A Single-Loop Reliability-based Design Optimization Method using Iteratively Updating Hessian

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Summary

Reliability-based design optimization (RBDO) is an efficient tool for solving engineering problems with uncertainty. There exist three types of analytical methods for solving RBDO problems such as double-loop, single-loop and decoupled-loop methods. Among them, the single-loop method is found to be computationally efficient because it approximates the most probable point (MPP) by using Karush-Kunn Tucker (KKT) conditions with the performance measurement approach. Although this method is efficient, but sometimes lacks in accuracy to achieve the target reliability. In this paper, a single-loop reliability-based design optimization method is proposed to improve the accuracy, which is achieved by approximating the MPP by including Hessian of the performance function. Further, this Hessian is updated iteratively to make the search direction descent. The proposed method is tested on two mathematical and two engineering RBDO problems. Results demonstrate its accuracy and computational efficiency over two methods from the literature.

Keywords: *Reliability-based design optimization, Single-loop method, Hessian, Performance measurement approach, KKT conditions, Most probable point, Reliability index, Standard normal variables*

1 Introduction

RBDO has been an important tool for solving those optimization problems which have uncertainty in design variables. This uncertainty can be handled by performing reliability analysis. However, the computational cost for solving reliability analysis is quite high. Therefore, various analytical methods have been developed to ease the computaional efficiency. Among them most probable point (MPP)-based method is widely used for reliability analysis. MPP-based methods include first-order reliability method (FORM)[1,2] and second-order reliability method (SORM),[3–5] which approximate the limit state function $g(\mathbf{X})$ by first-order and second-order Taylor series expansion, respectively. The reliability analysis using FORM produces less accurate solution than SORM, when the limit state function are highly non-linear. However, SORM requires more computation than FORM. It is because SORM requires second order derivative to approximate $g(\mathbf{X})$. To overcome these difficulties and maintain the efficiency, various methods have been developed for solving RBDO[6,7] problems. These methods can be broadly subdivided into three types: double-loop method, single-loop method and decoupled-loop method. A double-loop method[8–10] comprises of two loops in which the outer loop is for optimization and the inner loop is for reliability analysis. The probabilistic constraint in the double-loop method can be solved by either reliability index approach (RIA)[11] or performance measurement approach (PMA).[9] Although the double-loop method produces reliable solution, but needs high computation to solve the relativity analysis. Thus, the decoupling-loop methods are developed to reduce the computational cost and increase the efficiency.

The decoupling-loop method[12–15] decouples the nested loop structure of RBDO method into series of deterministic optimization and reliability analysis. This method shows a good convergence rate with less number of function evaluations. Among the decoupling methods, sequential optimization and reliability assessment (SORA)[12] is the most promising method. The main difficulty of decoupled-method is that it performs reliability assessment which requires a separate optimization. Thus the method has been further developed into single-loop method, where only single deterministic optimization was evaluated. Liang et al. proposed a single-loop single vector (SLSV)[16] method that approximates the reliability analysis and avoids the conventional approach for MPP. Chen et al.[17] transforms the probabilistic constraint into an approximate deterministic constraint. Instead of finding MPP, this method approximate the point on the basis of limit state sensitivities and target reliability. Further a semi single-loop method[18] is developed in which the sensitivity analysis is used to approximate MPP. From the above studies, it is found that the single-loop method produces most efficient results. The difficulty with this method is its convergence and accuracy. In this paper, the accuracy of single-loop method is improved by using Hessian matrix for approximating the MPP. Further, Hessian matrix is updated iteratively to eliminate the singularity. The proposed method is tested on four RBDO problems and the results are compared with two RBDO methods from the literature with same convergence criteria.

This paper is organised as follows. Section 2 describes the basic RBDO formulation. The single-loop method is discussed in brief in Section 3. The details of the proposed method are presented in Section 4. Examples are solved and discussed in Section 5 and finally, the conclusions are given in Section 6.

