RESEARCH ARTICLE

A Single-Loop Shifting Vector Method with Conjugate Gradient Search for Reliability based-Design Optimization

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ABSTRACT
Single-loop methods for reliability-based design optimization are characterized by high computational efficiency. However, the accuracy of these methods may be low for highly non-linear probabilistic constraints. A new hybrid method called "SLShV-CG" is proposed for improving the accuracy in which the single-loop method is coupled with the shifting vector approach of the sequential optimization and reliability assessment method. This shifting vector approach is incorporated with the probabilistic constraints without approximating them for reliability analysis. The most probable points for the probabilistic constraints are found using the Karush-Kuhn-Tucker conditions and the conjugate gradient search direction is used for determining the approximate most probable point in every iteration. The proposed method is tested on four mathematical and four engineering reliability-based design optimization problems, and the accuracy of solutions is verified using Monte-Carlo simulations. Results demonstrate better accuracy and computational efficiency of the proposed method over the six reliability-based design optimization methods from the literature.

KEYWORDS
Reliability-based design optimization; First-order reliability method; Single-loop method; Conjugate direction search

1. Introduction

Deterministic optimization is used for solving various constraint optimization problems in which the optimal solution is generally located on the constraint boundary. However, the design solution evolved by deterministic optimization can fail due to uncertainties from material properties, geometry, operational environment, and manufacturing process. Reliability-based design optimization (RBDO) is an efficient tool for solving such problems with desired target reliability and thus improving the quality of the design solution. RBDO model (Fiessler, Neumann, and Rackwitz 1979; Tu, Choi, and Park 1999; Aoues and Chateauneuf 2010) is generally expressed as a minimization of the objective function, which is subjected to probabilistic constraints or limit state functions. As a result, the design solution is safer and conservative due to the consideration of the probabilistic model. However, a trade-off for achieving the target reliable solution using the RBDO methods is the computational cost and the accuracy. Therefore, the development of RBDO methods is focused on achieving the desired reliability and computational efficiency.

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So far, various RBDO methods have been developed along with different types of reliability assessment such as sampling method, i.e., importance sampling and Monte Carlo simulation (MCS) (Lin et al. 1997), surrogate modeling (Dubourg, Sudret, and Bourinet 2011), the approximate integration method, and the most probable point (MPP)-based method. Among these methods, MCS is the most accurate method but it is computationally expensive. The MPP-based methods, on the other hand, gained popularity due to their computational efficiency. These methods can be further divided into three categories: single-loop, double-loop, and decoupled-loop methods.

The conventional double-loop method comprises of a nested loop. The outer-loop is the main optimization loop in the original design variable space within which it performs reliability analysis for each limit-state function in series. However, each reliability analysis itself requires an optimization loop in the standard normal space. In that case Rosenblatt transformation (Rosenblatt 1952) is used to map random variables from their original space to the standard normal space. Reliability analysis can either be performed by the reliability index approach (RIA) (Yu, Chang, and Choi 1998; Reddy, Grandhi, and Hopkins 1994) or the performance measurement approach (PMA) (Lee, Yang, and Ruy 2002; Tu, Choi, and Park 1999; Youn, Choi, and Du 2005). According to the literature, PMA is more stable and efficient than RIA, although both these approaches can use the first-order reliability method (FORM)(Rackwitz and Flessler 1978; M. Hasofer and Lind 1974) and the second-order reliability method (SORM) (Breitung 1984; Mansour and Olsson 2014). FORM linearizes the higher order non-linear limit-state functions using the first-order Taylor series that can make the probability of failure estimation erroneous. For this reason, SORM (Köylüoğlu and Nielsen 1994; Kiureghian, Lin, and Hwang 1987; Huang et al. 2018; Lee, Noh, and Yoo 2012) is developed, which has better accuracy than FORM. An asymptotically exact formulation is developed by Breitung (1984) using SORM that has been further modified by Tvedt (1983) using three-term approximation. On the other hand, a point-fitting second-order reliability approximation is proposed by Zhao and Ono (1999). An efficient SORM-based saddle point approximation is proposed by Hu and Du (2019). A quadratic problem is solved by bypassing the concept of MPP by Mansour and Olsson (2016). Further, an approximate Hessian analysis is proposed (Lim, Lee, and Lee 2014) to reduce the numerical efforts of SORM. A SORM-based RBDO with uncorrelated non-Gaussian variables has been derived by Strömberg (2017). However, both FORM and SORM need an iterative optimization procedure for reliability analysis which is combined with the main optimization loop in a double-loop method resulting in high computational cost.

In the single-loop methods, the equivalent deterministic constraints substitute the probabilistic constraints. The reliability assessment is approximated and integrated into the deterministic optimization loop. Karush-Kuhn-Tucker (KKT) optimality conditions are used to collapse the nested optimization loop into the single-loop method (Liang, P. Mourelatos, and Tu 2008; Madsen and Hansen 1992). A semi-single-loop method (Lim and Lee 2016) is also developed in which an approximate MPP is calculated by a sensitivity analysis of reliability analysis. Single-loop single vector (SLSV) method developed by Chen, Hasselman, and Neill (1997) also exists in the literature in which a quantile approximation of the limit-state functions is performed. The SLSV method is the first attempt in a truly single-loop method (Wang and Kodiyalam 2002; Yang and Gu 2004). In this method, the random variables are first transformed into the uncorrelated and standard normalized space and the MPP is found using the steepest descent search direction. Due to slow convergence of the steepest descent search, which is because of the orthogonality of the search directions, the conjugate gradient (CG) search direction is adopted with SLSV (Ezzati, Mammadov, and Kulkarni 2015; Jeong and Park 2016). The CG algorithm requires successive direction vectors of the previous iteration to calculate the MPP. This adds a little more calculations but it improves the stability and accuracy of the solution.

The decoupled-loop method is also developed to reduce the computational cost of RBDO methods. The formulation is developed by decoupling the optimization and reliability analysis loops. The decoupled-loop method comprises of direct decoupling approach (DDA) (Zou and Mahadevan 2006), sequential optimization and reliability assessment (SORA) (Du and Chen 2004), sequential approximate programming (SAP) (Cheng, Xu, and Jiang 2006). Further, approximate SORA (ASORA) (Yi, Zhu, and Gong 2016) is developed based on the previous information of approximate most probable target point (MPTP) and approximate performance measure approach. SAP is further developed by using PMA-based RBDO method (Yi and Cheng 2008). The adaptive decoupling approach (Chen et al. 2013) uses an update angle strategy and novel feasibility checking method to improve the efficiency of RBDO methods.

From the above literature, it can be observed that the single-loop methods are computationally efficient and the decoupled-loop methods are accurate RBDO methods. Moreover, FORM-based RBDO methods are generally simpler for practicing engineers compared to SORM-based RBDO methods. Furthermore, most existing SORMs may yield unphysical probabilities, such as negative or complex values. Motivated from the advantages of these methods, a hybrid method is proposed. Following are the contributions of the paper.

- A FORM-based single-loop method is coupled with the shifting vector approach of SORA in order to generate an accurate and computationally efficient solution. The shifting vector is incorporated with all probabilistic constraints that gets updated at every iteration.
- The KKT conditions are used to approximate the MPPs of these constraints and the approximate MPP is determined using the CG search direction in the normal variable space.
- The performance assessment of the proposed method, which is referred to as SLShV-CG, is tested on four mathematical and four engineering RBDO problems from the literature. SLShV-CG is also compared with other FORM-based RBDO methods such as PMA-AMV (Tu, Choi, and Park 1999), PMA-CGA (Ezzati, Mammadov, and Kulkarni 2015), SLSV (Liang, P. Mourelatos, and Tu 2008), SLSV-CG (Jeong and Park 2016), SORA (Du and Chen 2004) and ASORA (Yi, Zhu, and Gong 2016), and their results are verified using Monte-Carlo simulations.

The paper is organized into five sections. A basic RBDO formulation is described in Section 2 along with brief details of reliability assessment approaches and RBDO methods. In Section 3, the details of the proposed method are presented with a flow chart and optimization sequence. Results and discussion are presented in Section 4 and the paper is concluded in Section 5 with the scope of future work.

2. Reliability-based Design Optimization Preliminaries and Methods

2.1. *Basic formulation of RBDO*

In general, a deterministic optimization formulation can be written as

$$\begin{aligned} &\text{min. } f(\mathbf{d}), \\ &\text{s.t.: } g_i(\mathbf{d}) \leq 0, \qquad i = 1, \ldots, M, \\ &\quad \mathbf{d}^{\mathrm{L}} \leq \mathbf{d} \leq \mathbf{d}^{\mathrm{U}}, \end{aligned} \tag{1}$$

where $f(\mathbf{d})$ is the objective function, $g_i(\mathbf{d})$ is the $i-$th inequality constraints, $\mathbf{d} = [d_1, d_2, \ldots, d_N]^{\mathrm{T}}$ is the N-dimensional design vector, and $\mathbf{d}^{\mathrm{L}}$ and $\mathbf{d}^{\mathrm{U}}$ are the lower and upper limit of the design vector $\mathbf{d}$, respectively.

In order to introduce uncertainty into the formulation, the design variable vector is replaced by a vector of random variables. Therefore, the constraints are converted into the probabilistic constraints. The formulation of RBDO (Tu, Choi, and Park 1999; Youn, Choi, and Park 2003) is given in equation (2).

$$\begin{aligned} &\text{min. } f(\boldsymbol{\mu}_{\mathbf{x}}),\\ &\text{s.t.: } p_f = Pr[g_i(\mathbf{X}) \le 0] \le \Phi(-\beta_i^t), \quad i = 1, \ldots, M,\\ &\quad \boldsymbol{\mu}_{\mathbf{x}}^{\mathrm{L}} \le \boldsymbol{\mu}_{\mathbf{x}} \le \boldsymbol{\mu}_{\mathbf{x}}^{\mathrm{U}}, \end{aligned} \tag{2}$$

where $\mathbf{X} = [X_1, X_2, \ldots, X_N]^{\mathrm{T}}$ is the N-dimensional random variable vector, $\boldsymbol{\mu}_{\mathbf{x}}$ is the vector of mean values of random variable vector $\mathbf{X}$, $\boldsymbol{\mu}_{\mathbf{x}}^{\mathrm{L}}$ and $\boldsymbol{\mu}_{\mathbf{x}}^{\mathrm{U}}$ are the lower limit and upper limit of $\boldsymbol{\mu}_{\mathbf{x}}$, respectively. β_i^t is the target reliability index of $i-$th limit state function $g_i(\mathbf{X})$, $Pr[\cdot]$ is the probability operator of limit-state function, and $\Phi(\cdot)$ represents the standard normal cumulative distribution function. p_f represents the failure probability which can be expressed as a multidimensional integral (Madsen, Krenk, and Lind 1986) and is given in equation (3).

$$p_f = Pr[g_i(\mathbf{X}) \le 0] = F_{g_i}(0) = \int \cdots \int_{g_i(\mathbf{x}) \le 0} f_{\mathbf{X}}(x) \mathrm{d}\mathbf{X}, \tag{3}$$

where $f_{\mathbf{X}}(x)$ is the joint probability distribution function of the random variable in the original space $\mathbf{X}$ and $F_{g_i}(0)$ is the representation of the cumulative distribution function of $g_i(\mathbf{X})$.

