LECTURE 24

14-MAR-2016

DIFFERENTIAL RELATIONS FOR FLUID FLOW

We were discussing on integral or control volume relations for fluid flow for last many classes.

→ We were using Reynolds Transport Theorem throughout.

For any extensive property B

$$\left.\frac{dB}{dt}\right|_{system} = \frac{d}{dt}\left[\iiint_{cv} \beta \rho \, d\upsilon\right] + \iint_{cs} \beta \rho (\vec{V}_r \cdot \hat{n}) \, dA$$

where β is intensive property

→ In the beginning, we have mentioned that we will be following three main approaches to discuss about fluid flow:

- ⋆ Integral or Control Volume Approach
- ⋆ Differential Approach
- ⋆ Experimental or Dimensional Analysis.

→ Today, we will start discussing about the Differential Approach or Infinitesimal Approach

Recall about the Eulerian and Lagrangian description of motion.

Velocity is vector function of position and time

i.e. $\vec{v}(x,y,z,t) = \hat{i}\, u(x,y,z,t) + \hat{j}\, v(x,y,z,t) + \hat{k}\, w(x,y,z,t)$

where you know u, v, w are components of $\vec{v}$ in x, y, z - directions

This velocity vector is the most important variable in fluid mechanics.

→ We are considering the co-ordinates to be fixed in space and will be using Eulerian frame of reference.

⋆ Recall, in one of earlier lectures, we discussed the particle approach.

→ The total time derivative of velocity vector will give you the velocity vector.

$$\vec{a}(x,y,z,t) = \frac{d}{dt}\vec{V}(x,y,z,t) = \hat{i}\frac{du}{dt} + \hat{j}\frac{dv}{dt} + \hat{k}\frac{dz}{dt}$$

As $u(x,y,z,t)$, $v(x,y,z,t)$ and $w(x,y,z,t)$

we can use the chain rule of differentiation for total time derivative.

i.e. $$\frac{d}{dt}u(x,y,z,t) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

But you know that $\frac{dx}{dt}$ is u, $\frac{dy}{dt} = v$, $\frac{dz}{dt} = w$

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$= \frac{\partial u}{\partial t} + (\vec{v}\cdot\nabla)u = a_x$$

IIIy $$a_y = \frac{\partial v}{\partial t} + (\vec{v}\cdot\nabla)v$$

$$a_z = \frac{\partial w}{\partial t} + (\vec{v}\cdot\nabla)w$$

and Total acceleration:

$$\vec{a}(x,y,z,t) = \frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y} + w\frac{\partial \vec{v}}{\partial z}$$

i.e. $$\vec{a} = \frac{d\vec{v}}{dt} = \underbrace{\frac{\partial \vec{v}}{\partial t}}_{\text{Local Acceleration}} + \underbrace{(\vec{v}\cdot\nabla)\vec{v}}_{\text{Convective Acceleration}}$$

⇒ The term $\frac{d}{dt}$ is called Total Derivative or Substantial Derivative

$$\frac{d()}{dt} = \frac{\partial ()}{\partial t} + (\vec{v}\cdot\nabla)()$$

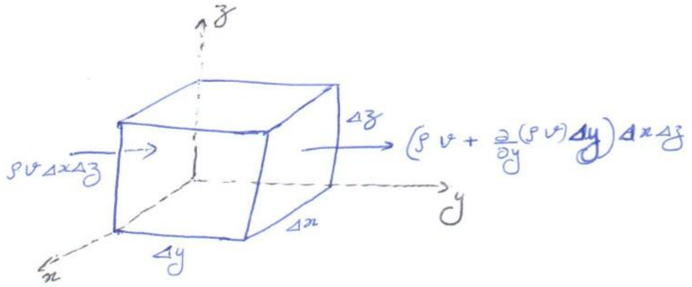
e.g.: If pressure, p is the variable:
Then substantial derivative of p will be:

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} + (\vec{v}\cdot\nabla)p$$

Again as done in integral approach, we will be dealing with conservation of mass, linear momentum, angular momentum, and energy in differential approach as well.

