Introduction to Algorithms

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Organization

Outline

O, Upper bound

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Exercise

- What is $O(1)$?
- Is $2^n = O(2^{n-1})$?
- What is $1 + r + r^2 + \ldots + r^n$ in terms of $O(r^{??})$?

Some basic idea

Geometric series sum

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- If $r = 1$, then the sum is $a(n+1)$; else if $r > 1$, the last term dominates; and if $r < 1$, the first term dominates.

Geometric series sum and throwing away a constant fraction

- An algorithm discards $1/3$ of whatever objects it is processing and recurses on the remaining $2/3$rd and suppose that in each call, the algorithm does not spend more than a constant time for each element. What is the overall time?

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The guiding divide and conquer recurrence

The recursion $T(n) = T(c_1 n) + T(c_2 n) + O(n)$ where c_1 and c_2 are constants and $c_1 + c_2 < 1$ solves to $T(n) = O(n)$.

Ω, Lower bound

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Relation between O and Ω

$f(n) = \Omega(g(n))$ if and only if $g(n) = O(f(n))$.

Exercises on O and Ω

Exercise on O

$$\log(n!) = \sum_{i=1}^{n} \log i \leq \sum_{i=1}^{n} \log n = O(n \log n)$$

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$$\log(n!) = \sum_{i=1}^{n} \log i \geq \sum_{i=1}^{\lceil \frac{n}{2} \rceil} \log\left(\frac{n}{2}\right) = \left\lceil \frac{n}{2} \right\rceil \log\left(\frac{n}{2}\right) = \Omega(n \log n)$$

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Exercise

- How many constants you need for Θ?
- Any constant function is $O(1)$, $\Omega(1)$ and $\Theta(1)$.

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Sums of functions

Let k be a fixed constant, and let $f_1, f_2, \ldots, f_k$ and h be functions such that $f_i = O(h)$, $\forall i$. Then $f_1 + f_2 + \cdots + f_k = O(h)$.

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Exercise

Suggest a function that is $o(1)$.

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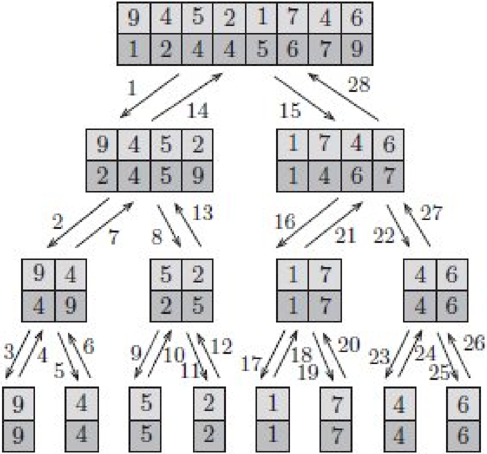
- This recurrence solves to ?

Merge Sort


9 4 5 2 1 7 4 6
1 2 4 4 5 6 7 9
9 4 5 2
2 4 5 9
1 7 4 6
1 4 6 7
9 4
4 9
5 2
2 5
1 7
1 7
4 6
4 6
9
9
4
4
5
5
2
2
1
1
7
7
4
4
6
6
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28

Mergesort

What is the guiding recurrence?

$$T(n) \leq \begin{cases} 0 & \text{if } n = 1 \\ 2T(n/2) + n - 1 & \text{if } n \geq 2 \end{cases}.$$

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- $xy = (2^{n/2}x_L + x_R)(2^{n/2}y_L + y_R)$
  $= 2^n x_L y_L + 2^{n/2}(x_L y_R + x_R y_L) + x_R y_R$
  (Significant operations are 4 $n/2$-bit multiplications and $O(n)$ left shifts and additions; recurrence is $T(n) \leq 4T(n/2) + O(n)$.)

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- $= 2^n x_L y_L + 2^{n/2}\{(x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R\} + x_R y_R$
  (Significant operations are 3 $n/2$-bit multiplications and $O(n)$ left shifts and additions; recurrence is $T(n) \leq 3T(n/2) + O(n)$.)

A few recursions

A divide and conquer recursion

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- At depth k, there are 3^k subproblems, each of size $n/2^k$.
- Total time spent at depth k is $3^k O(\frac{n}{2^k}) = (\frac{3}{2})^k O(n)$.
- The above is a geometric series with common ratio greater than 1. So, the last term $(k = \log_2 n)$ should matter. The last term is $O(3^{\log_2 n}) \approx O(n^{\log_2 3}) \approx O(n^{1.59})$.