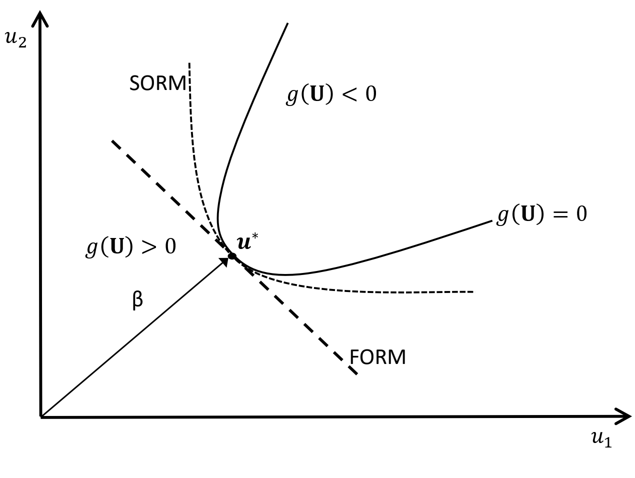

Figure 1: Approximation with FORM and SORM. Here, $g(\mathbf{U})$ is the limit state function, β is the target reliability, and $\mathbf{u}^*$ is the MPP

2 RBDO Formulation

The mathematical expression for RBDO is as follows

$$\begin{aligned} \text{min}: \quad & f(\mu_{\mathbf{x}}) \\ \text{s.t.}: \quad & P_f[g_i(\mathbf{X}) \leq 0] \leq \Phi(-\beta_i^t), \qquad i = 1, \ldots, nc \\ & {\mu_{\mathbf{x}}}^L \leq \mu_{\mathbf{x}} \leq {\mu_{\mathbf{x}}}^U, \end{aligned} \tag{1}$$

where f represents the objective function, $\mathbf{X}$ represents the vector of random variables with mean value $\mu_{\mathbf{X}}$. $\mu_{\mathbf{X}}^L$ and $\mu_{\mathbf{X}}^U$ are the lower and upper limit of mean value $\mu_{\mathbf{X}}$. P_f is failure probability of i^{th} performance function g_i. Φ represents the standard normal cumulative distribution function and β_i^t is the target reliability index of i^{th} performance function and nc is the number of constraints.

The failure probability of equation (1) can be evaluated by solving a multidimensional integral as given below

$$P_f[g(\mathbf{X}) \leq 0] = F_g(0) = \int \cdots \int_{g(\mathbf{x}) \leq 0} f_{\mathbf{X}}(\mathbf{x}) \mathrm{d}\mathbf{X}. \tag{2}$$

To ease the computational difficulty for solving equation (2) the integrand joint probability density function $f_{\mathbf{X}}(\mathbf{x})$ is simplified and the performance function $g(\mathbf{X})$ is approximated. It is done by transforming random variables from the original space ($\mathbf{X}$) to standard normal space ($\mathbf{U}$). This is achieved by Rosenblatt transformation, which is expressed by

$$\mathbf{U} = \Phi^{-1}[F_{\mathbf{X}}(\mathbf{X})] \tag{3}$$

$F_{\mathbf{X}}(\mathbf{X})$ is the representation of cumulative distribution function of $g(\mathbf{X})$.

Some approximate probability integration methods are developed to provide efficient solutions. Among these methods FORM and SORM are widely used. The formulation of FORM and SORM are described in the following subsection.

2.1 *First-order reliability method (FORM)*

The performance function $g(\mathbf{X})$ is approximated by first-order Taylor series expansion in standard normal space, which is given as

$$g(\mathbf{U}) \approx g(\mathbf{u}^*) + \nabla g(\mathbf{u}^*)(\mathbf{U} - \mathbf{u}^*)^T, \tag{4}$$

where $\mathbf{u}^*$ is the expansion point such that, $\mathbf{u}^* = [u_1^*, u_2^*, \ldots, u_n^*]^T$.

2.2 *Second-order reliability method (SORM)*

SORM uses Taylor series expansion up to second term at the MPP, $\mathbf{u}^*$. The performance function approximation is given as

$$g(\mathbf{U}) \approx g(\mathbf{u}^*) + \nabla g(\mathbf{u}^*)(\mathbf{U} - \mathbf{u}^*)^T + \frac{1}{2}(\mathbf{U} - \mathbf{u}^*)^T \mathrm{H}(\mathbf{u}^*)(\mathbf{U} - \mathbf{u}^*), \tag{5}$$

where $H(\mathbf{u}^*)$ is the Hessian matrix calculated at MPP $\mathbf{u}^*$, which can be given as

$$H(\mathbf{u}^*) = \begin{bmatrix} \frac{\partial^2 g}{\partial U_1^2} & \frac{\partial^2 g}{\partial U_1 \partial U_2} & \cdots & \frac{\partial^2 g}{\partial U_1 \partial U_n} \\ \frac{\partial^2 g}{\partial U_2 \partial U_1} & \frac{\partial^2 g}{\partial U_2^2} & \cdots & \frac{\partial^2 g}{\partial U_2 \partial U_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 g}{\partial U_n \partial U_1} & \frac{\partial^2 g}{\partial U_n \partial U_2} & \cdots & \frac{\partial^2 g}{\partial U_n^2} \end{bmatrix} \quad (6)$$

