21.03.2017

Lecture 26

Differential Analysis of Flow (Contd..)

In the last class, we derived the partial differential equation for *conservation of mass* principle.

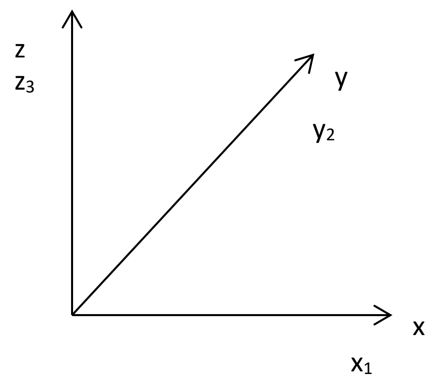


Fig. 1: Reference Coordinate axes

Recall that equation:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 \qquad \rightarrow (1)$$

where the velocity vector is $\vec{v}=u\hat{i}+v\hat{j}+w\hat{k}$ *or* $\vec{v}=v_1\hat{e}_1+v_2\hat{e}_2+v_3\hat{e}_3$

Looking into equation 1, it is obvious to you that it can be expressed as

$$\frac{\partial \rho}{\partial t}+\nabla.(\rho\vec{v})=0 \qquad \rightarrow (2)$$

- By this time, you should be knowing, how to write equation 2 in index notation.

  $$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_i}(\rho v_i)=0$$

- If the fluid is incompressible, then $\frac{\partial \rho}{\partial t} = 0$.
- Therefore, the conservation of mass equation becomes

  $\nabla.(\rho\vec{v}) = 0$

  i.e. $\rho(\nabla.\vec{v}) = 0$

  $\nabla.\vec{v} = 0 \qquad \rightarrow (3)$

  That is, for incompressible flow, $\nabla.\vec{v} = 0$

- If for any case, you need to use cylindrical polar coordinates, the principle will still be valid.

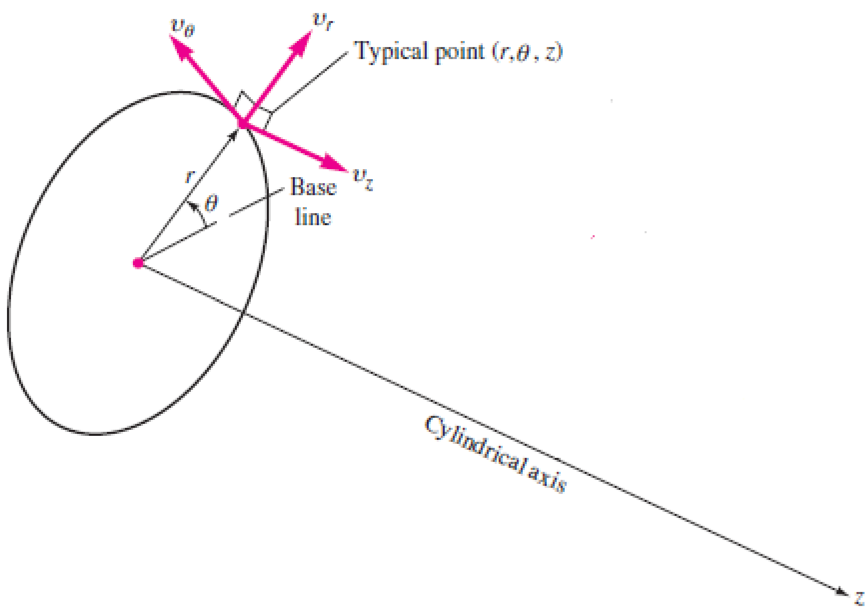


Fig.2: Definition sketch for cylindrical coordinate system
(Source: Fluid Mechanics by F.M. White)

The velocity components are:

Axial velocity : v_z

Radial velocity : v_r

Circumferential velocity : v_θ

Like in Cartesian coordinates (x_1, x_2, x_3 or x, y, z) in cylindrical polar coordinates any fluid property will be function of (r, θ, z, t).

- How are you going to write the conservation of mass equation in cylindrical polar coordinates ?

  Note the relation between cylindrical polar coordinates and Cartesian coordinates :

  $$r = \sqrt{x^2 + y^2}$$

  $$\theta = \tan^{-1}(\frac{y}{x})$$

  $$z = z$$

- The divergence of any vector $\vec{B}$ in cylindrical polar coordinates is:

  $$\vec{B} = B_r \hat{r} + B_\theta \hat{\theta} + B_z \hat{z}$$

  $$\nabla.\vec{B} = \frac{1}{r}\frac{\partial(rB_r)}{\partial r} + \frac{1}{r}\frac{\partial(B_\theta)}{\partial \theta} + \frac{\partial(B_z)}{\partial z}$$

  Therefore, the continuity equation will be :

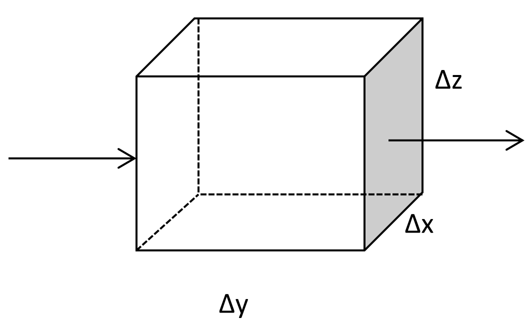
  $$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(v_z)}{\partial z}$$

Conservation of Linear Momentum

The principle of conservation of linear momentum can also be differentially analyzed for fluids.

