29 – March – 2017

Lecture 31

Conservation of Energy (Contd…)

The last class, we started discussing about the conservation of energy principle.

From the Reynolds Transport Theorem on Energy principle,

$$\frac{DE}{Dt} = \frac{d}{dt}\iiint_{cv} e\rho dU + \iint_{cs} e\rho(\vec{v}.\hat{n})dA$$

where E → energy in the system

e → intensive property (energy per unit mass),

We had earlier derived for the elementary control volume $\Delta x\Delta y\Delta z$:

$$\frac{dQ}{dt} - \frac{dW_v}{dx} = \Delta x\Delta y\Delta z\left[\rho\frac{de}{dt} + \left(\overrightarrow{v}.\nabla\right)p + p\nabla.\vec{v}\right]$$

(Note there is no shaft work rate)

Subsequently, we decided to describe $\frac{dQ}{dt}$ and $\frac{dW_v}{dx}$.

To describe $\frac{dQ}{dt}$:

Assuming that transmission of heat in the elementary volume is only through conduction,

- → Fourier's heat law is applied to describe the heat flux per unit (i.e., the heat energy per unit area per unit time).
  The vectorial description is
  $$\vec{q} = -k\nabla T$$
  $\vec{q}$ → heat flux transmitted per unit area (or heat energy transmitter per unit area per unit time).
  T → temperature

⇨ We also earlier described, the reason behind providing negative sign.

➔ Subsequently, we described that

- The net heat outflux through the control surfaces of the elementary volume

  $=\left[\frac{\partial}{\partial x} q_x + \frac{\partial}{\partial y} q_y + \frac{\partial}{\partial z} q_z\right] \Delta x \Delta y \Delta z$

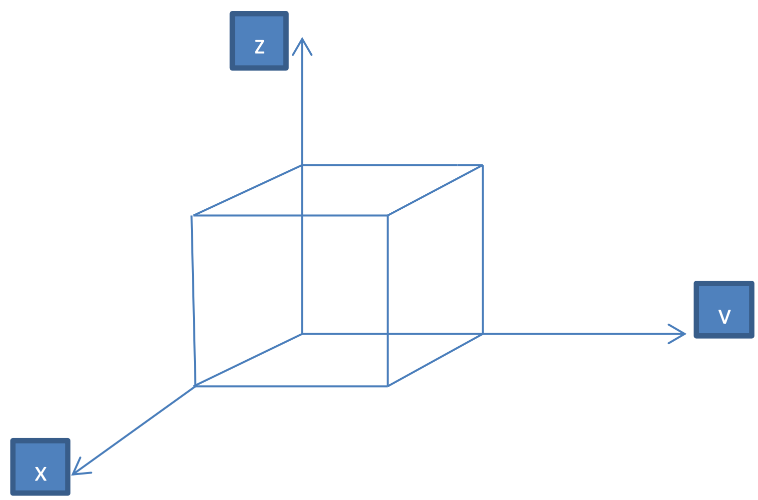
  $=(\nabla . \vec{q}) \Delta x \Delta y \Delta z$

  Hence, the term $\frac{dQ}{dt}$ → Rate of heat energy provided to the system

$$\frac{dQ}{dt} = -(\nabla . \vec{q}) \Delta x \Delta y \Delta z = (\nabla . k \nabla T) \Delta x \Delta y \Delta z$$

<u>To describe viscous work rate</u> ($\frac{dW_v}{dt}$):

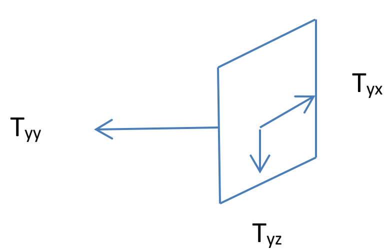
The viscous work rate has to be described now based on the elementary volume shape as shown below.



➔ In the y-direction, there are two planes perpendicular.

➔ On the left side and on the right side.

