22/03/2017

LECTURE – 27

Conservation of Linear Momentum

Yesterday, we started formulating the conservation of linear momentum principle for a rectangular elemental volume.

- After applying RTT to the elemental volume, we came up with the expression:

$$\sum \vec{F} = \rho \Delta x \Delta y \Delta z [\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v}]$$

$$\sum F_i = \rho \Delta x \Delta y \Delta z [\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}]$$

You also know that the term $\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v}$ is the material derivative of velocity & is also called acceleration.

- The term $\sum \vec{F}$ should account all types of forces.

You have studied that forces are of two types:

1. Body forces
2. Surface forces

You have also seen that <u>body forces</u> may be due to gravity, magnetism, electric potential etc.

- For the time being, we consider only the <u>gravitational force</u> as body force.
- For the given rectangular elemental volume $\Delta x \Delta y \Delta z$, the gravitational force may be given in elemental form

$$d\vec{F}_{gravity} = \rho \vec{g} \Delta x \Delta y \Delta z$$

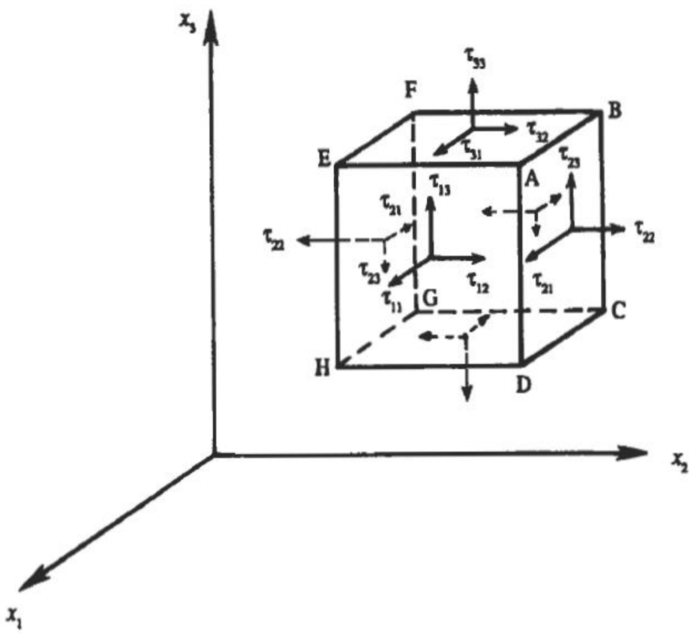
where gravitational acceleration vector can be given as:

$$\vec{g} = 0\hat{i} + 0\hat{j} - g\hat{k}$$

or $\quad 0\hat{e}_1 + 0\hat{e}_2 - g\hat{e}_3$

- The surface forces are due to pressure & various stresses, like viscous stress etc.

- As the elemental volume considered here is rectangular prism, there are six sides for the volume. The stresses on each sides of the volume will constitute the surface forces on the volume.
- The stresses considered here are hydrostatic pressure & viscous stresses.



(Source: Fluid Mechanics by Kundu & Cohen)

- As you see on the sides of this rectangular element, there can be normal & shear stresses.

- There are <u>normal stress in three directions</u> & <u>two shear stresses</u> on each plane.

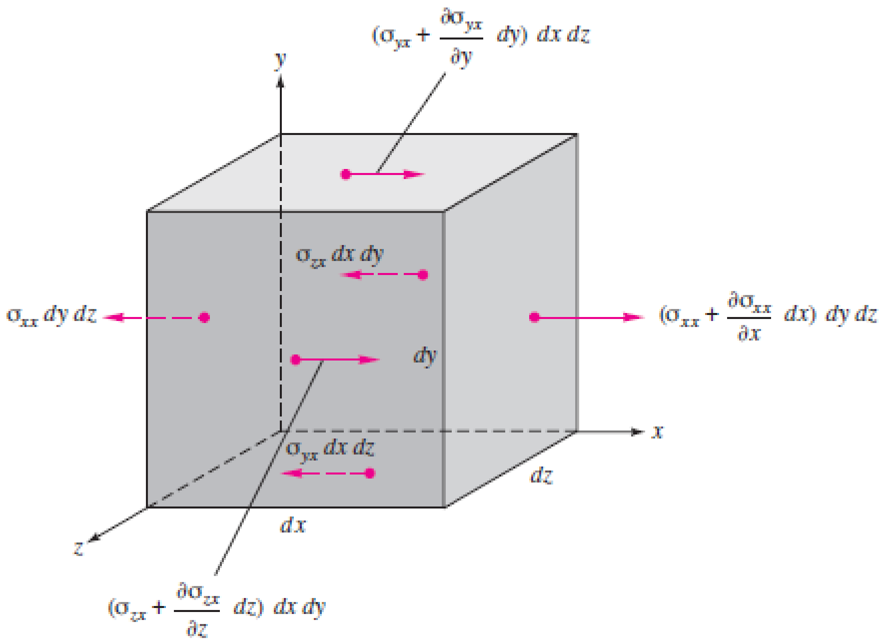
Therefore, mathematically to describe stress at a point, you require three normal components & six shear components. So, stress is a second rank tensor.

