07 April 2017

Lecture - 35

Frictionless Irrotational Flows:-

Recall, earlier we had described stream function for 2-D flows.

e.g. for a flow in x-y plane, you had defined

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x}$$

where, $\psi = \psi(x, y)$ is a stream function in x-y plane.

Also, for incompressible, inviscid, irrotational flows the continuity equation will be

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Again recalling, the linear momentum equation,

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y} + w\frac{\partial \vec{v}}{\partial z}\right] = \rho\vec{g} - \vec{\nabla}p + \mu\nabla^2\vec{v} \qquad (1)$$

i.e. $\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v}\right] = \rho\vec{g} - \vec{\nabla}p + \mu\nabla^2\vec{v}$

For steady flow

$$\rho\left[(\vec{v}.\nabla)\vec{v}\right] = \rho\vec{g} - \vec{\nabla}p + \mu\nabla^2\vec{v} \qquad (2)$$

If you take the curl of equation (2)

$$\vec{\nabla}\times\left[u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y}\right] = \vec{\nabla}\times\left[\vec{g} - \frac{1}{\rho}\vec{\nabla}p + \frac{\mu}{\rho}\left[\frac{\partial^2 \vec{v}}{\partial x^2} + \frac{\partial^2 \vec{v}}{\partial y^2}\right]\right]$$

For inviscid flow the above equation become

$$\vec{\nabla}\times\left[(\vec{v}.\nabla)\vec{v}\right] = \vec{\nabla}\times\left[\vec{g} - \frac{1}{\rho}\vec{\nabla}p\right]$$

There is a vector identity that suggest

$$(\vec{v}.\nabla)\vec{v} = \vec{\nabla}\times\left(\frac{1}{2}|\vec{v}|^2\right) + \vec{q}\times\vec{v}$$

Where $\vec{q} = \vec{\nabla}\times\vec{v}$ = curl of a velocity vector or vorticity vector

Therefore, for frictionless flows, or inviscid flows:

The momentum equation:

$$\vec{\nabla}\times\left[\vec{\nabla}\times\left(\frac{1}{2}|\vec{v}|^2\right)\right] + \vec{\nabla}\times(\vec{q}\times\vec{v}) = \vec{\nabla}\times\vec{g} - \vec{\nabla}\times\left(\frac{1}{\rho}\vec{\nabla}p\right) \tag{3}$$

At this stage, we are not further proceeding with the expansion as well as explanation of equation (3)

So coming back to Euler's inviscid equation (frictionless)

$$\rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v}\right] = \rho\vec{g} - \vec{\nabla}p \tag{4}$$

Using the above vector identity in this equation (4)

$$\frac{\partial\vec{v}}{\partial t} + \vec{\nabla}\times\left(\frac{1}{2}|\vec{v}|^2\right) + \vec{q}\times\vec{v} - \vec{g} + \frac{1}{\rho}\vec{\nabla}p$$

Or else, if we dot product this equation with position vector $d\vec{r} = (dx)\hat{i} + (dy)\hat{j}$

$$\left[\frac{\partial\vec{v}}{\partial t} + \vec{\nabla}\times\left(\frac{1}{2}|\vec{v}|^2\right) + \vec{q}\times\vec{v} - \vec{g} + \frac{1}{\rho}\vec{\nabla}p\right]\bullet d\vec{r} = 0$$

For such relation to exist

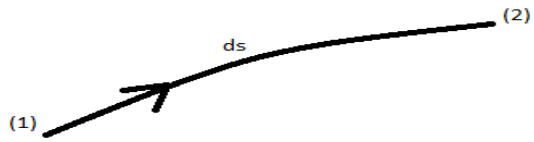
$(\vec{q}\times\vec{v})\bullet d\vec{r} = 0$

This is possible, if

i. $\vec{v} = 0$, i.e. purely hydrostatic condition
ii. $\vec{q} = 0$, the flow is irrotational
iii. $d\vec{r} = 0$, is perpendicular to $(\vec{q}\times\vec{v})$
iv. $d\vec{r} = 0$, is parallel to $\vec{v}$