➔ Consider the plane on the left side as shown below:



The viscous stresses are τ_{yy} , τ_{yz} , τ_{yx}, all in the negative directions of their coordinate axes.

The velocity vector : $\vec{v} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$

The rate of work done by the viscous stress was earlier defined

= Viscous stress component × Velocity component × Area of element face

Therefore, on this left side (LS) plane, the rate of viscous work will be:

$$\frac{dW}{dt}_{LS} = -\tau_{yy}v\Delta x\Delta z - \tau_{yx}u\Delta x\Delta z - \tau_{yz}w\Delta x\Delta z$$

(The negative sign shows, force and velocity in opposite direction).

That is, this work rate

$$= \left[-\tau_{yy}v - \tau_{yx}u - \tau_{yz}w\right]\Delta x\Delta z$$

Define this term as **viscous work rate**

On that plane per unit area, w_{v_y}

i.e., $w_{v_y} = -\left(u\tau_{yx} + v\tau_{yy} + w\tau_{yz}\right)$

(Rate of change of viscous work per unit area on a plane perpendicular to y-direction).

So, viscous work rate on the left plane done by the system:

$$\frac{dW}{dt}_{LS} = w_{v_y}\Delta x\Delta z$$

- ➔ In a similar way, we can define viscous work rate on the right side plane :

$$= \left[w_{v_y} + \frac{\partial w_{v_y}}{\partial y}\Delta y\right]\Delta x\Delta z$$

Therefore, the net viscous work rate in y-direction

$$= \frac{\partial(w_{v_y})}{\partial y}\Delta y \Delta x \Delta z$$

Or, the net viscous work rate by the system

$$\frac{dW_v}{dt} = [\frac{\partial(w_{v_x})}{\partial x} + \frac{\partial\left(w_{v_y}\right)}{\partial y} + \frac{\partial(w_{v_z})}{\partial z}]\Delta x \Delta y \Delta z$$

$$= -\left[\frac{\partial}{\partial x}\left(u\tau_{xx} + v\tau_{xy} + w\tau_{xz}\right) + \frac{\partial}{\partial y}\left(u\tau_{yx} + v\tau_{yy} + w\tau_{yz}\right) + \frac{\partial}{\partial z}\left(u\tau_{zx} + v\tau_{zy} + w\tau_{zz}\right)\right]\Delta x \Delta y \Delta z$$

$$or, \frac{dW_v}{dt} = -\nabla.(\vec{v}.\bar{\bar{\tau}})\Delta x \Delta y \Delta z$$

Therefore, the energy equation

$$\frac{dQ}{dt} - \frac{dW_v}{dx} = \Delta x \Delta y \Delta z\left[\rho\frac{de}{dt} + \left(\overrightarrow{v}.\nabla\right)p + p\nabla.\vec{v}\right]$$

Becomes

$$\left[\vec{\nabla}.k\vec{\nabla}T\right]\Delta x \Delta y \Delta z + \left[\overrightarrow{\nabla}.(\vec{v}.\bar{\bar{\tau}})\right]\Delta x \Delta y \Delta z = \Delta x \Delta y \Delta z\left[\rho\frac{de}{dt} + \left(\overrightarrow{v}.\vec{\nabla}\right)p + p(\vec{\nabla}.\overrightarrow{v})\right]$$

As the volume $\Delta x \Delta y \Delta z$ is arbitrary , the differential equation for energy is obtained:

$$\vec{\nabla}.\left(k\vec{\nabla}T\right) + \vec{\nabla}.(\vec{v}.\bar{\bar{\tau}}) = \rho\frac{de}{dt} + (\overrightarrow{v}.\vec{\nabla})p + p(\vec{\nabla}.\overrightarrow{v})$$

<u>In index notation:</u>

$$\frac{\partial}{\partial x_p}\left[k\frac{\partial T}{\partial x_p} + v_m\tau_{pm}\right] = \rho\frac{de}{dt} + v_p\frac{\partial p}{\partial x_p} + p\frac{\partial v_p}{\partial x_p}$$