An exact evaluation of equation (3) is difficult to obtain as multidimensional integral is involved. Therefore, $g_i(\mathbf{X})$ is approximated by using the first-order or second-order Taylor series. The MPP is then obtained either by using RIA or PMA. In the following subsections, these approaches are described briefly.

2.2. *Reliability index approach (RIA)*

The probabilistic constraint of equation (2) can be transformed into a reliability index as given in equation (4).

$$\hat{\beta}_i = -\Phi^{-1}(F_{g_i}(0)) \ge \beta_i^t, \tag{4}$$

where $\hat{\beta}_i$ and β_i^t are the reliability index and target reliability index for $i-$th limit state function, respectively. By replacing the probabilistic constraint of equation (2) by equation (4), the RBDO formulation can be written as

$$\begin{aligned} &\text{min. } f(\boldsymbol{\mu}_{\mathbf{x}}),\\ &\text{s.t.: } \hat{\beta}_i \ge \beta_i^t, \qquad i = 1, \ldots, M,\\ &\quad \boldsymbol{\mu}_{\mathbf{x}}^{\mathrm{L}} \le \boldsymbol{\mu}_{\mathbf{x}} \le \boldsymbol{\mu}_{\mathbf{x}}^{\mathrm{U}}. \end{aligned} \tag{5}$$

RIA uses FORM to calculate the reliability index, $\hat{\beta}_i$. For this calculation, a random variable vector $\mathbf{X}$ is transformed to the standard normal random variable vector $\mathbf{U}$ which can be performed by Rosenblatt transformation (Rosenblatt 1952) or Nataf transformation (Liu and Kiureghian 1986), $\mathbf{U} = \mathrm{T}(\mathbf{X})$ or $\mathbf{X} = \mathrm{T}^{-1}(\mathbf{U})$. The transformation from the original design

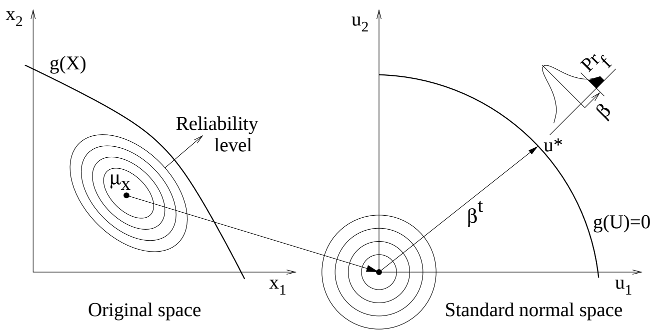

Figure 1. Transformation from the original design variable space to the standard normal space.

space to the standard normal space for two variables is shown in figure 1. The reliability index $\hat{\beta}_i$ is evaluated by solving an optimization problem (Rackwitz and Flessler 1978), which is given in equation (6).

$$\begin{aligned} &\text{find } \mathbf{U}^* \\ &\text{min. } \|\mathbf{U}\| = \beta_i, \\ &\text{s.t.: } g_i(\mathbf{U}) = 0. \end{aligned} \tag{6}$$

The optimum point $\mathbf{U}^*$ is called as the most probable failure point (MPFP) in the standard normal space. The flowchart of the reliability index approach is shown in figure 2. In RIA, at every iteration the value of β is updated until the target reliability is achieved by the MPP for the particular constraint.

2.3. *Performance measure approach (PMA)*

The probabilistic constraint of RBDO formulation given in equation (2) can also be transformed using equation (7).

$$G_i^p = F_{g_i}^{-1}(\Phi(-\beta_i^t)) \geq 0, \tag{7}$$

where G_i^p is the $i-$th probabilistic performance measure. Therefore, the RBDO method based on PMA can be formulated as

$$\begin{aligned} &\text{min. } f(\boldsymbol{\mu}_{\mathbf{x}}), \\ &\text{s.t.: } G_i^p \geq 0, \qquad\qquad i = 1, \ldots, M, \\ &\boldsymbol{\mu}_{\mathbf{x}}^{\mathrm{L}} \leq \boldsymbol{\mu}_{\mathbf{x}} \leq \boldsymbol{\mu}_{\mathbf{x}}^{\mathrm{U}}. \end{aligned} \tag{8}$$

The value of the performance measure can be calculated by solving the optimization problem given in equation (9).

$$\begin{aligned} &\text{find } \mathbf{U}^* \\ &\text{min. } g_i(\mathbf{U}), \\ &\text{s.t.: } \|\mathbf{U}\| = \beta_i^t, \end{aligned} \tag{9}$$

where the optimum point $\mathbf{U}^*$ is known as the most probable target point (MPTP) with target reliability β_i^t. The optimum value of $g_i(\mathbf{U}^*)$ is used as a performance measure of G_i^p in equation

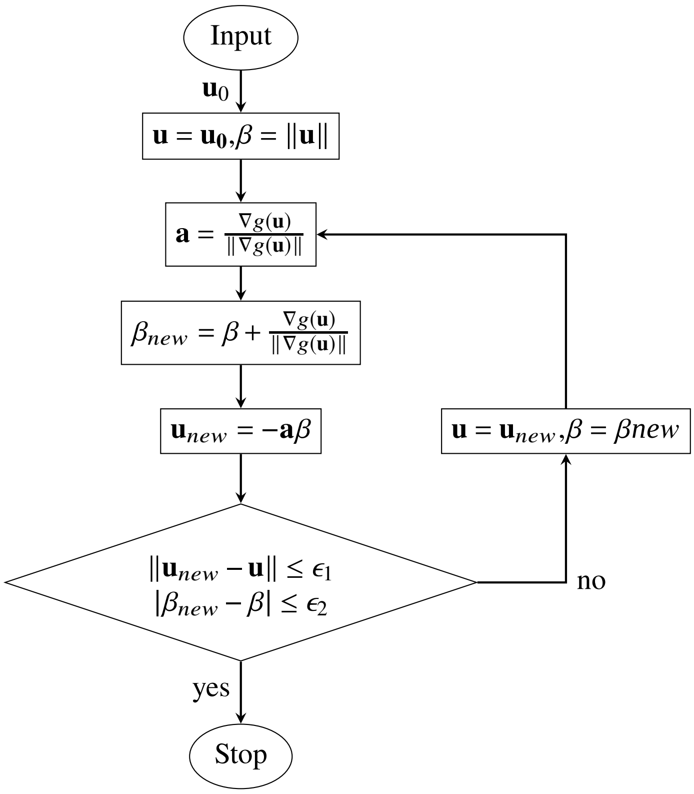

Figure 2. Flowchart of RIA

(8). The probabilistic performance measure can be calculated as

$$G_i^p = g_i(\mathbf{U}^*) = G_i(\mathbf{X}^*) = G_i(\mathbf{X}(\mathbf{U}^*)). \tag{10}$$

By applying Lagrangian multiplier method in equation (9), the MPTP point (Wu, Millwater, and Cruse 1990) can be calculated using equation (11).

$$\mathbf{U}^{k+1} = -\beta_i^t \cdot \frac{\nabla g(\mathbf{U}^k)}{\|\nabla g(\mathbf{U}^k)\|}, \tag{11}$$

where ∇ is the gradient of function with respect to $\mathbf{U}$ and $\mathbf{U}^k$ is the MPTP at $k-$th iteration. The flowchart for PMA is shown in figure 3. In PMA, the value of target reliability β is given and $\mathbf{u}_{new}$ is updated based on the gradient of the constraint. The PMA-based method gets terminated when the target reliability is achieved.

2.4. *Single-loop single vector (SLSV) method*

Liang, P. Mourelatos, and Tu (2008) proposed SLSV in which the MPP is updated by the approximate PMA-based reliability analysis. The loop of reliability analysis is eliminated and the approximate MPP is estimated by using KKT conditions. The formulation of SLSV is given in equation (12).

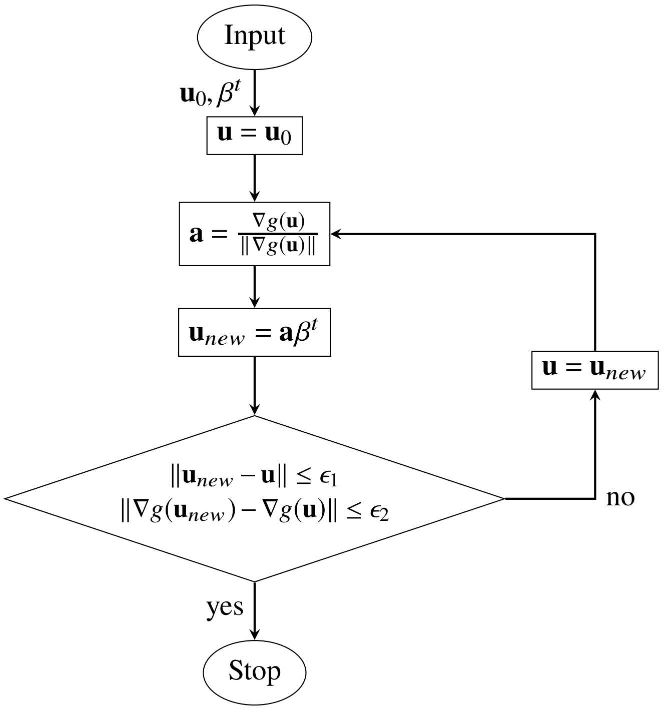

Figure 3. Flowchart for PMA

$$
\begin{aligned}
&\text{min. } f(\boldsymbol{\mu}_{\mathbf{x}}),\\
&\text{s.t.: } g_i(\mathbf{X}) \leq 0, \qquad\qquad i = 1,\ldots,M,\\
&\text{where } \mathbf{X}^{(k)} = \boldsymbol{\mu}_{\mathbf{x}}^{(k)} + \beta_i^t \boldsymbol{\sigma}\boldsymbol{\alpha}_i^{(k-1)},\\
&\boldsymbol{\alpha}_i^{(k-1)} = \frac{\boldsymbol{\sigma}\ \nabla g_{i(\mathbf{X})}(\mathbf{X}^{k-1})}{\|\boldsymbol{\sigma}\ \nabla g_{i(\mathbf{X})}(\mathbf{X}^{k-1})\|},
\end{aligned}
\tag{12}
$$

where $\mathbf{X}^{(k)}$ represents the vector of random variable at $k-$th iteration, $\boldsymbol{\mu}_{\mathbf{x}}^{(k)}$ is the mean value vector of $\mathbf{X}^{(k)}$, $\boldsymbol{\sigma}$ represents the vector of standard deviation of random variable $\mathbf{X}^{(k)}$, β_i^t is the target reliability of $i-$th constraint and $\boldsymbol{\alpha}_i^{(k)}$ is the steepest descent direction vector for $i-$th constraint.

