (dy)\hat{j} \)

\[
\left[ \frac{\partial \vec{v}}{\partial t} + \vec{\nabla} \times \left( \frac{1}{2} |\vec{v}|^2 \right) + \vec{q} \times \vec{v} - \vec{g} + \frac{1}{\rho} \vec{\nabla} p \right] \cdot d\vec{r} = 0
\]

For such relation to exist

\( (\vec{q} \times \vec{v}) \cdot d\vec{r} = 0 \)

This is possible, if

i. \( \vec{v} = 0 \), i.e. purely hydrostatic condition

ii. \( \vec{q} = 0 \), the flow is irrotational

iii. \( d\vec{r} = 0 \), is perpendicular to \( (\vec{q} \times \vec{v}) \)

iv. \( d\vec{r} = 0 \), is parallel to \( \vec{v} \)

Considering situation (iv), where \( d\vec{r} \) is parallel to \( \vec{v} \). That means, we are following a streamline. You are integrating along streamline. On integrating along a streamline for frictionless compressible flow

\[
\frac{\partial \vec{v}}{\partial t} \cdot d\vec{r} + \left( \frac{1}{2} |\vec{v}|^2 \right) + gdz + \frac{dp}{\rho} = 0
\]
\[ d\mathbf{r} = (dx)\mathbf{i} + (dy)\mathbf{j} + (dz)\mathbf{k} \]

Note: \[-g = -g\mathbf{k}\]

\[ \nabla p \cdot d\mathbf{r} = \left( \left. \frac{dp}{dx} \right|_i + \left. \frac{dp}{dx} \right|_j \right) \cdot (dx)\mathbf{i} + (dy)\mathbf{j} \]

\[ \nabla p \cdot d\mathbf{r} = \frac{dp}{dx} dx + \frac{dp}{dx} dy \]

\[ \nabla p \cdot d\mathbf{r} = dp \]

\[ \mathbf{v} \cdot d\mathbf{r} = vds \]

Therefore, along a stream line integrating between two point (1) and (2)

\[ \int_{(1)}^{(2)} \frac{\partial |\mathbf{v}|}{\partial t} ds + \frac{1}{2} \int_{(1)}^{(2)} d\left(|\mathbf{v}|^2\right) + g \int_{(1)}^{(2)} dz + \int_{(1)}^{(2)} \frac{dp}{\rho} = 0 \]

\[ \int_{(1)}^{(2)} \frac{\partial |\mathbf{v}|}{\partial t} ds + \int_{(1)}^{(2)} \frac{dp}{\rho} + \frac{1}{2} (v_2^2 - v_1^2) + g(z_2 - z_1) = 0 \]  \hspace{1cm} (5)

ds is the length along the streamline.

Equation (5) is the Bernoulli’s equation for frictionless unsteady flow along a streamline.

Now: from equation (5) you can extract

For incompressible, steady flow:

\[ \frac{1}{\rho} (p_2 - p_1) + \frac{1}{2} (v_2^2 - v_1^2) + g(z_2 - z_1) = 0 \]

\[ \frac{p_2}{\rho} + \frac{v_2^2}{2} + gz_2 = \frac{p_1}{\rho} + \frac{v_1^2}{2} + gz_1 = \text{constant} \hspace{1cm} \text{for a streamline} \]

This is the world famous Bernoulli’s equation applicable to incompressible, inviscid, steady flows along streamline. The value of the constant...
\[ \frac{p}{\rho} + \frac{v^2}{2} + gz = \text{constant, varies streamline to streamline} \]

For streamline \( \psi_1 \),
\[ \frac{p}{\rho} + \frac{v^2}{2} + gz = C_1 \]

For streamline \( \psi_2 \),
\[ \frac{p}{\rho} + \frac{v^2}{2} + gz = C_2 \]

For streamline \( \psi_3 \),
\[ \frac{p}{\rho} + \frac{v^2}{2} + gz = C_3 \]

**For irrotational flows:**
You recall for irrotational flows,

\[ \vec{q} = \nabla \times \vec{v} = 0 \]

Using a vector principle i.e. if curl of a vector is zero that means curl of divergence of a scalar is also zero.

\[ \vec{q} = \nabla \times \vec{v} = 0 = \nabla \times (\nabla \phi) = 0 \]

We can now define a function \( \phi (x,y) \) for 2-D flow, such that

\[ u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y} \]

You can extend in 3D as well by suggesting \( w = \frac{\partial \phi}{\partial z} \)

Such type of flow are called irrotational flows and the function \( \phi (x,y) \) is called **Potential Function**.
So in 2D, if the fluid flow is irrotational as well as incompressible and inviscid, the both $\phi$ and $\psi$ exist.

You can draw streamline and potential lines as solution in the flow domain. (Please note that at stagnant points $\vec{v} = 0$, we cannot draw them)

\[
\begin{align*}
    u &= \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \\
    v &= \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}
\end{align*}
\]

For irrotational, incompressible, inviscid flows,

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0
\]

Similarly, you can also say

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0
\]