LINEAR MOMENTUM EQUATIONS

Last week, we started discussing on the differential approach in fluid flow analysis

First, we discussed on conservation of mass and came up with the differential equation:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \vec{V}) = 0} \quad \text{or} \quad \boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0}$$

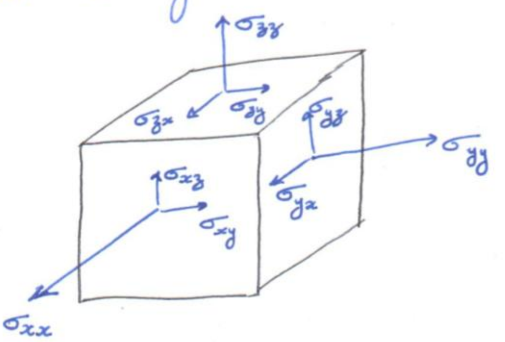
Subsequently, we started discussing on linear momentum

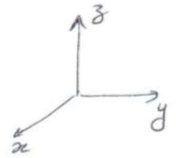
→ For the differential or elemental volume, we saw

$$\Sigma \vec{F} = \rho \frac{d\vec{V}}{dt}\, \Delta x \Delta y \Delta z$$

- → Gravitational force, $d\vec{F}_{grav} = \rho \vec{g}\, \Delta x \Delta y \Delta z$
- → Surface force
  - → Pressure force
  - → Viscous force

→ Let us give the stress notation as below:





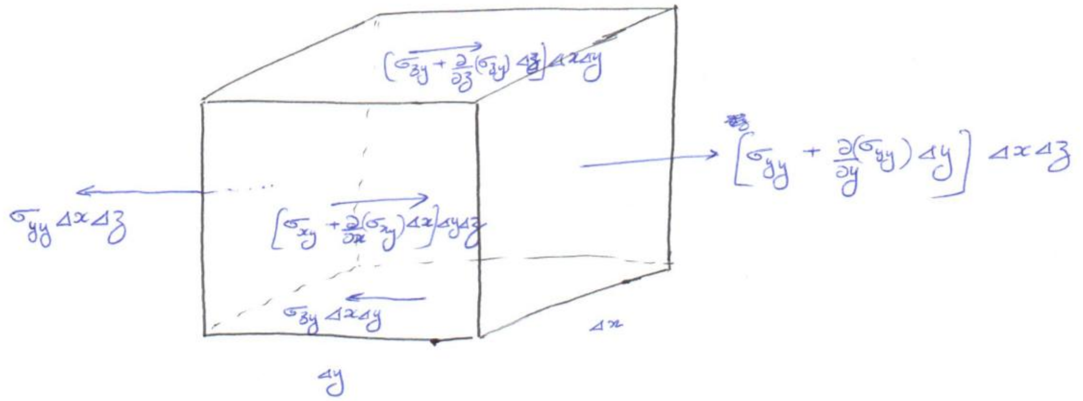
$\overline{\overline{\sigma}}$ → stress tensor

$$= \begin{pmatrix} -p+\tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p+\tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p+\tau_{zz} \end{pmatrix}$$

Recall shear stress can also be given as tensor $\overline{\overline{\tau}}$.

To find the surface forces on the element $\Delta x \Delta y \Delta z$, we thought of first splitting the force in component wise in each of x, y, z- directions.

Therefore, we thought of first finding the net surface force in y- direction.



and we got at that time:

$$dF_{y,\,surface} = \frac{\partial}{\partial x}(\sigma_{xy}) \Delta x \Delta y \Delta z + \frac{\partial}{\partial y}(\sigma_{yy}) \Delta x \Delta y \Delta z + \frac{\partial}{\partial z}(\sigma_{zy}) \Delta x \Delta y \Delta z$$

$$= \left[-\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}(\tau_{xy}) + \frac{\partial}{\partial y}(\tau_{yy}) + \frac{\partial}{\partial z}(\tau_{zy})\right] \Delta x \Delta y \Delta z$$

Recall, $\sigma_{yy} = -p + \tau_{yy}$, $\sigma_{xy} = \tau_{xy}$, $\sigma_{zy} = \tau_{zy}$