2.5. *Sequential optimization and reliability assessment (SORA) method*

Du and Chen (2004) proposed a decoupled-loop method in which the optimization and the reliability assessment are performed sequentially. The MPP is estimated by PMA for each probabilistic constraint. The shifting vector ($\mathbf{s}_i^{k+1}$) is used to push the violated deterministic constraint, i.e., having reliability less than the specified target toward the feasible direction. The shift vector is calculated as

$$
\mathbf{s}_i^{(k+1)} = \boldsymbol{\mu}_{\mathbf{X}}^{(k)} - \mathbf{X}_{\text{MPP}}^{(k)}, \tag{13}
$$

where $\boldsymbol{\mu}_{\mathbf{X}}^{(k)}$ is the mean value of random variable $\mathbf{X}$ at $k-$th iteration and $\mathbf{X}_{\text{MPP}}^{(k)}$ is the MPP for $i-$th constraint. Deterministic optimization is performed with the shifted constraints until the convergence is achieved. This method uses the reliability information from the previous iteration to shift the violated constraint. This idea of shifting vector is used to reduce the computational requirement of SORA for generating a reliable solution.

3. Single-Loop Shifting Vector Method with Conjugate Gradient Search (SLShV-CG)

In single-loop method, the reliability analysis is performed by approximating the MPP using KKT conditions. Since the update of the MPP requires gradient informations, SLSV shows inefficiency and inaccuracy for RBDO problems with highly non-linear constraints. On the other hand, SORA decouples optimization and reliability analysis and thus, improves computational efficiency while solving RBDO problems with any type of constraints. In this paper, SLSV is coupled with the shifting vector approach of SORA in which the constraints with less reliability is shifted toward the feasible direction. In this method, the constraints are not required to be approximated using Taylor series. The proposed method is explained in the following subsections.

3.1. *Conjugate gradient method*

In this paper, the conjugate gradient (CG) (Reeves and Fletcher 1964) method is used to calculate the approximate MPP as it improves the stability as well as efficiency. The basic idea of the CG method is to move in a non-intersecting direction, unlike the steepest descent method. Initially, the CG method uses the steepest descent direction, i.e., negative of the gradient of the performance function. The algorithm for unconstrained deterministic optimization is as follows.

Step 1 Set the initial guess $\boldsymbol{\mu}^0$, $k = 1$ and termination parameters ϵ.

Step 2 Calculate direction vector $\mathbf{S}$

$$\mathbf{S}^0 = -\nabla f(\boldsymbol{\mu}^0)$$

Step 3 Calculate λ^0, such that $f(\boldsymbol{\mu}^0 + \lambda^0\mathbf{S}^0)$ is minimum. Update $\boldsymbol{\mu}^k = \boldsymbol{\mu}^0 + \lambda\mathbf{S}^0$. Also calculate $\nabla f(\boldsymbol{\mu}^k)$.

Step 4 Set conjugate direction

$$\mathbf{S}^k = -\nabla f(\boldsymbol{\mu}^k) + \frac{||\nabla f(\boldsymbol{\mu}^k)||^2}{||\nabla f(\boldsymbol{\mu}^{k-1})||^2} \cdot \mathbf{S}^{k-1}$$

Step 5 Find λ^k, such that $f(\boldsymbol{\mu}^k + \lambda^k\mathbf{S}^k)$ is minimum. Set $\boldsymbol{\mu}^{k+1} = \boldsymbol{\mu}^k + \lambda^k\mathbf{S}^k$.

Step 6 If $\frac{||\boldsymbol{\mu}^{k+1} - \boldsymbol{\mu}^k||}{||\boldsymbol{\mu}^k||} \leq \epsilon$ or $||\nabla f(\boldsymbol{\mu}^{k+1}||) \leq \epsilon$ holds true, then terminate, else set $k = k + 1$ and goto step 4.

3.2. *Details of SLShV-CG method*

Since the shift vector approach of SORA is coupled with SLSV, the RBDO formulation is modified as

$$\begin{aligned}&\text{min. } f(\boldsymbol{\mu}_{\text{x}}),\\&\text{s.t.: } g_i(\boldsymbol{\mu}_{\text{x}}^{(k)} - \mathbf{s}_i^{(k)}) \leq 0, \qquad i = 1, 2, \ldots, M,\\&\quad \mathbf{s}_i^{(k)} = \boldsymbol{\mu}_{\mathbf{x}}^{(k)} - \mathbf{X}_{(MPP)}^{(k)}.\end{aligned} \tag{14}$$

The shifting vector $\mathbf{s}_i^{(k)}$ gets updated in every iteration using the approximate MPP, which is given as

$$\mathbf{X}_{(MPP)}^{(k)} = \boldsymbol{\mu}_{\mathbf{x}}^{(k)} + \beta_i^t \boldsymbol{\sigma}_{\mathbf{x}} \boldsymbol{\alpha}_i^{(k-1)}, \tag{15}$$

where $\mathbf{X}_{(MPP)}^{(k)}$ represents the approximate MPP vector. Note that the vector $\boldsymbol{\sigma}_{\mathbf{x}}$ is multiplied with the constraint direction vector $\boldsymbol{\alpha}_i^{(k-1)}$. This multiplication is performed by multiplying each component of vector $\boldsymbol{\sigma}_{\mathbf{x}}$ with the corresponding component of vector $\boldsymbol{\alpha}_i^{(k-1)}$. The direction $\boldsymbol{\alpha}_i^{(k-1)}$ is found using the conjugate gradient direction, which is given as

$$\begin{aligned}\boldsymbol{\alpha}_i^{(k-1)} &= \frac{\mathbf{D}_i^{(k-1)}}{\|\mathbf{D}_i^{(k-1)}\|},\\ \mathbf{D}_i^{(k-1)} &= \boldsymbol{\sigma}_{\mathbf{x}} \nabla g_i^{(k-1)} + \frac{(\boldsymbol{\sigma}_{\mathbf{x}} \nabla g_i^{(k-1)})^T (\boldsymbol{\sigma}_{\mathbf{x}} \nabla g_i^{(k-1)})}{(\boldsymbol{\sigma}_{\mathbf{x}} \nabla g_i^{(k-2)})^T (\boldsymbol{\sigma}_{\mathbf{x}} \nabla g_i^{(k-2)})} \cdot \mathbf{D}_i^{(k-2)},\end{aligned} \tag{16}$$

where $\mathbf{D}_i^{(k)}$ signifies the direction vector of $i-$th constraint at the $k-$iteration. Since the conjugate gradient direction is used to update the MPP of the proposed hybrid single-loop method with the shifting vector approach of SORA method, it is referred to as SLShV-CG. Following are the steps of SLShV-CG.

Step 1 Set $k = 0$, initial design variables $\mathbf{X}^0 = \boldsymbol{\mu}_{\mathbf{x}}^0$, $\mathbf{X}_{MPP}^{(0)} = \boldsymbol{\mu}_{\mathbf{x}}^0$ and the given standard deviation $\boldsymbol{\sigma}_{\mathbf{x}}$ and target reliability index β^t.

Step 2 Calculate the shifting vector of the $i-$th constraint as

$$\mathbf{s}_i^{(k)} = \boldsymbol{\mu}_{\mathbf{x}}^{(k)} - \mathbf{X}_{(MPP)}^{(k)}, \tag{17}$$

The initial shifting vector will be zero as $\mathbf{X}_{(MPP)}^{(0)} = \boldsymbol{\mu}_{\mathbf{x}}^0$. Therefore, the first optimization will be a deterministic evaluation of optimal solution.

Step 3 Perform deterministic optimization of the design problem with the shifted constraints.

$$\begin{aligned}&\text{min } f(\boldsymbol{\mu}_{\text{x}}),\\&\text{s.t.: } g_i(\boldsymbol{\mu}_{\text{x}}^{(k)} - \mathbf{s}_i^{(k)}) \leq 0, \qquad i = 1, 2, \ldots, M,\\&\quad \mathbf{s}_i^{(k)} = \boldsymbol{\mu}_{\mathbf{x}}^{(k)} - \mathbf{X}_{(MPP)}^{(k)}.\end{aligned} \tag{18}$$

Step 4 Calculate the conjugate direction vector

$$\alpha_i^{(k)} = \begin{cases} \left.\dfrac{\sigma_{\mathbf{x}} \nabla g_i}{\|\sigma_{\mathbf{x}} \nabla g_i\|}\right|_{\mu_{\mathbf{x}}^0}, & \text{if } k = 0, \\ \left.\dfrac{\mathbf{D}_i}{\|\mathbf{D}_i\|}\right|_{\mathbf{X}_{(MPP)}^{(k)}} & \text{otherwise,} \end{cases} \tag{19}$$

where

$$\mathbf{D}_i^{(k)} = \begin{cases} \sigma_{\mathbf{x}} \nabla g_i, \quad k \le 1 \\ \sigma_{\mathbf{x}} \nabla g_i^{(k)} + \frac{(\sigma_{\mathbf{x}} \nabla g_i^{(k)})^T (\sigma_{\mathbf{x}} \nabla g_i^{(k)})}{(\sigma_{\mathbf{x}} \nabla g_i^{(k-1)})^T (\sigma_{\mathbf{x}} \nabla g_i^{(k-1)})} \cdot \mathbf{D}_i^{(k-1)}, & \text{otherwise.} \end{cases}$$

Step 5 Set $k = k + 1$ and update $\mathbf{X}_{MPP}^{(k)}$ in the original space as

$$\mathbf{X}_{(MPP)}^{(k)} = \sigma_{\mathbf{x}}^{(k)} + \beta^t \sigma_{\mathbf{x}}^T \alpha_i^{(k-1)}. \tag{20}$$

Step 6 If the convergence criterion
$\| f(\mu_{\mathbf{x}}^{k+1}) - f(\mu_{\mathbf{x}}^k)\| / \| f(\mu_{\mathbf{x}}^k)\| \le 0.001$
or $\|\mu_{\mathbf{x}}^{k+1} - \mu_{\mathbf{x}}^k\| \le 0.001$ is satisfied, terminate. Otherwise go to Step 2.

The flowchart of SLShV-CG is shown in figure 4 in which all the steps are shown. It can be observed that SLShV-CG does not require a transformation of the original variable space to the standard normal variable space.

4. Results and Discussion

In this section, four mathematical and four engineering RBDO examples are solved to demonstrate the accuracy and computational efficiency of SLShV-CG method. The accuracy of the obtained solutions is evaluated through Monte Carlo simulation (MCS) with one million sample size and the target reliability index for each probabilistic constraint is determined. The computational efficiency is measured through the number of function calls (NFC) in which f_{FC} is the measure of objective function calls and g_{FC} is the measure of probabilistic function calls. `Iter` is used to denote the total number of iterations required by the optimization algorithm for convergence. The performance of SLShV-CG is also compared with PMA-AMV (Tu, Choi, and Park 1999), PMA-CGA (Ezzati, Mammadov, and Kulkarni 2015), SLSV (Liang, P. Mourelatos, and Tu 2008), SLSV-CG (Jeong and Park 2016), SORA (Du and Chen 2004), and ASORA (Yi, Zhu, and Gong 2016). It is noted that all methods are initialized with the same initial point and get terminated using the same termination conditions given at Step 6 of SLShV-CG algorithm. For optimization, the tool *fmincon* is used as an optimizer.

