Conservation of Linear Momentum

Yesterday, we started discussing on differential approach to analyse fluid motion.

→ The first principle we discussed was the conservation of mass.

We got the following relations.

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \vec{v}) = 0} \rightarrow (1)$$

→ We also stated that above equation (1) can be further simplified for various situations.

i.e. (i) For steady flow: $\boxed{\nabla\cdot(\rho \vec{v}) = 0}$

(ii) For incompressible flow, as we are taking a small elemental volume, or rather, a mathematical point we can take

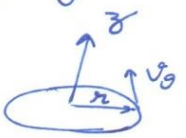
$\frac{\partial \rho}{\partial t} \approx 0$. Also $\frac{\partial \rho}{\partial x} = 0$, $\frac{\partial \rho}{\partial y} = 0$, $\frac{\partial \rho}{\partial z} = 0$

$\therefore$ We have $\boxed{\nabla\cdot\vec{v} = 0}$ for incompressible flow.

⇒ Let us take a quick quiz.

As for conservation of mass, $\frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{v}) = 0$ is vectorial notation, one can use any co-ordinate system.

⇒ If you want to use cylindrical polar coordinates, you have velocity components



Axial velocity → v_z

Radial velocity → v_r

Circumferential velocity → v_θ

That is, in cylindrical polar coordinates, all properties have to be continuous function of (r, θ, z, t).

⇒ How to write conservation of mass equation in cylindrical-polar co-ordinates ?!

Note: $r = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1} y/x$

$z = z$

Also the divergence of any vector $\vec{A}$ in cylindrical polar coordinates can be given as

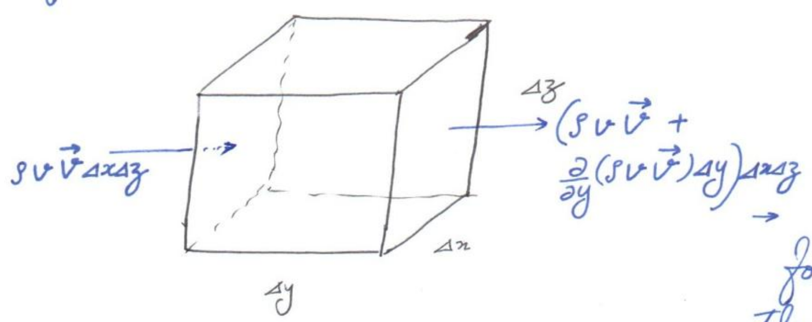
$$\vec{\nabla}.\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(A_\theta) + \frac{\partial}{\partial z}(A_z)$$

Now upto to you to form mass conservation equation.

Linear Momentum

In today's class, we will start discussing on conservation of linear momentum principle through differential approach.

⇒ Again, we will take the same elemental or differential volume discussed for conservation of mass.



→ This differential element has six faces

→ As described for mass, let there be one-dimensional inlet for momentum fluxes and one-dimensional outlet for momentum fluxes.

⇒ The volume is fixed in space and stationary

Recall RTT:

B = momentum, $\therefore \left.\frac{dB}{dt}\right|_{system} = \sum \vec{F}$,
$\beta = \vec{V}$

$$\therefore \sum \vec{F} = \frac{\partial}{\partial t}\left[\iiint_{cv} \vec{V} \rho \, dU\right] + \iint_{cs} \vec{V} \rho (\vec{V} \cdot \hat{n}) \, dA$$

For one-dimensional inlets and outlets

$$\sum \vec{F} = \frac{\partial}{\partial t}\left[\iiint_{cv} \vec{V} \rho \, dU\right] + \sum_{outlet}\left(\rho_i V_i \vec{V} A_i\right) - \sum_{inlet}\left(\rho_i V_i A_i \vec{V}\right)$$

As this is elemental volume $\Delta x \Delta y \Delta z$, we can write

$$\frac{\partial}{\partial t}\left[\iiint_{cv} \vec{V} \rho \, dU\right] \approx \frac{\partial(\rho \vec{V})}{\partial t} \Delta x \Delta y \Delta z$$

Similarly the net momentum outflow in y-direction will be

$$\left[\rho v \vec{V} + \frac{\partial}{\partial y}(\rho v \vec{V}) \Delta y\right] \Delta x \Delta z - \rho v \vec{V} \Delta x \Delta z$$

$$= \frac{\partial}{\partial y}(\rho v \vec{V}) \Delta y \Delta x \Delta z$$

Similarly in x-direction

$$\rightarrow \frac{\partial}{\partial x}(\rho u \vec{V}) \Delta x \Delta y \Delta z$$

z-direction

$$\rightarrow \frac{\partial}{\partial z}(\rho w \vec{V}) \Delta z \Delta x \Delta y$$

$\therefore$ The RTT becomes:

$$\Sigma \vec{F} = \frac{\partial}{\partial t}(\rho \vec{V}) \Delta x \Delta y \Delta z + \frac{\partial}{\partial x}(\rho u \vec{V}) \Delta x \Delta y \Delta z + \frac{\partial}{\partial y}(\rho v \vec{V}) \Delta x \Delta y \Delta z + \frac{\partial}{\partial z}(\rho w \vec{V}) \Delta x \Delta y \Delta z$$

i.e. $$\Sigma \vec{F} = \Delta x \Delta y \Delta z \left[\frac{\partial}{\partial t}(\rho \vec{V}) + \frac{\partial}{\partial x}(\rho u \vec{V}) + \frac{\partial}{\partial y}(\rho v \vec{V}) + \frac{\partial}{\partial z}(\rho w \vec{V})\right]$$

i.e. $$\Sigma \vec{F} = \Delta x \Delta y \Delta z \left[\vec{V}\left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V}\right) + \rho\left(\frac{\partial \vec{V}}{\partial t} + u \frac{\partial \vec{V}}{\partial x} + v \frac{\partial \vec{V}}{\partial y} + w \frac{\partial \vec{V}}{\partial z}\right)\right]$$

i.e. $$\sum \vec{F} = \rho \Delta x \Delta y \Delta z \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right]$$

Recall the acceleration of fluid was given as

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v}$$

$$\therefore \quad \sum \vec{F} = \rho \frac{d\vec{V}}{dt} \Delta x \Delta y \Delta z$$

You can see Mass × Acceleration here

We have to now see, what all types of forces are there that contribute to $\sum \vec{F}$

→ Body forces

→ Surface forces

Among body forces you can have - gravity, magnetism, electric potential, etc. type forces.

For our simplicity, let us consider the fluid that will have only gravitational force as ~~ga~~ body force.

$$\therefore \quad d\vec{F}_{grav} = \rho \vec{g} \Delta x \Delta y \Delta z$$

You may represent $\vec{g} = 0\hat{i} + 0\hat{j} - g\hat{k}$

→ The surface forces will be due to pressure and viscous stresses.