Similarly,

$$dF_{x,\,surface} = \left[-\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx})\right] \Delta x \Delta y \Delta z$$

$$dF_{z,\,surface} = \left[-\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}(\tau_{xz}) + \frac{\partial}{\partial y}(\tau_{yz}) + \frac{\partial}{\partial z}(\tau_{zz})\right] \Delta x \Delta y \Delta z$$

$$\therefore \; d\vec{F}_{surface} = dF_{x_{surface}}\,\hat{i} + dF_{y_{surface}}\,\hat{j} + dF_{z_{surface}}\,\hat{k}$$

Note that, it is gradient of pressure and viscous stress that cause surface forces.

Therefore, in the expression

$$\Sigma \vec{F} = \rho \frac{d\vec{v}}{dt} \Delta x \Delta y \Delta z$$

i.e. $$d\vec{F}_{gravity} + d\vec{F}_{surface} = \rho \frac{d\vec{v}}{dt} \Delta x \Delta y \Delta z$$

i.e. $$\Big\{ \rho\vec{g} + \left(-\frac{\partial p}{\partial x}\hat{i} - \frac{\partial p}{\partial y}\hat{j} - \frac{\partial p}{\partial z}\hat{k} \right) + \left[\frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx}) \right]\hat{i}$$
$$+ \left[\frac{\partial}{\partial x}(\tau_{xy}) + \frac{\partial}{\partial y}(\tau_{yy}) + \frac{\partial}{\partial z}(\tau_{zy}) \right]\hat{j} + \left[\frac{\partial}{\partial x}(\tau_{xz}) + \frac{\partial}{\partial y}(\tau_{yz}) + \frac{\partial}{\partial z}(\tau_{zz}) \right]\hat{k} \Big\}$$
$$= \rho \left[\frac{\partial \vec{v}}{\partial t} + u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y} + w\frac{\partial \vec{v}}{\partial z} \right] \longrightarrow ①$$

See, we have cancelled $\Delta x \Delta y \Delta z$ on both sides.

⇒ We know that force is a vector. Therefore, the above equation should match for each components in x, y, z directions.

i.e. $$\vec{v} = u\hat{i} + v\hat{j} + w\hat{k}$$
$$\vec{g} = g_x\hat{i} + g_y\hat{j} + g_z\hat{k}$$

In x- direction, you have:

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx} = \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right]$$

llly $$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{zy} = \rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right]$$

and $$\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz} = \rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right]$$

$\Rightarrow ②$

The set of three partial differential equations are the differential equations that are used in full form for linear momentum principle.

→ One need to solve these partial differential equations ~~to~~ in the respective domain with appropriate boundary and initial conditions. to interpret the fluid flow.

→ This set is valid for any type and any fluid.

If the flow is inviscid ??

If the fluid flow is inviscid, it means that effects of viscous stresses are negligible.

Then your momentum equation becomes:

$$\boxed{\rho \vec{g} - \vec{\nabla} p = \rho \frac{d\vec{V}}{dt}} \longrightarrow (3)$$

Equation (3) is called Euler's equation for inviscid flow.

⇒ If you integrate, Euler's equation for a streamline, you will get the Bernoulli's equation.

For Newtonian fluids

We have studied earlier for Newtonian fluids, the viscous stresses are proportional to strain rates. (Recall Newton's theory on viscosity).

Extending those relations to the entire components of viscous stress tensor

$$\bar{\bar{\tau}} = \begin{pmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{pmatrix}$$

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} \quad , \quad \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

From higher theories on fluid mechanics, it is observed that $\tau_{yx} = \tau_{xy}$, ~~and~~ $\tau_{zx} = \tau_{xz}$, and

$$\tau_{zy} = \tau_{yz}$$

$$\tau_{yy} = 2\mu \frac{\partial v}{\partial y} \quad , \quad \tau_{zz} = 2\mu \frac{\partial w}{\partial z} .$$

$$\tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

Recall, μ was your viscosity coefficient

Therefore, the momentum equations for a constant density fluid will be:

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu \frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]$$

$$= \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right]$$