4.1. *Mathematical example 1*

The first example given in equation (21) is a non-linear mathematical problem (Jeong and Park 2016; Yi, Zhu, and Gong 2016), which has linear objective function and three non-linear constraints. Constraints $g_1(\mathbf{X})$ and $g_3(\mathbf{X})$ are convex functions and $g_2(\mathbf{X})$ is slightly concave in nature. Random variables x_1 and x_2 are normally distributed and statistically independent. Both these design variables have the lower and upper bound set on their mean values to 0 and

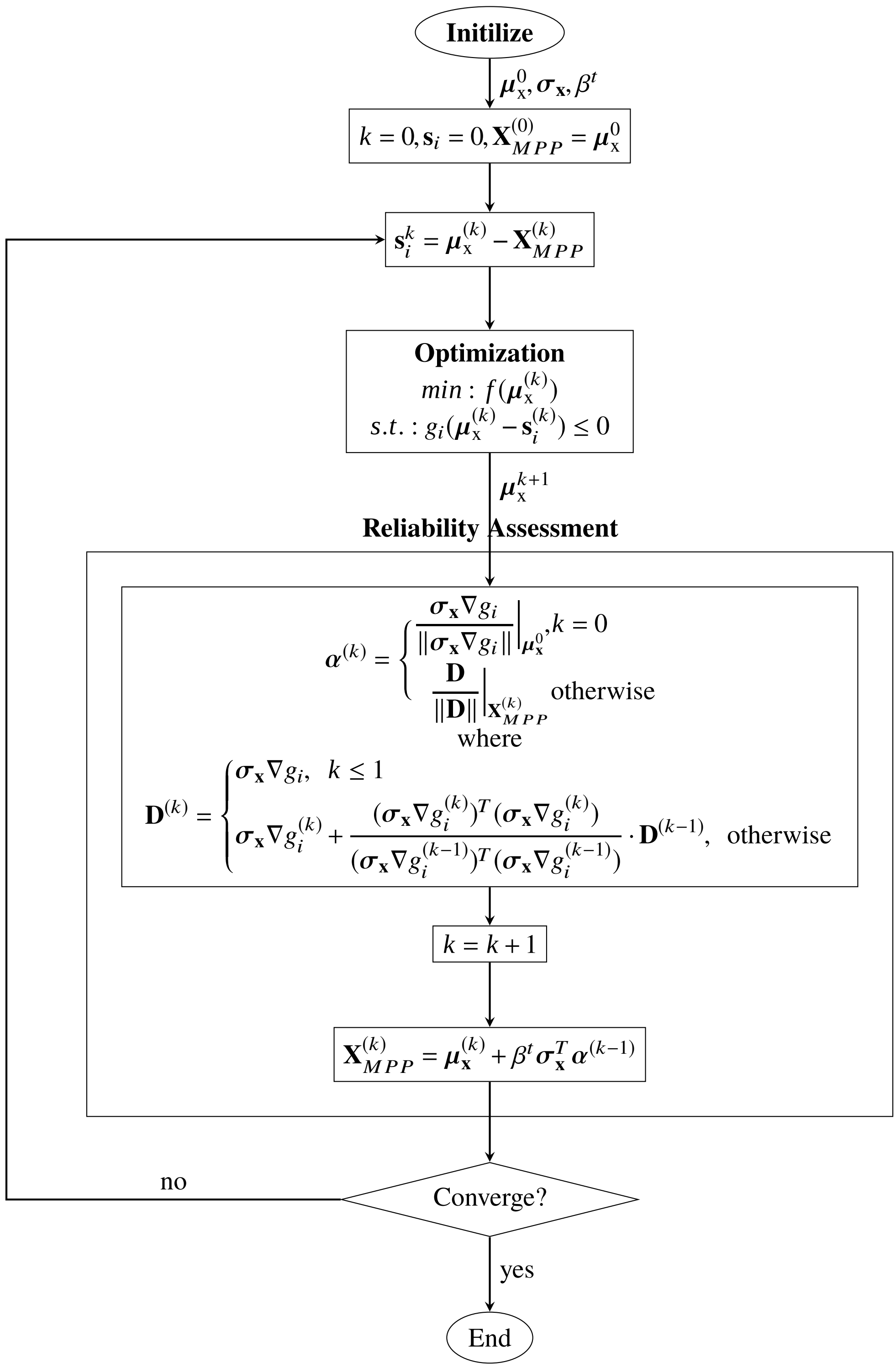

Figure 4. Flowchart of SLShV-CG

Table 1. RBDO results for Example 1 with $\beta_t = 3.0$

Methods	f^*	μ_x^*	NFC		β^t_{MCS}			Iter
			$\mathrm{f_{FC}}$	$\mathrm{g_{FC}}$	g_1	g_2	g_3	
DO	5.172	(3.111, 2.061)	-	-	-	-	-	-
PMA-AMV	6.7219	(3.4363, 3.2855)	27	5193	3.0045	3.0307	Inf	8
PMA-CGA	6.7256	(3.4391, 3.2865)	25	6157	2.9698	3.0068	Inf	8
SLSV	6.7199	(3.4419, 3.2779)	67	191	2.9402	2.9719	Inf	3
SLSV-CG	6.7253	(3.4393, 3.2861)	76	225	2.9576	3.0459	Inf	4
SORA	6.7226	(3.4369, 3.2857)	76	1137	2.9989	3.0617	Inf	4
ASORA	6.7256	(3.4391, 3.2865)	96	312	2.9698	3.0068	Inf	6
SLShV-CG	6.7224	(3.4371, 3.2853)	124	402	3.0258	3.0332	Inf	9

10, respectively with standard deviation of 0.3. The initial point of this problem is taken as, $\mu_x^{(0)} = [5.0, 5.0]^T$ and the target reliability index β_i^t for all constraints is set to 3.0.

$$\begin{aligned}
&\text{Find: } [\mu_{x_1}, \mu_{x_2}]^T \\
&\text{min: } \mu_{x_1} + \mu_{x_2}, \\
&\text{s.t.: } Pr\left[g_1(\mathbf{X}) = 1 - \frac{x_1^2 x_2}{20} > 0\right] \le \phi(-\beta_1^t), \\
&Pr\left[g_2(\mathbf{X}) = 1 - \frac{(x_1 + x_2 - 5)^2}{30} - \frac{(x_1 - x_2 - 12)^2}{120} > 0\right] \\
&\le \phi(-\beta_2^t), \\
&Pr\left[g_3(\mathbf{X}) = 1 - \frac{80}{(x_1^2 + 8x_2 + 5)} > 0\right] \le \phi(-\beta_3^t), \\
&0 \le \mu_{x_i} \le 10,\ x_i \sim N(\mu_{x_i}, 0.3^2) \text{ for } = 1, 2, \\
&\beta_j^t = 3.0,\ \mu_x^{(0)} = [5.0, 5.0]^T,\ j = 1, 2, 3.
\end{aligned} \tag{21}$$

Table 1 presents the optimal solutions obtained from the methods. The target reliability achieved by the solutions for each probabilistic constraint is shown from the sixth column to the eighth column, which is evaluated through MCS. The constraint function calls $\mathrm{g_{FC}}$ denote the total number of times the constraint function is called by "fmincon" ($g_{fmincon}$), plus the number of times it is called for its gradient calculation (grf). Meaning, $\mathrm{g_{FC}}$ is equal to $g_{fmincon} + grf \times nc \times nv \times 2$, where nc and nv represent the number of constraints and number of variables, respectively and the numeric 2 is multiplied as the central difference method is used for gradient calculation. Similarly, the number of the cost function calls (f_{FC}) is calculated by determining the number of function called by "fmincon".

Regarding efficiency, SLShV-CG is found to be computationally efficient than PMA-AMV when comparing their $\mathrm{f_{FC}}$ and $\mathrm{g_{FC}}$ in the fourth and the fifth columns of the table. Comparing the methods based on NFC, SLSV, SLSV-CG and ASORA are found to be efficient than SLShV-CG. However, SLShV-CG generates the reliable solution by compromising with NFC.

Regarding accuracy, it can be seen from β^t_{MCS} columns of Table 1 that only SLShV-CG and PMA-AMV methods evolve the optimal solutions with the desired target reliability. Other methods like DLM-CGA, SLSV-CG, SORA and ASORA are unable to achieve the target reliability for $g_1(\mathbf{X})$. It can also be seen that SLSV is unable to achieve the target reliability for both $g_1(\mathbf{X})$ and $g_2(\mathbf{X})$. It is noted that the constraint $g_3(\mathbf{X})$ is found to be inactive and thus, its target reliability is infinite for the solutions obtained by all methods. Figure 5 shows the convergence plots for all methods with respect to the number of iterations. The maximum number of iterations required to evolve the optimal solution is also mention in the last column

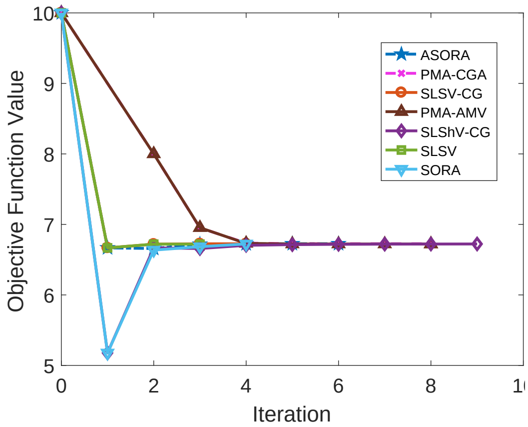

Figure 5. Convergence plot of example 1.

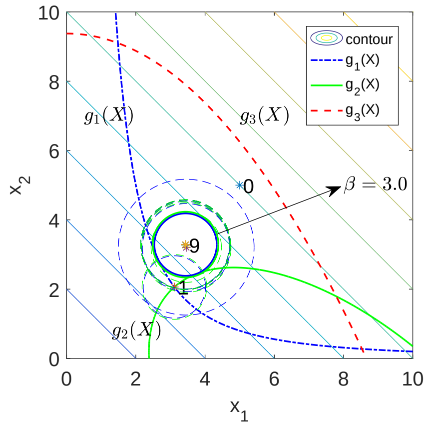

Figure 6. Contour plot of example 1.

of table 1. A smooth convergence of SLShV-CG can be observed with respect to other methods.

The progress of the solution obtained through SLShV-CG in two-variable space is shown in figure 6. The β circles for constraints $g_1(\mathbf{X})$ and $g_2(\mathbf{X})$ are shown, whereas for constraint $g_3(\mathbf{X})$ is not shown because the target reliability index is infinity. Graphically, it can be seen that SLShV-CG evolved the optimal solution with the desired target reliability.

