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Lecture – 28

Navier – Stoke Momentum equations Development

In the last class, we were deriving the conservation of momentum equation. You can recall, we came up with the expression:

$$\rho g - \vec{\nabla} p + \nabla \bar{\bar{\tau}} = \rho \frac{D\vec{v}}{Dt} \tag{1}$$

where, $\dfrac{D\vec{v}}{Dt} = \dfrac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v}$

As is being repeatedly discussed, the conservation of linear momentum equation is a first ranked tensor or a vector.

Therefore, equation 1 can be written in component wise (or in expanded form) in a three-dimension orthogonal Cartesian coordinate system:

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx} = \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] \quad \textbf{..........} \text{ in x-direction}$$

$$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{zy} = \rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right] \quad \textbf{..........} \text{ in y-direction} \tag{2a}$$

$$\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz} = \rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right] \quad \textbf{..........} \text{ in z-direction}$$

You can also write the above equation in index notation as:

$$\rho g_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\tau_{ji} = \rho\left[\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right] \tag{2b}$$

The gradient of viscous stress tensor cause the viscous force.

Inviscid Fluid

A simple situation of flow is, it is frictionless.

That is the effect of viscosity will not resist the fluid flow. For such fluid flow the viscous stress tensor will be zero.

i.e. $\bar{\bar{\tau}} = 0$

Then equation (2b) become

$$\rho g_i - \frac{\partial p}{\partial x_i} = \rho\left[\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right]$$

$$\rho\vec{g} - \vec{\nabla}p = \rho\frac{D\vec{v}}{Dt}$$

This is **Euler's inviscid flow** equation. In inviscid flow you need to solve only Euler's equation to get the information of dependent variable 'p' and $\vec{v}$. For general fluid flow, the momentum equation is :-

$$\rho\vec{g} - \vec{\nabla}p + \nabla.\bar{\bar{\tau}} = \rho\frac{D\vec{v}}{Dt}$$

Recall the stress tensor

$$\bar{\bar{\sigma}} = -p\bar{\bar{\delta}} + \bar{\bar{\tau}} \quad \text{or} \quad \sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

where δ_{ij} is Kronecker delta

- You know for static fluid $\bar{\bar{\tau}} = 0$.
- That is, the stress is decomposed into static stress and fluid dynamic stress.
- The fluid dynamic viscous stress is dependent on velocity gradient tensor.
- The Newtonian fluids follow the simplest possible linear constitutive equation with velocity gradient.
- Recall we had earlier studied, the velocity gradient tensor as:-

$$\frac{\partial v_i}{\partial x_j} = \underbrace{\frac{1}{2}\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right]}_{Symmeteric} + \underbrace{\frac{1}{2}\left[\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right]}_{Anti-symmeteric}$$

Theoretically, stress will develop only in fluid element that change shape.

Rotation does not change shape.

Therefore, the viscous stress depends on strain rates only.

The most general linear constitutive form of viscous stress with strain rate is:

$$\tau_{pq} = K_{pqmn}S_{mn}$$

where, K_{pqmn} is a fourth order tensor that have 81 components.

However, the stress tensor is symmetric and assuming the fluid to be isotropic, the fourth order tensor has to become isotropic.

i.e. $K_{pqmn} = \lambda\delta_{pq}\delta_{mn} + \mu\delta_{pm}\delta_{qn} + \gamma\delta_{pn}\delta_{qm}$
where, λ, μ and γ are scalar values.

As τ_{pq} is symmetric, K_{pqmn} also have to be symmetric in 'p' and 'q'.

Then $\gamma = \mu$

You will get $\tau_{pq} = 2\mu S_{pq} + \lambda S_{mm}\delta_{pq}$

where $S_{pq} = \frac{1}{2}\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right]$ and $S_{mm} = \frac{\partial v_m}{\partial x_m}$ (S_{mm} suggest summation of diagonal elements $\frac{\partial v_m}{\partial x_m}$)

i.e. $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla.\vec{v}$

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \nabla.\vec{v}$$

The summation of $\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \nabla.\vec{v}$ is actually the volumetric strain rate.

<u>For incompressible fluids</u> :-

$$\tau_{pq} = 2\mu S_{pq} + \lambda S_{mm}\delta_{pq}$$

$$\tau_{pq} = 2\mu\frac{1}{2}\left[\frac{\partial v_p}{\partial x_q} + \frac{\partial v_q}{\partial x_p}\right] + \lambda\left[\frac{\partial v_m}{\partial x_m}\right]\delta_{pq}$$

$$\tau_{pq} = \mu\left[\frac{\partial v_p}{\partial x_q} + \frac{\partial v_q}{\partial x_p}\right]$$

(Recall for incompressible fluid $\nabla.\vec{v} = 0$)

Expanding the expression for system of equation in (2a)

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x}$$

(Please note that here x is not a dummy index)

$$\tau_{yy} = 2\mu\frac{\partial v}{\partial y}$$

$$\tau_{zz} = 2\mu\frac{\partial w}{\partial z}$$

$$\tau_{yx} = \mu\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]$$
$$\tau_{xz} = \mu\left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right]$$
$$\tau_{zy} = \mu\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right]$$

The coefficient μ is here the dynamic viscosity coefficient.

Hence, the momentum equation for incompressible fluid in x, y and z directions are as follows:-

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] = \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right]$$
$$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[2\mu\frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] = \rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right]$$
$$\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z}\right] = \rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right]$$