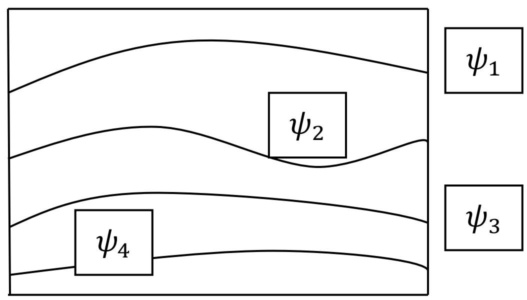
LECTURE 34

05-APRIL-2017

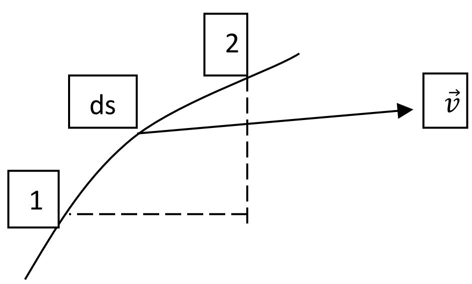
<u>STREAM FUNCTIONS (CONTD…)</u>

Yesterday, we were giving geometric interpretations to stream functions.

- For incompressible, inviscid, irrotational flow in two-dimensions (i.e., horizontal plane), you saw that the fluid flow can be described in steady state conditions as:
  $$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0 \qquad \rightarrow \boxed{1}$$
- So, subsequently you were told that in any xy plane, where the above fluid flow is valid, you can solve equation $\boxed{1}$ and get values of ψ(x,y).
- You can do contour plotting and develop streamlines for the flow as shown below:



- You were also informed that along a streamline, the value of ψ is constant.
- If we take any arbitrary control surface through which streamlines go 1
- Consider two points 1 and 2 on an elemental control surface of length 'ds'. The streamlines, (i.e., line in direction of velocity vector) is as shown below:



- Let this elemental length on control surface be represented as vector ($\overrightarrow{ds}$) x 1 (Unit direction perpendicular to paper, here).
  $\overrightarrow{ds}=dx\hat{\imath}+dy\hat{\jmath}$
  (Note: We are talking about two-dimensional horizontal flow)

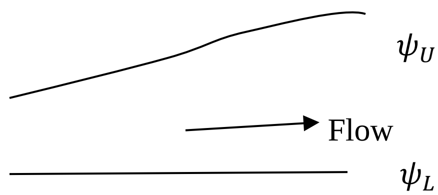
- The unit vector normal to the control surface is given by $\hat{n} = \frac{dy}{ds}\hat{\imath} - \frac{dx}{ds}\hat{\jmath}$
  i.e., the relation $ds.\hat{n} = 0$
  $(dx\hat{\imath} + dy\hat{\jmath}).\left(\frac{dy}{ds}\hat{\imath} - \frac{dx}{ds}\hat{\jmath}\right) = 0$ (is true).

- Thr volumetric flux across this elemental control surface is given as dQ
  $$dQ = (\vec{v}.\hat{n})dA$$
  where $dA = dsx1$
  ⇨ $dQ = (u\hat{\imath} + v\hat{\jmath})\left(\frac{dy}{ds}\hat{\imath} - \frac{dx}{ds}\hat{\jmath}\right).ds = \left(\frac{\partial\psi}{\partial y}\hat{\imath} - \frac{\partial\psi}{\partial x}\hat{\jmath}\right)\left(\frac{dy}{ds}\hat{\imath} - \frac{dx}{ds}\hat{\jmath}\right)ds$
  $$= \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy$$
  $$i.e., dQ = d\psi$$

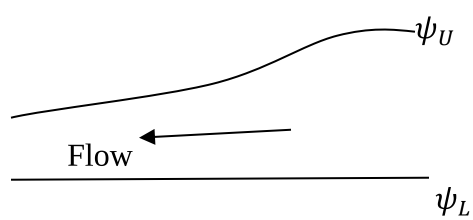
  That is, change in ψ across the elemental surface is equal to the volumetric flux through the surface.

  Therefore, volumetric flux(or discharge) between any two streamlines ψ_1 and ψ_2 is: $\int_1^2 dQ = \int_1^2 d\psi = \psi_2 - \psi_1$

  If $\psi_U > \psi_L$, the flow is:



  If $\psi_U < \psi_L$, then



Example (As adopted from FM White's Fluid Mechanics)

In a flow field, the velocity components were evaluated as u=a(x²-y²), v=-2axy, w=0

- ***Check whether you can form a stream function for this flow field. If so, what is the stream function?***

Solution: Given: $u = a(x^2-y^2)$

$v = -2axy$
$w = 0$

As w=0, the flow is 2-dimensional,we need to check whether the flow is incompressible. (Note: For incompressible fluid $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$).

$$\frac{\partial u}{\partial x} = 2ax, \qquad \frac{\partial v}{\partial y} = -2ax$$

$$So, \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

That is fluid is incompressible and the continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ is satisfied.