- Here again, we will take the same rectangular prismoidal element that was discussed for conservation of mass principle.

$$\rho v \vec{v} \Delta x \Delta z$$



$$[\rho v \vec{v} + \frac{\partial(\rho v \vec{v})}{\partial y}\Delta y]\Delta x \Delta z$$

- This differential element has six rectangular faces.
- As per the shape we are assuming 1-D input and output faces.

Since the volume is fixed in space and stationary, the momentum principle using RTT for this elementary volume will be:

$$\frac{D(m\vec{v})}{Dt} = \frac{\partial}{\partial t}[\iiint_{cv} \vec{v}\rho dU] + \iint_{cs} \vec{v}\rho(\vec{v}.\hat{n})dA$$

i.e. $$\sum \vec{F} = \frac{\partial}{\partial t}[\iiint_{cv} \vec{v}\rho dU] + \sum_{outlet}[\rho|\vec{v}|A\vec{v}] - \sum_{inlet}\rho|\vec{v}|A\vec{v}$$

Since there is 1-dimensional input and output, assumed.

- The surface integral term is Net momentum outflux.

- The volumetric integral:
$$\frac{\partial}{\partial t}[\iiint_{cv} \vec{v}\rho dU] \approx \frac{\partial(\rho\vec{v})}{\partial t}\Delta x\Delta y\Delta z$$
(for the rectangular elemental volume)

- Let the momentum influx on the left side will be = $\rho v\vec{v}\Delta x\Delta z$

  And the momentum outflux will be = $[\rho v\vec{v} + \frac{\partial(\rho v\vec{v})}{\partial y}\Delta y]\Delta x\Delta z$

Therefore, the net momentum outflux in the y-direction will be outflux – influx

$$= [\rho v\vec{v} + \frac{\partial(\rho v\vec{v})}{\partial y}\Delta y]\Delta x\Delta z - \rho v\vec{v}\Delta x\Delta z$$

$$= \frac{\partial(\rho v\vec{v})}{\partial y}\Delta x\Delta y\Delta z$$

Similarly, in x-direction = $\frac{\partial(\rho u\vec{v})}{\partial x}\Delta x\Delta y\Delta z$

And, in z- direction = $\frac{\partial(\rho w\vec{v})}{\partial z}\Delta x\Delta y\Delta z$

Therefore, the RTT becomes:

$$\sum \vec{F} = \frac{\partial(\rho\vec{v})}{\partial t}\Delta x\Delta y\Delta z + \frac{\partial(\rho u\vec{v})}{\partial x}\Delta x\Delta y\Delta z + \frac{\partial(\rho v\vec{v})}{\partial y}\Delta x\Delta y\Delta z + \frac{\partial(\rho w\vec{v})}{\partial z}\Delta x\Delta y\Delta z$$

$$\sum \vec{F} = [\frac{\partial(\rho\vec{v})}{\partial t} + \frac{\partial(\rho u\vec{v})}{\partial x} + \frac{\partial(\rho v\vec{v})}{\partial y} + \frac{\partial(\rho w\vec{v})}{\partial z}]\Delta x\Delta y\Delta z$$

$$\sum \vec{F} = [\vec{v}\frac{\partial\rho}{\partial t} + \vec{v}(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}) + \rho\frac{\partial\vec{v}}{\partial t} + \rho\,\mathrm{u}\frac{\partial\vec{v}}{\partial x} + \rho v\frac{\partial\vec{v}}{\partial y} + \rho w\frac{\partial\vec{v}}{\partial z}]\Delta x\Delta y\Delta z$$

$$\sum \vec{F} = [\vec{v}\{\frac{\partial\rho}{\partial t} + \nabla.(\rho\vec{v})\} + \rho\{\frac{\partial\vec{v}}{\partial t} + \mathrm{u}\frac{\partial\vec{v}}{\partial x} + v\frac{\partial\vec{v}}{\partial y} + w\frac{\partial\vec{v}}{\partial z}\}]\Delta x\Delta y\Delta z$$

Note $\frac{\partial\rho}{\partial t} + \nabla.(\rho\vec{v}) = 0$

i.e. $\sum \vec{F} = \rho\Delta\,\mathrm{x}\,\Delta\,\mathrm{y}\,\Delta\,\mathrm{z}[\frac{\partial\vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v}]$

Recall acceleration, $\vec{a} = \frac{D\vec{v}}{Dt} = \frac{\partial\vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v}$ (studied earlier in particle kinematics)

$$\sum \vec{F} = \rho\frac{D\vec{v}}{Dt}\Delta x\Delta y\Delta z$$

- As discussed earlier there can be:
  - Body forces
  - Surface forces

- The body forces can be:
  1. Gravitational
  2. Magnetic
  3. Electric potential, etc.

- For our usual fluid like water, etc. the magnetic and electric potential are neglected.

Quiz:

1. Write in index notation, the conservation of mass and conservation of momentum equation.
2. Write the conservation of mass equation in cylindrical polar coordinates.