Recurrence

Master theorem

If $T(n) = aT(\lceil n/b \rceil) + O(n^d)$ for some constants $a > 0$, $b > 1$, and $d \geq 0$, then

$$T(n) \leq \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}.$$

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Proof idea

The k-th level of the tree is made up of a^k subproblems, each of size n/b^k. The total work done is $a^k \cdot O(\frac{n}{b^k})^d = O(n^d) \cdot (\frac{a}{b^d})^k$. The three cases now follow from the idea of the sum of a geometric series.

Outline

Lower bound on comparison based sorting

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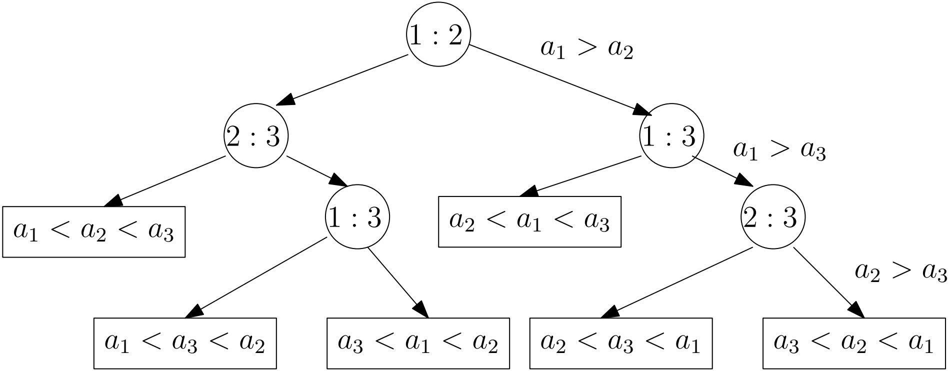

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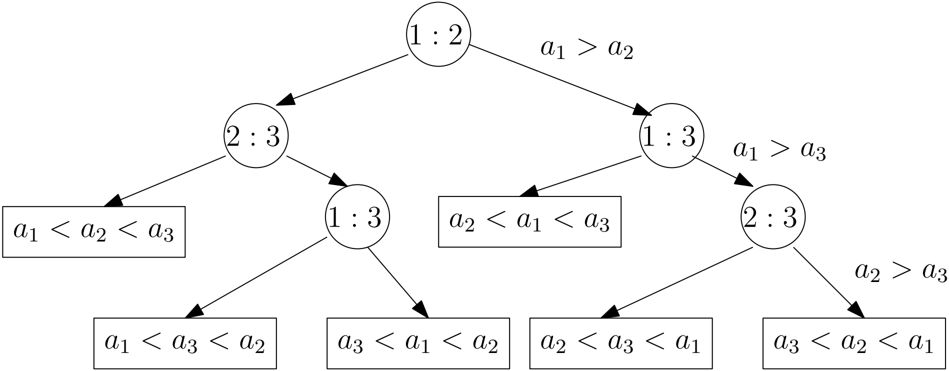

Observation

Time complexity in the worst case is the length of a longest path from the root to a leaf, which is the height of the decision tree.

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Lower bound on average case of comparison based sorting

Lower bound on average case of comparison based sorting is also $\Omega(n \log n)$

Quicksort and heapsort

Quicksort

The worst case time complexity of quicksort is $O(n^2)$.

Heapsort

The worst case time complexity of heapsort is $O(n \log n)$.

Average case analysis of quicksort

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- All permutations are equally likely. This ensures that any number in $\mathcal{A}$ is equally likely to be the pivot.
- Let $T(n)$ denote the number of comparisons done by the algorithm on an average on $\mathcal{A}$.
- We average over all possible inputs and the expression is

Average case analysis of quicksort

$$T(n) = (n-1) + \frac{1}{n}\sum_{i=1}^{n}(T(i-1) + T(n-i))$$

Since, $\sum_{i=1}^{n} T(n-i) = T(n-1) + T(n-2) + \cdots + T(0) = \sum_{i=1}^{n} T(i-1)$

We have $T(n) = (n-1) + \frac{2}{n}\sum_{i=1}^{n} T(i-1)$.

This recurrence solves to $O(n \log n)$

Average case time complexity of quicksort

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Linear time sort(!)

What happened to the lower bound of $\Omega(n \log n)$?

Counting sort

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- Counting sort is a stable sort.

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- Working on the example, pick up 3 from A; go to 3rd location of C; there is 7; go to 7th location of B, write 3. B and C looks like $B = \{x, x, x, x, x, x, 3, x\}$ and $C = \{2, 2, 4, 6, 7, 8\}$.