4.2. *Mathematical example 2*

The second mathematical problem (Jeong and Park 2016) as shown in equation (22) contains a non-linear objective function and a highly concave constraint. This problem is considered in order to check the robustness of the proposed method for concave function. The initial point is taken as $\boldsymbol{\mu}_{\mathbf{x}}^{(0)} = [5.0, 5.0]^T$ and the standard deviation of the problem is $\boldsymbol{\sigma} = [0.6, 0.6]^T$. The lower and upper bound of mean values of the random variables are 0.0 and 10.0, respectively. Table 2 presents the optimal solutions obtained from all methods.

$$\begin{aligned}
&\text{Find: } [\mu_{x_1}, \mu_{x_2}]^T \\
&\text{min.: } (\mu_{x_1}+2)^2 + (\mu_{x_2}+2)^2 - 2\mu_{x_1}\mu_{x_2}, \\
&\text{s.t.: } Pr\left[\frac{e^{(0.8x_1-1.2)} + e^{(0.7x_2-0.6)} - 5}{10} > 0\right] \le \phi(-\beta_1^t), \\
&\qquad 0 \le \boldsymbol{\mu}_{\mathbf{x}_i} \le 10,\ x_i \sim N(\boldsymbol{\mu}_{\mathbf{x}_i}, 0.6^2) \text{ for } i = 1, 2, \\
&\qquad \beta_1^t = 3.0,\ \boldsymbol{\mu}_{\mathbf{x}}^{(0)} = [5.0, 5.0]^T.
\end{aligned} \tag{22}$$

Regarding efficiency, SLShV-CG is found better than other methods, except for SLSV when comparing their NFC. However, SLSV has evolved the worst reliable solution among all methods. The convergence of all methods with respect to the number of iterations is shown in figure 7. A smooth convergence is not observed with any method. It is due to the non-linear concave probabilistic constraint.

The accuracy of all these methods can be investigated by observing β_{MCS}^t columns of Table 2. It can be seen that SLShV-CG and PMA-CGA are the only methods which can evolve the solution with the desired target reliability. This is because PMA-CGA and SLShV-CG both utilize the conjugate gradient update to locate the MPP. On the other hand, PMA-AMV, SLSV,

Table 2. RBDO results for Example 2 with $\beta_t = 3.0$

Methods	f^*	$\boldsymbol{\mu}_{\mathbf{x}}^*$	NFC		β^t_{MCS}	Iter
			f_{FC}	g_{FC}	g_1	
DO	27.0180	(3.1595, 3.8920)	–	–	–	–
PMA-AMV	36.7425	(3.1595, 3.8920)	53	10431	2.8538	12
PMA-CGA	37.3957	(3.5760, 3.7641)	33	993	3.0701	10
SLSV	32.3925	(1.8332, 3.5382)	116	127	1.3619	3
SLSV-CG	37.0741	(3.0822, 3.9833)	1248	1336	2.9517	42
SORA	36.7965	(2.9152, 3.9933)	190	2176	2.8109	8
ASORA	–	–	–	–	–	–
SLShV-CG	37.3956	(3.5715, 3.7677)	278	310	3.0729	10

SORA and ASORA use the steepest descent update for MPP calculation. Steepest descent works well for convex function but found unstable for concave function. Therefore, these methods shows poor convergence.

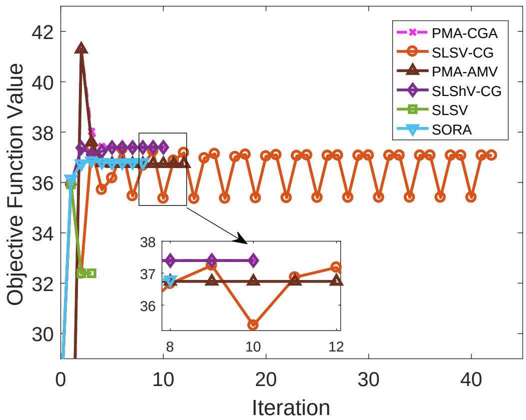

Figure 7. Convergence plot of problem 2.

4.3. *Mathematical example 3*

The third example (Jiang et al. 2017) as given in equation (23) is a highly non-linear mathematical example as illustrated in figure 9. The constraint $g_2(\mathbf{X})$ is concave in nature while $g_1(\mathbf{X})$ and $g_3(\mathbf{X})$ are convex in nature. The initial point is taken as $\boldsymbol{\mu}_{\mathbf{x}}^{(0)} = [5.0, 5.0]^T$ and the reliability for each constraint is increased to $\beta_i^t = 3.5$. Table 3 summarizes the obtained solutions from all methods.

Table 3. RBDO results for Example 3 with $\beta_t = 3.5$

Methods	f^*	$\boldsymbol{\mu}_\mathbf{x}^*$	NFC		β^t_{MCS}			Iter
			f_{FC}	g_{FC}	g_1	g_2	g_3	
DO	−2.292	(5.197, 0.741)	–	–	–	–	–	–
PMA-AMV	–	–	–	–	–	–	–	–
PMA-CGA	−1.6409	(4.5273, 2.1587)	51	16490	3.4808	3.6522	Inf	11
SLSV	−1.7289	(4.9177, 1.9488)	56	185	3.4808	2.3279	Inf	3
SLSV-CG	−1.6595	(4.5965, 2.1131)	92	290	3.4601	3.4227	Inf	4
SORA	–	–	–	–	–	–	–	–
ASORA	–	–	–	–	–	–	–	–
SLShV-CG	−1.6432	(4.5225, 2.1537)	198	699	3.5401	3.6949	Inf	13

$$
\begin{aligned}
&\text{Find: } [\mu_{x_1},\ \mu_{x_2}]^T \\
&\text{min: } -\frac{(\mu_{x_1}+\mu_{x_2}-10)^2}{30}-\frac{(\mu_{x_1}-\mu_{x_2}+10)^2}{120} \\
&\text{s.t.: } Pr\left[g_1(\mathbf{X}) = 1-\frac{x_1^2 x_2}{20} > 0\right] \le \phi(-\beta_1^t), \\
&\qquad Pr\big[g_2(\mathbf{X}) = -1+(Y-6)^2+(Y-6)^3 \\
&\qquad -0.6(Y-6)^4+Z > 0\big] \le \phi(-\beta_2^t), \\
&\qquad Pr\left[g_3(\mathbf{X}) = 1-\frac{80}{(x_1^2+8x_2+5)} > 0\right] \le \phi(-\beta_3^t), \\
&\qquad Y = 0.9063x_1+0.4226x_2, \\
&\qquad Z = 0.4226x_1-0.9063x_2, \\
&\qquad 0 \le \boldsymbol{\mu}_{\mathbf{x}_i} \le 10,\ x_i \sim N(\boldsymbol{\mu}_{\mathbf{x}_i}, 0.3^2) \text{ for } i = 1,2, \\
&\qquad \beta_j^t = 3.5,\ \boldsymbol{\mu}_\mathbf{x}^{(0)} = [5.0, 5.0]^T\ j = 1,2,3.
\end{aligned}
\tag{23}
$$

Regarding efficiency, SLShV-CG consumes more function evaluations than SLSV and SLSV-CG in order to generate a reliable optimum solution. This is because the conjugate gradient shows slow convergence for the convex function $g_1(\mathbf{X})$. PMA-CGA generates a better solution than SLSV and SLSV-CG but with the expense of g_{FC}.

Regarding accuracy, it can be seen from β^t_{MCS} columns of Table 3 that SLShV-CG again emerges as the only method that can evolve the optimal solution with the desired target reliability. Other methods like SLSV and SLSV-CG are unable to generate the reliable optimal solutions. On the other hand, PMA-CGA is able to generate a solution that satisfies target reliability for constraint $g_2(\mathbf{X})$. PMA-AMV, SORA and ASORA are unable to converge for this example because the MPP is updated by the steepest descent, which fails for concave functions.

The convergence of the solutions with respect to the number of iterations is shown in figure 8. It can be seen that SLShV-CG requires more iterations in order to generate a reliable optimal solution. The progress of solutions obtained through SLShV-CG is shown in figure 9, which indicates target reliability achieved by the optimal solution.

4.4. *Mathematical example 4*

The last mathematical example shown in equation (24) is the problem no. 113 of Hock and Schittkowski (1981). The original problem is a deterministic problem, thus this RBDO problem

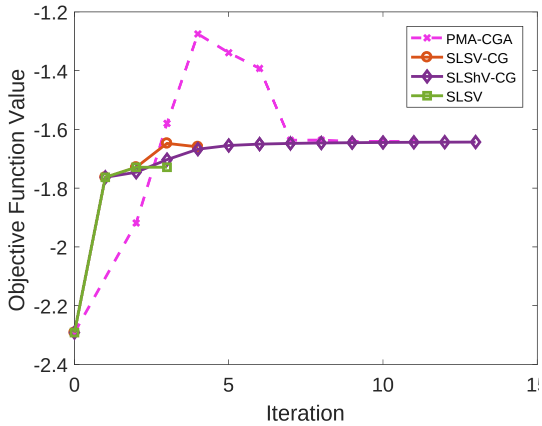

Figure 8. Convergence plot of example 3.

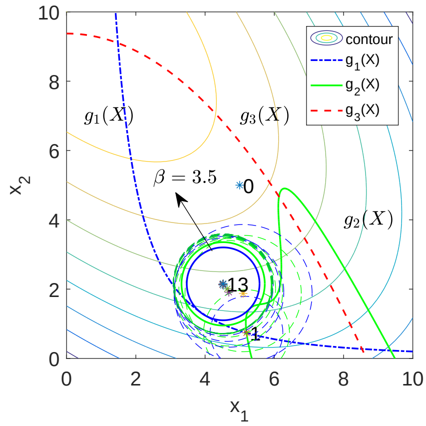

Figure 9. Contour plot of example 3.

is adopted from Lee and Lee (2005). There are 10 statistically independent random variables with normal distribution and eight probabilistic constraints with the desired reliability index of 3.0. The optimal solution obtained from deterministic optimization is selected as the initial design point.

Results obtained from all methods are summarized in table 4. It can be observed that all methods converge to a similar solution with an objective function value of 27.7466.

The efficiency of all methods is investigated by comparing their NFC. It can be seen that SLShV-CG has similar efficiency as SLSV and SLSV-CG and is better than other methods. It can also be observed that PMA-AMV and PMA-CGA require many NFC. This is because of the nested optimization in which the inner loop calls the constraint function several times at every iteration in order to calculate the gradient. However, SLShV-CG is the single-loop method thus requiring the minimum number of constraint function calls.