Differential Equation of Mass Conservation

Let us consider a small rectangular prism shaped element of fluid that is flowing.



→ The size, let us say, is $\Delta x \; \Delta y \; \Delta z$

→ Recall about mass flux that we were discussing in previous chapters.

→ In the y- direction, let the mass influx be $\rho v \, \Delta x \Delta z$

Let the mass outflux be $\left[\rho v + \frac{\partial}{\partial y}(\rho v)\Delta y\right]\Delta x \Delta z$

→ Similarly we can assign for all three directions:

i.e.

Direction	Mass Influx	Mass Outflux
x	$\rho u \, \Delta y \, \Delta z$	$\left[\rho u + \frac{\partial}{\partial x}(\rho u)\Delta x\right]\Delta y \Delta z$
z	$\rho w \, \Delta x \, \Delta y$	$\left[\rho w + \frac{\partial}{\partial z}(\rho w)\Delta z\right]\Delta x \Delta y$

As this rectangular prism is an elemental volume $\Delta x \Delta y \Delta z$, let us first apply the Reynolds Transport Theorem to this volume.

For mass,
$$\left.\frac{dB}{dt}\right|_{system} = \frac{dm}{dt} = 0 = \int_{cv} \frac{\partial \rho}{\partial t}\, dV + \sum_i \left[\rho_i A_i V_i\right]_{outlet} - \sum_i \left[\rho_i A_i V_i\right]_{inlet}$$

As the volume is elemental (or differential)

$$\int_{cv} \frac{\partial \rho}{\partial t}\, dV \approx \frac{\partial \rho}{\partial t} \Delta x \Delta y \Delta z$$

Also
$$\sum_i (\rho_i A_i V_i)_{outlet} = \left[\rho u + \frac{\partial}{\partial x}(\rho u)\Delta x\right]\Delta y \Delta z + \left[\rho v + \frac{\partial}{\partial y}(\rho v)\Delta y\right]\Delta x \Delta z + \left[\rho w + \frac{\partial}{\partial z}(\rho w)\Delta z\right]\Delta x \Delta y$$

$$\sum_i (\rho_i A_i V_i)_{inlet} = \rho u \,\Delta y \Delta z + \rho v \,\Delta x \Delta z + \rho w \,\Delta x \Delta y$$

$\therefore$ RTT becomes:

$$0 = \frac{\partial \rho}{\partial t} \Delta x \Delta y \Delta z + \frac{\partial}{\partial x}(\rho u)\Delta x \Delta y \Delta z + \frac{\partial}{\partial y}(\rho v)\Delta y \Delta x \Delta z + \frac{\partial}{\partial z}(\rho w)\Delta z \Delta x \Delta y$$

As the volume $\Delta x \Delta y \Delta z$ taken is fixed,

we get

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0}$$

This is the differential equation for conservation of mass.

We can also write that equation as:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0}$$

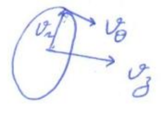
→ Imagine what will happen for incompressible flow. $\boxed{\nabla \cdot \vec{v} = 0}$

⇒ If someone is using cylindrical polar coordinates, the three independent velocity components will be:

Axial velocity → v_z

Radial velocity → v_r

Circumferential velocity → v_θ



So now we need all properties to be continuous function of (r, θ, z, t).

We can transform r, θ, z, t → to x, y, z, t

$$r = \sqrt{(x^2 + y^2)}$$

$$\theta = \tan^{-1} \frac{y}{x}$$

$$z = z$$

Divergence of any vector in cylindrical-polar coordinate will be:

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r} \frac{\partial}{\partial r}(r A_r) + \frac{1}{r} \frac{\partial}{\partial \theta}(A_\theta) + \frac{\partial}{\partial z}(A_z)$$

The continuity equation will be:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r}(r \rho v_r) + \frac{1}{r} \frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0}$$