The viscous stress is given by $\bar{\bar{\tau}}$ or τ_{ij}

$$\bar{\bar{\tau}} = \begin{pmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \tau_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \tau_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{pmatrix}$$

- You also recall strain rates that were described as second rank tensors in earlier chapter (ε_{ij})
- The combination of pressure & viscous stresses in the stress tensor σ_{ij} or ($\bar{\bar{\sigma}}$)

$$\bar{\bar{\sigma}} \rightarrow \sigma_{ij} = \begin{pmatrix} -p+\tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p+\tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p+\tau_{zz} \end{pmatrix}$$



(Image Source: Fluid Mechanics by F. M. White)

Let us first consider the y-direction & evaluate the surface force in y-direction

- The normal surface forces in y-direction on the rectangular element

$$= (\sigma_{yy} + \frac{\partial \sigma_{yy}}{\partial y}\Delta y)\Delta x\Delta z - \sigma_{yy}\Delta x\Delta z$$

$$= \frac{\partial \sigma_{yy}}{\partial y}\Delta x\Delta y\Delta z$$

- The tangential or shear force in y direction on the rectangular element

$$= (\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial x}\Delta x)\Delta y\Delta z - \sigma_{xy}\Delta x\Delta y + (\sigma_{zy} + \frac{\partial \sigma_{zy}}{\partial z}\Delta z)\Delta x\Delta y - \sigma_{zy}\Delta x\Delta y$$

Therefore, the net surface force in y direction

$$dF_{y,surface} = [\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z}]\Delta x \Delta y \Delta z$$

i.e. $$dF_{y,surface} = [-\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}]\Delta x \Delta y \Delta z$$

$$dF_{x,surface} = [-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}]\Delta x \Delta y \Delta z$$

$$dF_{z,surface} = [-\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}]\Delta x \Delta y \Delta z$$

Similarly, $$d\vec{F}_{surface} = dF_{x,surface}\hat{i} + dF_{y,surface}\hat{j} + dF_{z,surface}\hat{k}$$

$$d\vec{F}_{surface} = -\vec{\nabla} p + d\vec{F}_{viscous}$$

$$d\vec{F}_{surface} = -\vec{\nabla} p \Delta x \Delta y \Delta z + \nabla . \overline{\overline{\tau}} \Delta x \Delta y \Delta z$$

where $\overline{\overline{\tau}}$ is second ranked viscous stress tensor

So, for the elemental volume,

$$d\vec{F}_{surface} = \rho \vec{g} \Delta x \Delta y \Delta z$$

$$d\vec{F}_{surface} = -\vec{\nabla} p + \nabla . \overline{\overline{\tau}}$$

In the expression,

$$\sum \vec{F} = \rho \Delta \mathrm{x} \Delta \mathrm{y} \Delta \mathrm{z} [\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v}]$$

$$\rho \vec{g} \Delta x \Delta y \Delta z - \vec{\nabla} p \Delta x \Delta y \Delta z + \nabla . \overline{\overline{\tau}} \Delta x \Delta y \Delta z = \rho \Delta \mathrm{x} \Delta \mathrm{y} \Delta \mathrm{z} [\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v}]$$

As the elemental volume ΔxΔyΔz is arbitrary & non-zero quantity:

$$\rho\vec{g} - \vec{\nabla}p + \nabla.\overline{\overline{\tau}} = \rho[\frac{\partial\vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v}]$$

$$\rho\frac{D\vec{v}}{Dt} = \rho\vec{g} - \vec{\nabla}p + \nabla.\overline{\overline{\tau}}$$

The momentum equation is a vectorial expression & hence consists of three components in three directions.

Therefore, you can write,

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} = \rho[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}]$$

$$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\tau_{yy}}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} = \rho[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}]$$

$$\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\tau_{zz}}{\partial z} = \rho[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}]$$

These are three partial differential equations used in full form for linear momentum principle.

You need to solve these equations to interpret any fluid flow in any domain.

You can write the above in the index notation:

$$\rho g_i - \frac{\partial p}{\partial x_i} + \frac{\partial\tau_{ji}}{\partial x_j} = \rho[\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}]$$