The accuracy of these methods is investigated from β^t_{MCS} columns of Table 4. It can be seen that none of the methods are able to generate the optimal solution with the desired target reliability for all probabilistic constraints. SLShV-CG, SLSV, SLSV-CG, and ASORA evolve the same optimal solution having the desired target reliability index closer to 3.0 for $g_1(\mathbf{X})$. Other methods like PMA-AMV, PMA-CGA and SORA are unable to achieve the desired target reliability for both $g_3(\mathbf{X})$ and $g_4(\mathbf{X})$. Figure 10 shows the convergence of all methods with respect to the number of iterations which are found to be smooth for all methods.

$$
\begin{aligned}
&\text{Find: } [\mu_{x_1}, \mu_{x_2}, \mu_{x_3}, \mu_{x_4}, \mu_{x_5}, \mu_{x_6}, \mu_{x_7}, \mu_{x_8}, \mu_{x_9}, \mu_{x_{10}}]^T \\
&\text{min.: } \mu_{x_1}^2 + \mu_{x_2}^2 + \mu_{x_1}\mu_{x_2} - 14\mu_{x_1} - 16\mu_{x_2} + (\mu_{x_3} - 10)^2 \\
&\quad + 4(\mu_{x_4} - 5)^2 + (\mu_{x_5} - 3)^2 + 2(\mu_{x_6} - 1)^2 + 5\mu_{x_7}^2 \\
&\quad + 7(\mu_{x_8} - 11)^2 + 2(\mu_{x_9} - 10)^2 + (\mu_{x_{10}} - 7)^2 + 45 \\
&\text{s.t.: } Pr\left[g_j(\mathbf{X}) > 0\right] \le \phi(-\beta_j^t),\ \beta_j^t = 3.0,\ \text{for } j = 1, \ldots, 8, \\
&g_1(\mathbf{X}) = \frac{4x_1 + 5x_2 - 3x_7 + 9x_8}{105} - 1 > 0, \\
&g_2(\mathbf{X}) = 10x_1 - 8x_2 - 17x_7 + 2x_8 > 0, \\
&g_3(\mathbf{X}) = \frac{-8x_1 + 2x_2 + 5x_9 - 2x_{10}}{12} - 1 > 0, \\
&g_4(\mathbf{X}) = \frac{3(x_1 - 2)^2 + 4(x_2 - 3)^2 + 2x_3^2 - 7x_4}{120} - 1 > 0, \\
&g_5(\mathbf{X}) = \frac{5x_1^2 + 8x_2 + (x_3 - 6)^2 - 2x_4}{40} - 1 > 0, \\
&g_6(\mathbf{X}) = \frac{0.5(x_1 - 8)^2 + 2(x_2 - 4)^2 + 3x_5^2 - x_6}{30} - 1 > 0, \\
&g_7(\mathbf{X}) = x_1^2 + 2(x_2 - 2)^2 - x_1x_2 + 14x_5 - 6x_6 > 0, \\
&g_8(\mathbf{X}) = -3x_1 + 6x_2 + 12(x_9 - 8)^2 - 7x_{10} > 0, \\
&0 \le \boldsymbol{\mu}_{\mathbf{x}_i} \le 10,\ x_i \sim N(\boldsymbol{\mu}_{\mathbf{x}_i}, 0.02^2),\ \text{for } i = 1, 2, \ldots, 10, \\
&\boldsymbol{\mu}_{\mathbf{x}}^{(0)} = [2.17, 2.36, 8.77, 5.10, 0.99, 1.43, \\
&\ 1.32, 9.83, 8.28, 8.38]^T.
\end{aligned}
\tag{24}
$$

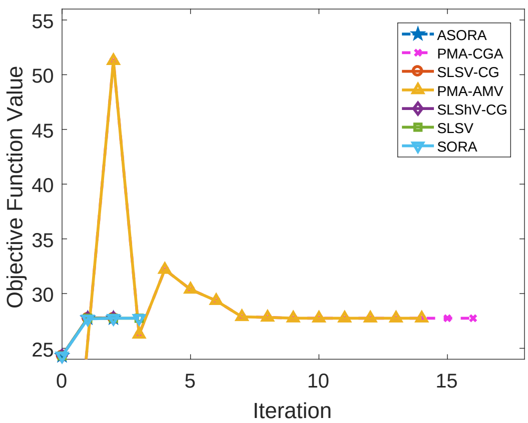

Figure 10. Convergence plot of example 4

4.5. *Speed Reducer*

A speed reducer problem (Lee and Lee 2005; Rao 2009), as illustrated in figure 11, is taken as the first engineering RBDO example. The design objective is to minimize the weight of the

Table 4. RBDO results for Example 4 with $\beta_t = 3.0$

Methods	f^*	$\boldsymbol{\mu}_{\mathbf{x}}^*$	NFC		β^t_{MCS}	Iter
			f_{FC}	g_{FC}	g_i	
DO	24.3062	(2.1720, 2.3637, 8.7739, 5.0960, 0.9907, 1.4306, 1.3216, 9.8287, 8.2801, 8.3759)	–	–	–	–
PMA-AMV	27.7466	(2.1350, 2.3309, 8.7094, 5.1021, 0.9225, 1.4452, 1.3885, 9.8094, 8.1556, 8.4755)	184	666272	3.0280, 3.0045, 2.9781, 2.9824, 3.0000, inf, 3.0332, inf	14
PMA-CGA	27.7466	(2.1350, 2.3309, 8.7094, 5.1021, 0.9225, 1.4452, 1.3885, 9.8094, 8.1556, 8.4755)	205	541500	3.0280, 3.0045, 2.9781, 2.9824, 3.0000, inf, 3.0332, inf	16
SLSV	27.7466	(2.1350, 2.3308, 8.7106, 5.1026, 0.9238, 1.4449, 1.3847, 9.8185, 8.1501, 8.4799)	287	1449	2.9957, 3.0332, 3.0013,, 3.0307, 3.0185, inf, 3.0407, inf	2
SLSV-CG	27.7466	(2.1350, 2.3308, 8.7094, 5.1021, 0.9225, 1.4452, 1.3885, 9.8094, 8.1556, 8.4755)	289	1449	2.9957, 3.0332, 3.0013, 3.0307, 3.0185, inf, 3.0407, inf	2
SORA	27.7466	(2.1350, 2.3309, 8.7094, 5.1021, 0.9225, 1.4452, 1.3885, 9.8094, 8.1556, 8.4755)	496	17031	3.0280, 3.0045, 2.9781, 2.9824, 3.0000, inf, 3.0332, inf	3
ASORA	27.7466	(2.1350, 2.3309, 8.7094, 5.1021, 0.9225, 1.4452, 1.3885, 9.8094, 8.1556, 8.4755)	376	2159	2.9957, 3.0332, 3.0013,, 3.0307, 3.0185, inf, 3.0407, inf	2
SLShV-CG	27.7466	(2.1350, 2.3308, 8.7094, 5.1021, 0.9225, 1.4452, 1.3885, 9.8094, 8.1556, 8.4755)	289	1439	2.9957, 3.0332, 3.0013, 3.0307, 3.0185, inf, 3.0407, inf	2

speed reducer which is subjected to the constraints on bending stress, contact stress, longitudinal displacement stress, stress of the shaft, transverse deflection, and geometric conditions. It contains seven independent random normal variables each with standard deviation $\sigma = 0.005$ and 11 probabilistic constraints with target reliability index of 3.0. The design variables are gear teeth (x_1), teeth module (x_2), number of teeth in pinion (x_3), the distance between two bearings (x_4, x_5), and axis diameter (x_6, x_7). The problem formulation is given in equation (25). Table 5 presents the optimal solutions obtained by the methods.

The efficiency of all methods is investigated by comparing their NFC. SLShV-CG is found to be better than other methods. Among all, PMA-AMV seems to be the least efficient. On the other hand, PMA-CGA is twice efficient than PMA-AMV. Methods like SLSV and SLSV-CG show similar computational efficiency.

The accuracy is investigated from β^t_{MCS} columns of Table 5. It can be seen that the constraints $g_5(\mathbf{x}), g_6(\mathbf{x}), g_8(\mathbf{x})$ and $g_{11}(\mathbf{x})$ are active and rest of them are inactive at the optimal

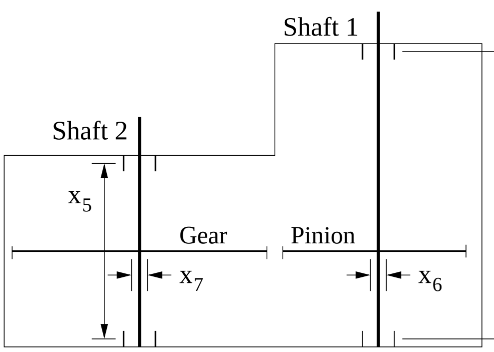

Figure 11. A Speed Reducer example

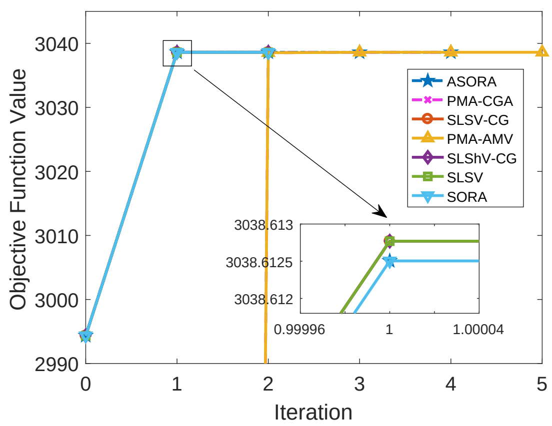

Figure 12. Convergence plot of Speed Reducer

solution. For this example, all RBDO methods are converged to the same optimal solution with the desired target reliability.

Figure 12 shows the convergence of all method with respect to the number of iterations. It can be noted that SLShV-CG requires only two iterations for convergence.

$$
\begin{aligned}
&\text{Find: } [\mu_{x_1}, \mu_{x_2}, \mu_{x_3}, \mu_{x_4}, \mu_{x_5}, \mu_{x_6}, \mu_{x_7}]^T \\
&\text{min.: } 0.7854\mu_{x_1}\mu_{x_2}^2(3.3333\mu_{x_3}^2 + 14.9334\mu_{x_3} - 43.0934) - \\
&\quad 1.508\mu_{x_1}(\mu_{x_6}^2 + \mu_{x_7}^2) + 7.477(\mu_{x_6}^3 + \mu_{x_7}^3) \\
&\quad + 0.7854(\mu_{x_4}\mu_{x_6}^2 + \mu_{x_5}\mu_{x_7}^2), \\
&\text{s.t.: } Pr\left[g_j(\mathbf{X}) > 0\right] \leq \phi(-\beta_j^t), \\
&\quad g_1(\mathbf{X}) = \frac{27}{x_1 x_2^2 x_3} - 1 > 0,\ g_2(\mathbf{X}) = \frac{397.5}{x_1 x_2^2 x_3^2} - 1 > 0, \\
&\quad g_3(\mathbf{X}) = \frac{1.93x_4^3}{x_2 x_3 x_6^4} - 1 > 0,\ g_4(\mathbf{X}) = \frac{1.93x_5^3}{x_2 x_3 x_7^4} - 1 > 0, \\
&\quad g_5(\mathbf{X}) = \frac{\sqrt{(\frac{745x_4}{x_2x_3})^2 + 16.9\times 10^6}}{0.1x_6^3} - 1100 > 0, \\
&\quad g_6(\mathbf{X}) = \frac{\sqrt{(\frac{745x_5}{x_2x_3})^2 + 157.5\times 10^6}}{0.1x_7^3} - 850 > 0, \\
&\quad g_7(\mathbf{X}) = x_2x_3 - 40 > 0,\ g_8(\mathbf{X}) = 5 - \frac{x_1}{x_2} > 0, \\
&\quad g_9(\mathbf{X}) = \frac{x_1}{x_2} - 12 > 0,\ g_{10}(\mathbf{X}) = \frac{1.5x_6 + 1.9}{x_4} - 1 > 0, \\
&\quad g_{11}(\mathbf{X}) = \frac{1.1x_7 + 1.9}{x_5} - 1 > 0, \\
&\quad 2.6 \leq x_1 \leq 3.6,\ 0.7 \leq x_2 \leq 0.8,\ 17 \leq x_3 \leq 28, \\
&\quad 7.3 \leq x_4 \leq 8.3,\ 7.3 \leq x_5 \leq 8.3,\ 2.9 \leq x_6 \leq 3.9, \\
&\quad 5 \leq x_7 \leq 5.5, \\
&\quad x_i \sim N(\boldsymbol{\mu}_{\mathbf{x}_i}, 0.005^2), \text{for } i = 1, 2, \ldots, 7, \\
&\quad \beta_j^t = 3.0,\ j = 1, 2, \ldots, 11, \\
&\quad \boldsymbol{\mu}_{\mathbf{x}}^{(0)} = [3.5, 0.7, 17, 7.3, 7.72, 3.35, 5.29]^T.
\end{aligned}
\tag{25}
$$

4.6. *Tension/Compression spring*

A tension/compression spring (Meng and Keshtegar 2019) is illustrated in figure 13 and the RBDO formulation is described in equation (26). The objective of the problem is to minimize the spring weight. This problem has three statistically independent random variables with normal distribution, namely mean coil diameter (x_1), the wire diameter (x_2), and the number of coils (x_3) and four probabilistic constraints. The target reliability is fixed to 3.0 for all constraints. The initial design point is selected as $\mu_x^{(0)} = [0.05, 0.5, 10]^T$. Table 6 summarizes the results obtained from all methods.