Therefore, we can define the scalar stream function ψ for the given problem.

As $u = \frac{\partial \psi}{\partial y} = a(x^2 - y^2)$

⇨ $\psi = \int u dy + f(x)$

⇨ $\psi = ax^2y - a\frac{y^3}{3} + f(x)$ → $\boxed{1}$

⇨ $\frac{\partial \psi}{\partial x} = 2axy + f'(x)$ → $\boxed{2}$

You know $v = -\frac{\partial \psi}{\partial x} = -2axy$

⇨ $\frac{\partial \psi}{\partial x} = 2axy$

Here in $\boxed{2}$, $f'(x) = 0$

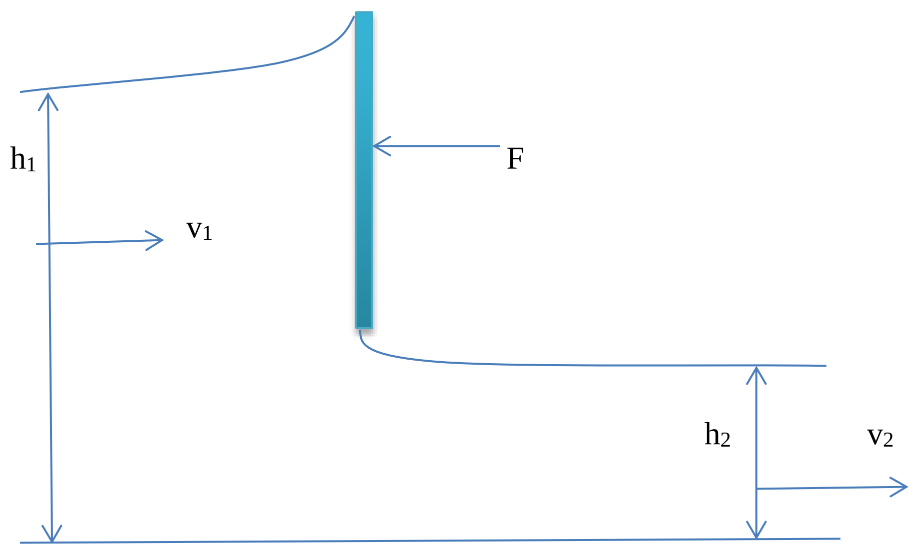
Or, f(x) = constant C

⇨ $\psi = ax^2y - a\frac{y^3}{3} + C$

Example (***As adopted from FM White's Fluid Mechanics***)

Long back in control volume analysis, you had worked out on a sluice gate problem, if you recall. Now list all the boundary conditions needed to solve this flow using differential analysis.

Solution:



In differential analysis, we need to find the values of the dependent variables in each and every point inside the domain. Here the domain can be assumed to be two-dimensional in y-z plane. That is u=0

- ➔ As open channel flow is considered, the fluid will be incompressible.
- ➔ Moreover, isothermal conditions can be assumed to prevail. Therefore, temperature variations are not included in the analysis here.

The governing equations are:

$$\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad \text{(Conservation of mass)}$$

$$\rho g_x-\frac{\partial p}{\partial x}+\mu(0)=0$$

$$\rho g_y-\frac{\partial p}{\partial y}+\mu\left[\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right]=\rho\left[\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right]$$

$$\rho g_z-\frac{\partial p}{\partial z}+\mu\left[\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right]=\rho\left[\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right]$$

Also recall, $\vec{g}=0\hat{\imath}+0\hat{\jmath}-g\hat{k}$, velocity is uniform at left and right sections (i.e., it is also steady).

Therefore, the equations are:

$$\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$-\frac{\partial p}{\partial y}+\mu\left[\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right]=\rho\left[v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right]$$

$$-\rho g-\frac{\partial p}{\partial z}+\mu\left[\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right]=\rho\left[v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right]$$

Again, the open channel flow is predominantly only in y-direction. The variations of velocities in vertical direction may be neglected.

i.e., $v=v_{avg}(z)$ (i.e., the velocity in the y-direction is averaged over vertical direction at any cross section).

i.e., The dependents you can see are: v, w, p

→ The boundary conditions are:

$$v(y=0,z)=v_1$$
$$v(y=L,z)=v_2$$
$$w(y,z=0)=0$$
$$p(y=0,z)=\rho g(h_1-z)$$
$$p(y=L,z)=\rho g(h_2-z)$$
$$w[0<y<L_1,z=h_1]=0$$
$$w[L_1<y<L,z=h_2]=0$$