Considering situation (iv), where $d\vec{r}$ is parallel to $\vec{v}$. That means, we are following a streamline. You are integrating along streamline. On integrating along a streamline for frictionless compressible flow

$$\frac{\partial\vec{v}}{\partial t}.d\vec{r} + d\left(\frac{1}{2}|\vec{v}|^2\right) + g\,dz + \frac{dp}{\rho} = 0$$



$$d\vec{r} = (dx)\hat{i} + (dy)\hat{j} + (dz)\hat{k}$$

Note: $\quad -\vec{g} = -g\hat{k}$

$$\vec{\nabla}p \bullet d\vec{r} = \left(\frac{dp}{dx}\hat{i} + \frac{dp}{dx}\hat{j}\right) \bullet \left((dx)\hat{i} + (dy)\hat{j}\right)$$

$$\vec{\nabla}p \bullet d\vec{r} = \frac{dp}{dx}dx + \frac{dp}{dx}dy$$

$$\vec{\nabla}p \bullet d\vec{r} = dp$$

$$\vec{v} \bullet d\vec{r} = vds$$

Therefore, along a stream line integrating between two point (1) and (2)

$$\int_{(1)}^{(2)} \frac{\partial|\vec{v}|}{\partial t} ds + \frac{1}{2}\int_{(1)}^{(2)} d\left(|\vec{v}|^2\right) + g\int_{(1)}^{(2)} dz + \int_{(1)}^{(2)} \frac{dp}{\rho} = 0$$

$$\int_{(1)}^{(2)} \frac{\partial|\vec{v}|}{\partial t} ds + \int_{(1)}^{(2)} \frac{dp}{\rho} + \frac{1}{2}\left(v_2^{\,2} - v_1^{\,2}\right) + g(z_2 - z_1) = 0 \qquad (5)$$

ds is the length along the streamline.

Equation (5) is the Bernoulli's equation for frictionless unsteady flow along a streamline

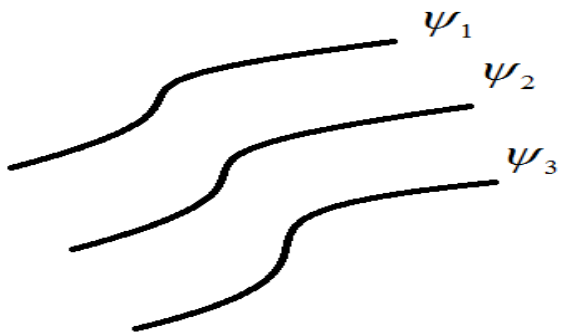
Now: from equation (5) you can extract
For incompressible, steady flow:

$$\frac{1}{\rho}(p_2 - p_1) + \frac{1}{2}\left(v_2^{\,2} - v_1^{\,2}\right) + g(z_2 - z_1) = 0$$

$$\frac{p_2}{\rho} + \frac{v_2^{\,2}}{2} + gz_2 = \frac{p_1}{\rho} + \frac{v_1^{\,2}}{2} + gz_1 = \text{constant} \quad \text{for a streamline}$$

This is the world famous Bernoulli's equation applicable to incompressible, inviscid, steady flows along streamline. The value of the constant

$\frac{p}{\rho}+\frac{v^2}{2}+gz$ = constant , varies streamline to streamline



For streamline ψ_1, $\frac{p}{\rho}+\frac{v^2}{2}+gz=C_1$

For streamline ψ_2, $\frac{p}{\rho}+\frac{v^2}{2}+gz=C_2$

For streamline ψ_3, $\frac{p}{\rho}+\frac{v^2}{2}+gz=C_3$

<u>For irrotational flows:</u>

You recall for irrotational flows,

$$\vec{q}=\vec{\nabla}\times\vec{v}=0$$

Using a vector principle i.e. if curl of a vector is zero that means curl of divergence of a scalar is also zero.

$$\vec{q}=\vec{\nabla}\times\vec{v}=0=\vec{\nabla}\times(\vec{\nabla}\phi)=0$$

We can now define a function ϕ (x,y) for 2-D flow, such that

$$u=\frac{\partial\phi}{\partial x} \text{ and } v=\frac{\partial\phi}{\partial y}$$

You can extend in 3D as well by suggesting $w=\frac{\partial\phi}{\partial z}$

Such type of flow are called irrotational flows and the function ϕ (x,y) is called **Potential Function**.

So in 2D, if the fluid flow is irrotational as well as incompressible and inviscid, the both ϕ and ψ exist.

You can draw streamline and potential lines as solution in the flow domain.
(Please note that at stagnant points $\vec{v} = 0$, we cannot draw them)

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$
$$v = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}$$

For irrotational, incompressible, inviscid flows,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Similarly, you can also say

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$