Outline

Lower bounds by reduction

Convex hull problem

Given a set of points $P = \{p_1, \ldots, p_n\}$, compute the smallest convex set that contains P.

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- Thus, convex hull has a lower bound of $\Omega(n \log n)$.

Outline

Selection

Median finding

We can find the median or any k-th largest element of a set of elements in $\Theta(n)$ time.

Deterministic selection

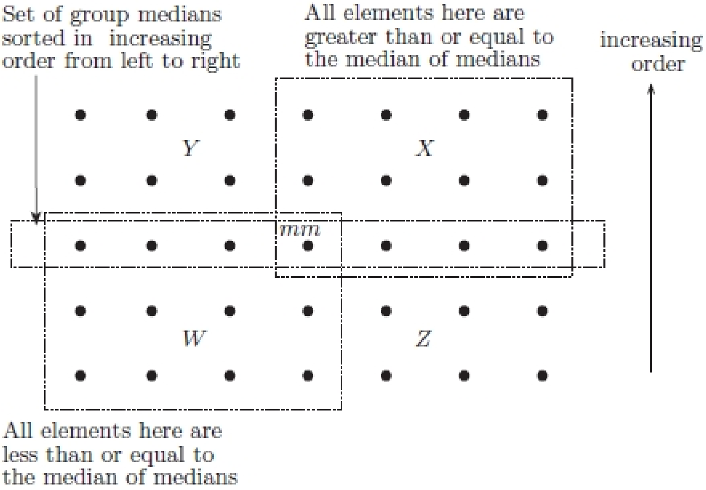

Figure: The schematic for selection algorithm.

A digression into Probability

Expectation

Random variable

A function defined on a sample space is called a random variable. Given a random variable X, $Pr[X = j]$ means X's probability of taking the value j.

Expectation – "the average value"

The expectation of a random variable X is defined as: $E[X] = \sum_{j=0}^{\infty} j \cdot Pr[X = j]$

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- So, the expectation of X, $E[X] = \sum_{j=0}^{\infty} j \cdot Pr[X = j] = \frac{1}{p}$.

Randomized Quick Sort

RandQSORT(A, p, q)

1: If $p \geq q$, then EXIT.

2: While no central splitter has been found, execute the following steps:

   2.1: Choose uniformly at random a number $r \in \{p, p+1, \ldots, q\}$.

   2.2: Compute s = number of elements in A that are less than $A[r]$, and
   t = number of elements in A that are greater than $A[r]$.

   2.3: If $s \geq \frac{q-p}{4}$ and $t \geq \frac{q-p}{4}$, then $A[r]$ is a central splitter.

3: Position $A[r]$ in $A[s+1]$, put the members in A that are smaller than the central splitter in $A[p \ldots s]$ and the members in A that are larger than the central splitter in $A[s+2 \ldots q]$.

4: RandQSORT(A, p, s);

5: RandQSORT(A, $s+2$, q).

Analysis of RandQSORT

Fact: One execution of Step 2 needs $O(q - p)$ time.

Question: How many times Step 2 is executed for finding a central splitter ?

Result:

The probability that the randomly chosen element is a central splitter is $\frac{1}{2}$.

Recall "Waiting for success"

If p be the probability of success of a random experiment, and we continue the random experiment till we get success, the expected number of experiments we need to perform is $\frac{1}{p}$.

Implication in Our Case

- The expected number of times Step 2 needs to be repeated to get a central splitter (success) is 2 as the corresponding success probability is $\frac{1}{2}$.
- Thus, the expected time complexity of Step 2 is $O(n)$

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- Number of levels of recursion $= \log_{\frac{4}{3}} n = O(\log n)$.
- Thus, the expected running time is $O(n \log n)$.

Finding the k-th largest

Median Finding

Similar ideas of getting a central splitter and waiting for success bound applies for finding the median in $O(n)$ time.

Outline

Longest common subsequence

Given two strings A and B of lengths n and m respectively over an alphabet set Σ, determine the length of the longest subsequence that is common to both A and B.

Let $A = zxyxyz$ and $B = xyyzx$ and $\Sigma = \{x, y, z\}$. The LCS is $xyyz$ and the length is 4.

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- If $i, j > 0$ and $a_i \neq b_j$, then $L[i, j] = \max\{L[i, j - 1], L[i - 1, j]\}$.
- The algorithm takes $O(nm)$ time by filling up a table of size nm.