Table 5. RBDO results for speed reducer problem with $\beta_t = 3.0$

Methods	f^*	$\mu_{\mathbf{x}}^*$	NFC		β^t_{MCS}	Iter
			f_{FC}	g_{FC}	g_i	
PMA-AMV	3038.612	(3.5765, 0.7000, 17.0000, 7.3000, 7.7541, 3.3652, 5.3017)	48	229680	Inf,Inf,Inf,Inf,3.0307, 3.3082, Inf,3.0000,Inf,Inf,3.0068	5
PMA-CGA	3038.612	(3.5765, 0.7000, 17.0000, 7.3000, 7.7541, 3.3652, 5.3017)	48	120240	Inf,Inf,Inf,Inf,3.0307, 3.3082, Inf,3.0000,Inf,Inf,3.0068	5
SLSV	3038.612	(3.5765, 0.7000, 17.0000, 7.3000, 7.7541, 3.3652, 5.3017)	77	1038	Inf,Inf,Inf,Inf,3.0307, 3.3082, Inf,3.0000,Inf,Inf,3.0068	2
SLSV-CG	3038.612	(3.5765, 0.7000, 17.0000, 7.3000, 7.7541, 3.3652, 5.3017)	77	1038	Inf,Inf,Inf,Inf,3.0307, 3.3082, Inf,3.0000,Inf,Inf,3.0068	2
SORA	3038.612	(3.5765, 0.7000, 17.0000, 7.3000, 7.7541, 3.3652, 5.3017)	77	14874	Inf,Inf,Inf,Inf,3.0307, 3.3082, Inf,3.0000,Inf,Inf,3.0068	3
ASORA	3038.612	(3.5765, 0.7000, 17.0000, 7.3000, 7.7541, 3.3652, 5.3017)	112	1520	Inf,Inf,Inf,Inf,3.0307, 3.3082, Inf,3.0000,Inf,Inf,3.0068	4
SLShV-CG	3038.612	(3.5765, 0.7000, 17.0000, 7.3000, 7.7541, 3.3652, 5.3017)	76	1014	Inf,Inf,Inf,Inf,3.0307, 3.3082, Inf,3.0000,Inf,Inf,3.0068	2

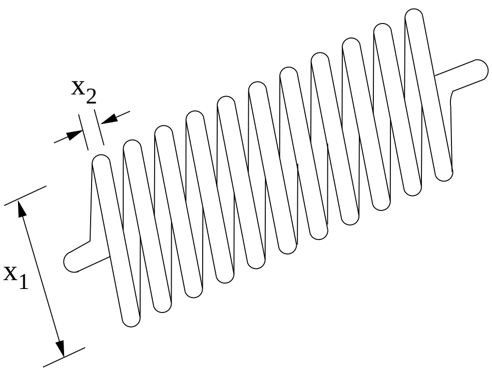

Figure 13. A Spring tension/compression example

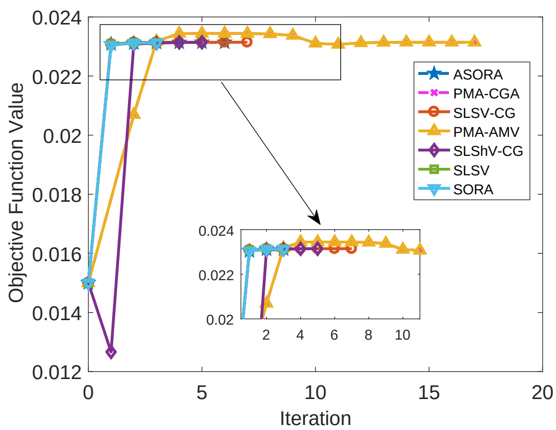

Figure 14. Convergence plot of spring tension/compression

Table 6. RBDO results for spring tension/compression problem with $\beta_t = 3.0$

Methods	f^*	$\boldsymbol{\mu}_{\mathbf{x}}^*$	NFC		β^t_{MCS}	Iter
			f_{FC}	g_{FC}	g_i	
PMA-AMV	0.02314298	(0.0589, 0.4620, 12.4319)	72	24480	2.9803, 3.0091 Inf,Inf	17
PMA-CGA	0.02314203	(0.0589, 0.4621,) 12.4286)	72	28744	2.9760, 3.0000 Inf,Inf	17
SLSV	0.02313374	(0.0586, 0.4545, 12.7981)	202	653	3.0000, 2.9576, Inf,Inf	3
SLSV-CG	0.02314299	(0.0589, 0.4607, 12.4970)	344	1281	2.9867, 3.0433, Inf,Inf	7
SORA	0.02312068	(0.0593, 0.4730, 11.9008)	210	1979	2.9781, 3.0091, Inf,Inf	3
ASORA	0.02314015	(0.0590, 0.4654, 12.2680)	321	1155	2.9824, 3.0045, Inf,Inf	6
SLShV-CG	0.02314287	(0.0590, 0.4649, 12.2908)	256	919	3.0068, 3.0282, Inf,Inf	5

$$
\begin{aligned}
&\text{Find: } [\mu_{x_1}, \mu_{x_2}, \mu_{x_3}]^T \\
&\text{min.: } (\mu_{x_3}+2)\mu_{x_2}\mu_{x_1}^2, \\
&\text{s.t.: } Pr\left[g_j(\mathbf{X}) > 0\right] \le \phi(-\beta_j^t),\ j = 1,2,3,4, \\
&\quad g_1(\mathbf{X}) = \frac{x_2^3 x_3}{71785 x_1^4} - 1 > 0, \\
&\quad g_2(\mathbf{X}) = 1 - \frac{4x_2^2 - x_1 x_2}{12566(x_2 x_1^3 - x_1^4)} - \frac{1}{5108 x_1^2} > 0, \\
&\quad g_3(\mathbf{X}) = \frac{140.45 x_1}{x_2^2 x_3} - 1 > 0, \\
&\quad g_4(\mathbf{X}) = 1 - \frac{x_1 + x_2}{1.5} > 0, \\
&\quad 0.01 \le \mu_{x_1} \le 0.1,\ 0.1 \le \mu_{x_2} \le 1.0,\ 5.0 \le \mu_{x_3} \le 15.0, \\
&\quad x_1 \sim N(\boldsymbol{\mu}_{x_1}, 0.001^2),\ x_2 \sim N(\boldsymbol{\mu}_{x_2}, 0.01^2), \\
&\quad x_3 \sim N(\boldsymbol{\mu}_{x_3}, 0.8^2),\ \beta_j^t = 3.0,\ \boldsymbol{\mu}_x^{(0)} = [0.05, 0.5, 10]^T.
\end{aligned}
\tag{26}
$$

The efficiency of all methods is investigated by comparing their NFC. SLShV-CG is found to be better than all methods, except for SLSV. However, SLSV is unable to generate a reliable solution. Methods like SLSV-CG and ASORA have comparable efficiency. PMA-AMV and PMA-CGA are the least efficient method because of the double loop structure.

Regarding accuracy, it can be seen from β^t_{MCS} columns of Table 6 that only SLShV-CG converges to the optimal solution with the desired target reliability and other methods fail to generate a reliable solution. Methods like SLSV and SORA converge to a solution with a poor reliability as compared to other methods.

Figure 14 shows the convergence plot of all methods. It can be seen from the figure that SLShV-CG converges to the optimal solution within five iterations.

4.7. *Welded beam design*

A welded beam (Cho and Lee 2011; Lee and Lee 2005) as shown in figure 15 is taken as the next engineering RBDO example. The objective function is to minimize the welding cost.

Table 7. Fixed parameters for the welded beam problem

z_1 :	2.6688×10^4 (N)
z_2 :	3.556×10^2 (mm)
z_3 :	2.0685×10^5 (MPa)
z_4 :	8.274×10^4 (MPa)
z_5 :	6.35 (mm)
z_6 :	9.377×10 (MPa)
z_7 :	2.0685×10^2 (MPa)
c_1 :	6.74135×10^{-5} (\$/mm^3)
c_2 :	2.93585×10^{-6} (\$/mm^3)

There are five probabilistic constraints related to the physical quantities such as shear stress, bending stress, bucking, and tip deflection. Design variables such as depth (x_1) and length (x_2) of the welding, and height (x_3) and thickness (x_4) of the beam are statistically independent random variables with normal distribution. The RBDO model of the welded beam is given in the equation (27). The fixed system parameters of equation (27) are listed in table 7. Table 8 presents the results obtained from all methods.

Regarding efficiency, it can be seen from the fifth column of the table that among all the methods, SLShV-CG is the most efficient method. Other methods like SLSV, SLSV-CG and ASORA show better efficiency than PMA-AMV and PMA-CGA.

Comparing the accuracy for this engineering RBDO example from β^t_{MCS} columns of Table 8, all methods converge to the same optimal solution with the desired target reliability. It can be noted from the sixth column of the table that $g_4(\mathbf{X})$ is an inactive constraint.

