04/04/2017

LECTURE – 33

Geometric Interpretation of Stream Function:

In the last class, you came to know about the different types of boundary conditions that needs to be applied to solve the governing equations for fluid flow:

i.e. the conservation of mass, the conservation of linear momentum, the conservation of energy.

You were told that, the most general way of solving a fluid flow problem is to simultaneously solve the above equations on conservation principles & get the values of the unknown or dependent variables (i.e. ρ, p, u, v, w, T).

However, we as human beings, it may be difficult for us to solve all three simultaneously even for a simple fluid flow.

Moreover, considering engineering applications, there may no need to solve all of them simultaneously.

You can reduce the dependent variables & the equations as per the situation.

In that light, we introduced you the concept of stream function for horizontal two dimensional flow.

i.e. you know that flow varies only in x & y directions & assuming isothermal conditions as well as incompressible flow, the continuity equation is:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

(As, $\frac{\partial w}{\partial z}$ = 0 ; no variations in vertical direction)

For solving benefit, you were introduced a function $\psi=\psi(x,y)$ such that

$$\frac{\partial}{\partial x}(\frac{\partial \psi}{\partial y})+\frac{\partial}{\partial y}(-\frac{\partial \psi}{\partial x})=0$$

i.e, this means that

$$u = \frac{\partial \psi}{\partial y} \;\&\; v = -\frac{\partial \psi}{\partial x}$$

So, the governing equation for motion becomes unknown in only one quantity ψ. **How??**

Rather than giving in most general form:

Recall the velocity gradient term.

It consisted of strain rate tensor & vorticity tensor.

That is, a fluid flow consist of rate of deformation & rate of rotation.

i.e, vorticity tensor = $\frac{1}{2}[\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}]$

Associated with vorticity tensor, we can define a vector called vorticity vector $\vec{q} = \nabla \times \vec{v}$

Or, $\overrightarrow{q_i} = \varepsilon_{ijk} \frac{\partial v_k}{\partial x_j}$

That is, the vorticity vector is actually the curl of the velocity vector.

So, in the given two dimensional flow, if the flow is irrotational, that means

$\vec{\nabla} \times \vec{v} = 0$

i.e, $\det \begin{pmatrix} \hat{i} & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \psi}{\partial y} & -\frac{\partial \psi}{\partial x} & 0 \end{pmatrix} = 0$ or, $\frac{\partial \psi^2}{\partial x^2} + \frac{\partial \psi^2}{\partial y^2} = 0$

That is, for irrotational flows of inviscid fluids we can have the governing equation in steady-state conditions as:

$$\frac{\partial \psi^2}{\partial x^2} + \frac{\partial \psi^2}{\partial y^2} = 0$$

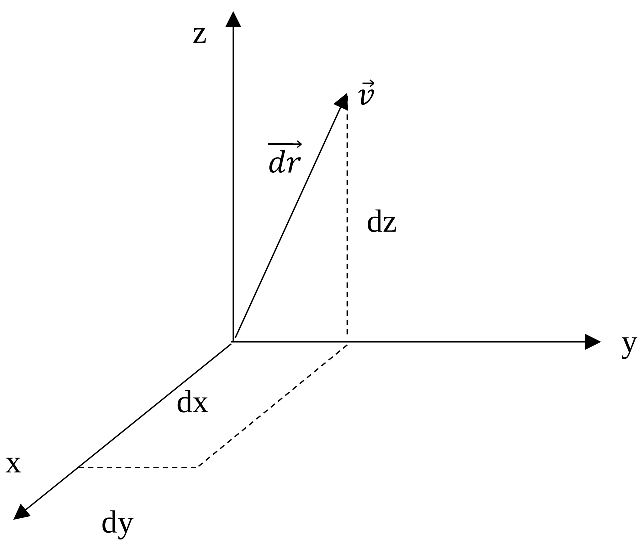
You need to now solve only for ψ in the given x, y domain.

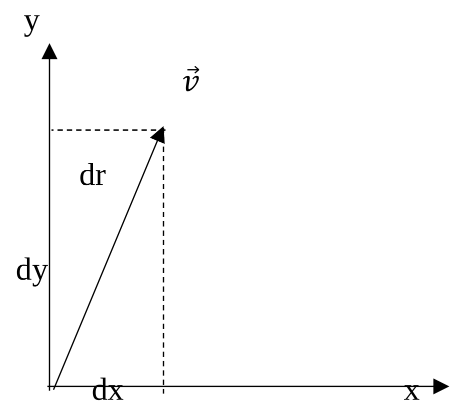
(Please note that this is possible only for irrotational, inviscid flows.)

<u>Geometric Interpretation of ψ</u>:

Recall, we had earlier defined streamlines.

A streamline is a line, which is tangent to velocity vector $\vec{v}$ everywhere at a given instant.





The position vector is $\overrightarrow{dr}$ for a particle or at a mathematical point.

$\overrightarrow{dr} = dx\hat{i} + dy\,j + dzk$

Now the velocity vector is tangent to position vector or parallel.

The components of velocity vector & position vector should match.

$\overrightarrow{dr} = dx\hat{i} + dy\,j + dzk$

$\vec{v} = u\hat{i} + v\,j$

i.e, $\frac{dx}{u} = \frac{dy}{v} = \frac{|dr|}{|\vec{v}|}$ (for both of them to be parallel)

As per streamline definition, $u = \frac{\partial \psi}{dy}$ & $v = -\frac{\partial \psi}{dx}$

As, $\frac{dx}{u} = \frac{dy}{v}$

i.e., vdx – udy = 0 (equation for a streamline)

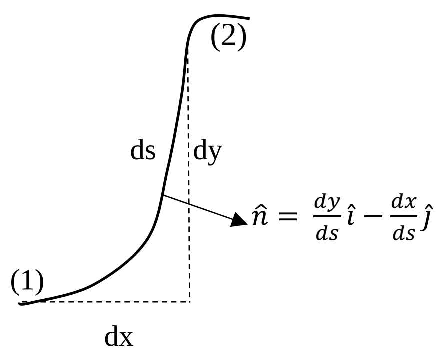
i.e., $-\frac{\partial \psi}{dx}dx - \frac{\partial \psi}{dy}dx = 0$

i.e., $\frac{\partial \psi}{dx}dx + \frac{\partial \psi}{dy}dx = 0 = d\psi$ (Calculus Principle)

this means that $d\psi = 0$

Or, along a streamline, ψ = constant

So, for a given flow problem, if we solve for ψ, then subsequently you can plot streamlines or lines of constant ψ's.



For a <u>control surface</u> (Not stream line)

The net outflow through the control surface between two point (1) & (2) :

$$dQ = (\vec{v}.n)dA$$

Taking unit width into the paper:

dA = ds *1 = ds

$$dQ = (u\hat{i} + v\hat{j})(\frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j})(ds \times 1)$$

$$= (\frac{\partial \psi}{dy}\hat{i} - \frac{\partial \psi}{dx}\hat{j})(\frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j})(ds \times 1)$$

i.e, $dQ = (\frac{\partial \psi}{dx}dx + \frac{\partial \psi}{dy}dy) = d\psi$

$$dQ = (\frac{\partial \psi}{dx}dx + \frac{\partial \psi}{dy}dy) = d\psi$$

That is change in ψ across the element surface is equal to the volume flow through the element.

Volume flow between any two stream lines ψ_1 & ψ_2 = $\psi_2 - \psi_1 = \int_1^2 d\psi = \int_1^2 dQ$