The convergence plot is shown in figure 16 which indicates a smooth convergence of all methods, except for PMA-AMV and PMA-CGA.

$$
\begin{aligned}
&\text{Find: } [\mu_{x_1}, \mu_{x_2}, \mu_{x_3}, \mu_{x_4}]^T \\
&\text{min.: } c_1\mu_{x_1}^2\mu_{x_2} + c_2\mu_{x_3}\mu_{x_4}(z_2 + \mu_{x_2}), \\
&\text{s.t.: } Pr\left[g_1(\mathbf{X}) = \frac{\tau(\mathbf{X},\mathbf{z})}{z_6} - 1 > 0\right] \le \phi(-\beta_1^t), \\
&\quad Pr\left[g_2(\mathbf{X}) = \frac{\sigma(\mathbf{X},\mathbf{z})}{z_7} - 1 > 0\right] \le \phi(-\beta_2^t), \\
&\quad Pr\left[g_3(\mathbf{X}) = \frac{x_1}{x_4} - 1 > 0\right] \le \phi(-\beta_3^t), \\
&\quad Pr\left[g_4(\mathbf{X}) = \frac{\delta(\mathbf{X},\mathbf{z})}{z_5} - 1 > 0\right] \le \phi(-\beta_4^t), \\
&\quad Pr\left[g_5(\mathbf{X}) = 1 - \frac{P_c(\mathbf{X},\mathbf{z})}{z_1} > 0\right] \le \phi(-\beta_5^t), \\
&\quad 3.175 \le x_1 \le 50.8,\ 0 \le x_2 \le 254,\ 0 \le x_3 \le 254, \\
&\quad 0 \le x_4 \le 50.8, \\
&\quad x_{1,2} \sim N(\boldsymbol{\mu}_{x_{1,2}}, 0.1693^2),\ x_{3,4} \sim N(\boldsymbol{\mu}_{x_{3,4}}, 0.0107^2), \\
&\quad \beta_j^t = 3.0,\ j = 1, 2, \ldots, 5, \qquad\qquad (27) \\
&\quad \boldsymbol{\mu}_{\mathbf{x}}^{(0)} = [6.208, 157.82, 210.62, 6.208]^T, \\
&\quad t(\mathbf{X},\mathbf{z}) = \frac{z_1}{\sqrt{2}x_1x_2},\ tt(\mathbf{X},\mathbf{z}) = M(\mathbf{X},\mathbf{z})\frac{R(\mathbf{X},\mathbf{z})}{J(\mathbf{X},\mathbf{z})}, \\
&\quad M(\mathbf{X},\mathbf{z}) = z_1\left(z_2 + \frac{x_2}{2}\right),\ R(\mathbf{X},\mathbf{z}) = \frac{\sqrt{x_2^2 + (x_1 + x_3)^2}}{2}, \\
&\quad J(\mathbf{X},\mathbf{z}) = \sqrt{2}x_1x_2\left\{\frac{x_2^2}{12} + \frac{(x_1 + x_3)^2}{4}\right\}, \\
&\quad \sigma(\mathbf{X},\mathbf{z}) = \frac{6z_1z_2}{x_3^2x_4},\ \delta(\mathbf{X},\mathbf{z}) = \frac{4z_1z_2^3}{z_3x_3^3x_4}, \\
&\quad P_c(\mathbf{X},\mathbf{z}) = \frac{4.013x_3x_4^3\sqrt{z_3z_4}}{6z_2^2}\left(1 - \frac{x_3}{4z_2}\sqrt{\frac{z_3}{z_4}}\right), \\
&\quad \tau(\mathbf{X},\mathbf{z}) = \left\{t(\mathbf{X},\mathbf{z})^2 + 2t(\mathbf{X},\mathbf{z})tt(\mathbf{X},\mathbf{z})\left(\frac{x_2}{2R(\mathbf{X},\mathbf{z})}\right) + tt(\mathbf{X},\mathbf{z})^2\right\}^{1/2}.
\end{aligned}
$$

4.8. *Cantilever beam problem*

A cantilever beam problem (Liang, P. Mourelatos, and Tu 2008) adopted in this example is shown in figure 17. The problem formulation is given in equation (28). The beam is loaded with two point loads, lateral load p_z and vertical load p_y at the tip. The width and thickness of the beam are 'w' and 't' respectively and the length L of the beam is equal to 100 inches. The objective of the problem is to minimize the weight of the beam. The first constraint represents that the maximum stress at the fixed end should be less than the yield strength S_y. The second constraint represents the displacement that should not exceed the allowable value of D_0. The

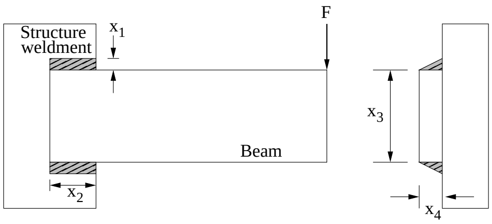

Figure 15. A welded beam structure example

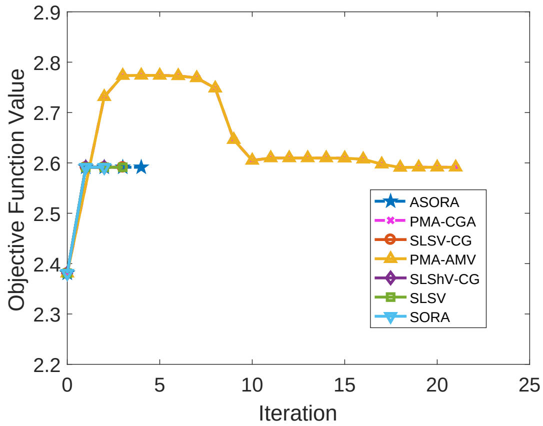

Figure 16. Convergence plot of welded beam

Table 8. RBDO results for Welded beam design with $\beta_t = 3.0$

Methods	f^*	μ_x^*	NFC f_{FC}	NFC g_{FC}	β^t_{MCS} g_i	Iter
PMA-AMV	2.5913	(5.7300, 200.8982, 210.5977, 6.2389)	115	57750	3.0233, 3.0233 3.0091,Inf,3.0091	21
PMA-CGA	2.5913	(5.7300, 200.8981, 210.5977, 6.2389)	110	37290	3.0233, 3.0233 3.0091,Inf,3.0091	21
SLSV	2.5913	(5.7300, 200.8982, 210.5977, 6.2389)	177	847	3.0233, 3.0233 3.0091,Inf,3.0091	3
SLSV-CG	2.5913	(5.7300, 200.8982, 210.5977, 6.2389)	177	847	3.0233, 3.0233 3.0091,Inf,3.0091	3
SORA	2.5913	(5.7300, 200.8982, 210.5977, 6.2389)	160	2155	3.0233, 3.0233 3.0091,Inf,3.0091	3
ASORA	2.5913	(5.7300, 200.8982, 210.5977, 6.2389)	193	905	3.0233, 3.0233 3.0091,Inf,3.0091	4
SLShV-CG	2.5913	(5.7300, 200.8982, 210.5977, 6.2389)	164	740	3.0233, 3.0233 3.0091,Inf,3.0091	3

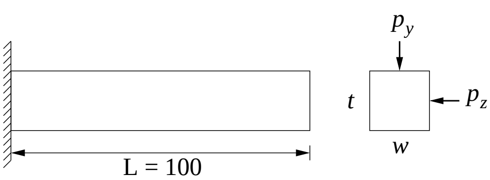

Figure 17. A cantilever beam example

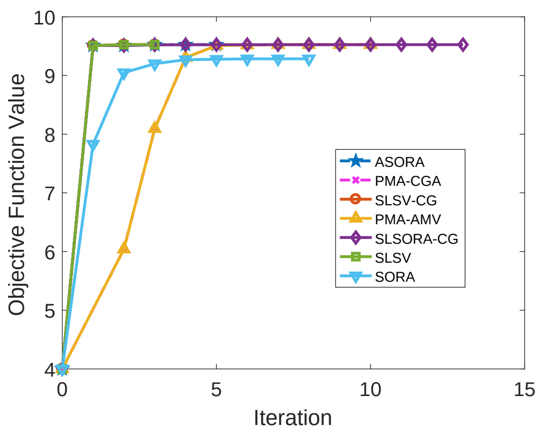

Figure 18. Convergence plot of cantilever beam

Table 9. RBDO results for cantilever beam with $\beta_t = 3.0$

Methods	f^*	$\boldsymbol{\mu}^*$	NFE		β^t_{MCS}	Iter
			f_{FE}	g_{FE}	g_1 g_2	
PMA-AMV	9.5253	(2.4538, 3.8819)	77	13930	3.0025, 3.0320	10
PMA-CGA	9.5253	(2.4531, 3.8829)	77	26936	3.0357, 3.0045	10
SLSV	9.5253	(2.4538, 3.8819)	175	370	3.0025, 3.0320	3
SLSV-CG	9.5253	(2.4538, 3.8819)	182	384	3.0025, 3.0320	3
SORA	9.2840	(2.5802, 3.5981)	325	2498	2.5696, 3.0000	3
ASORA	9.5253	(2.4538, 3.8819)	260	570	3.0022, 3.0357	5
SLShV-CG	9.5252	(2.4538, 3.8819)	443	1122	3.0185, 3.0433	13

desired reliability index of $\beta^t_i = 3.0$ is kept for both the constraints. The results for this example are summarized in table 9.

$$
\begin{aligned}
\text{Find: } & \mu_w,\ \mu_t \\
\text{min: } & w \times t, \\
\text{s.t.: } & Pr\left[g_j(\mathbf{X}) > 0\right] \leq \phi(-\beta^t_j),\ j = 1,2, \\
& g_1(S_y, p_y, p_z, t) = S_y - \left(\frac{600}{wt^2}p_y + \frac{600}{wt^2}p_z\right) > 0, \\
& g_2(E, w, t, p_y, p_z) = D_0 - \frac{4L^3}{Ewt}\sqrt{\left(\frac{p_y}{t^2}\right)^2 + \left(\frac{p_z}{w^2}\right)^2} > 0, \\
& 0 \leq w \leq 5,\ 0 \leq t \leq 5,\ 500 \leq p_z \leq 800, \\
& 1000 \leq p_y \leq 1500,\ 35000 \leq S_y \leq 45000, \\
& 25 \times 10^6 \leq E \leq 30 \times 10^6, \\
& w \sim N(\boldsymbol{\mu}_w, 0.01^2),\ t \sim N(\boldsymbol{\mu}_t, 0.01^2), \\
& p_z \sim N(\boldsymbol{\mu}_{p_z}, 100^2), \\
& p_y \sim N(\boldsymbol{\mu}_{p_y}, 100^2),\ S_y \sim N(\boldsymbol{\mu}_{S_y}, 2000^2), \\
& E \sim N(\boldsymbol{\mu}_E, (1.45 \times 10^6)^2), \\
& \beta^t_j = 3.0,\ \boldsymbol{\mu}^{(0)} = [2, 2, 500, 1000, 40000, 29 \times 10^6]^T \ .
\end{aligned} \tag{28}
$$

Regarding efficiency, SLShV-CG is found to be better than PMA-AMV, PMA-CGA and SORA when comparing their NFC. However, SLShV-CG needs improvement over SLSV, SLSV-CG and ASORA, which are computationally efficient.

Regarding accuracy, all methods evolve the same reliable solution for this problem, except SORA. The convergence plot for all methods is shown in figure 18. It can be seen that SLShV-CG requires more iterations than SLSV, SLSV-CG, ASORA, and SORA.

5. Conclusion

An accurate and computationally efficient hybrid RBDO method has been proposed in this paper. The method was developed by coupling the single-loop method with the shifting vector approach of SORA. The approximate MPP was updated using the conjugate gradient search direction in each iteration. Based on the results obtained after solving four mathematical and four engineering RBDO problems, it can be concluded that SLShV-CG is able to generate the optimal solution with the desired target reliability. Mathematical examples 2 and 3 demonstrated that SLShV-CG converged to the desired reliable solution irrespective of the convex or concave nature of the limit state functions, whereas other methods failed to converge. Also, SLShV-CG was found to be computationally efficient than those methods which were able to generate the reliable optimal solution. In future work, SLShV-CG can be tested on other RBDO problems in which the variables follow non-normal distribution. Moreover, it can be tested on large scale RBDO problems.

Disclosure statement

No potential conflict of interest was reported by the authors.

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