The asymptotic solution of probability of failure, when target reliability β is large, given as

$$P_f = P\{g(\mathbf{X}) \leq 0\} = \Phi(-\beta) \prod_{i=1}^{n-1} (1+\beta\kappa_i)^{1/2}, \quad (7)$$

where κ denotes the curvature of the performance function. Figure 1 shows the approximation and accuracy of FORM and SORM with a desired target reliability in the $\mathbf{U}$-space.

2.3 Performance measure approach (PMA)

The value of performance measure can be calculated by solving the following optimization problem

$$\begin{aligned} &\text{find} && \mathbf{U}^*, \\ &\min && g_i(\mathbf{U}), \\ &s.t.: && \|\mathbf{U}\| = \beta_i^t, \end{aligned} \quad (8)$$

where the optimum point $\mathbf{U}^*$ is known as the most probable target point (MPTP). The optimum value of $g_i(\mathbf{U}^*)$ is used as performance measure. The performance measure can be calculated as

$$g_i(\mathbf{U}^*) = G_i(\mathbf{X}^*). \quad (9)$$

3 Single-Loop Single Vector

A single-loop single vector method for solving RBDO was suggested by Liang et. al.[19] The formulation is obtained by solving the probabilistic constraint of equation (1) with PMA and KKT optimality conditions are imposed to get the approximate MPP. The formulation can be expressed as

$$\begin{aligned} \min: \quad & f(\mu_{\mathbf{X}}), \\ \text{s.t.}: \quad & G_i(\mathbf{X}) \geq 0, \qquad i = 1,2,\ldots,nc \\ & \mu_{\mathbf{x}}^L \leq \mu_{\mathbf{x}} \leq \mu_{\mathbf{x}}^U, \end{aligned} \quad (10)$$

where

$$\mathbf{X}^{(k)} = \mu_{\mathbf{x}} + \sigma\beta_i^t \alpha^{(k-1)}, \quad (11)$$

$$\alpha^{(k)} = \left. \frac{\sigma \nabla G_i(\mathbf{X})}{\|\sigma \nabla G_i(\mathbf{X})\|} \right|_{\mathbf{X}^{(k)}}, \quad (12)$$

where $\alpha^{(k)}$ is the normalized gradient vector of i^{th} constraint at k^{th} iteration, $\mathbf{X}^{(k)}$ is the approximate MPP in the original space and *sigma* is the standard deviation of the random variable. This method decreases the computational cost by eliminating the exact MPP search.

4 The Proposed Single-Loop Method

The proposed single-loop method with iteratively updating Hessian (SLM-MH) has similar formulation as given in equation (10). However, MPP $\mathbf{X}^{*(k)}$ in the original space is updated as

$$\mathbf{X}^{*(k)} = \mu_{\mathbf{x}}^{(k)} + \sigma\beta_i^t \alpha^{(k-1)}, \quad (13)$$

where

$$\alpha^{(k)} = \left[\frac{\sigma[\mathbf{H}_i]^{-1} \nabla G_i}{\|\sigma[\mathbf{H}_i]^{-1} \nabla G_i\|} \right]_{\mathbf{x}^{*(k)}}, \quad (14)$$

where $\mathbf{H}_i$ is iteratively modified as

$$\mathbf{H}_i = [\mathbf{H} + \lambda^{(k)} \mathbf{I}], \quad (15)$$

where $\mathbf{H}$ is the Hessian matrix and $\mathbf{I}$ is the identity matrix of same size of $\mathbf{H}$. The value of constant parameter $\lambda^{(k)}$ reduces to its half value in every iteration ($\lambda^{(k)} = \lambda^{(k-1)}/2$). $\alpha^{(k)}$ is the direction vector and is updated by updating Hessian matrix as shown in equation (15).

Initially, $\lambda^{(k)}$ is kept high so that $\mathbf{H}_i$ is equivalent to $\mathbf{I}$, which is convex. After a number of iterations when a solution is in the vicinity of the exact MPP, $\mathbf{H}_i$ becomes $\mathbf{H}$ that can be positive definite. In that case, the proposed search direction can be descent to locate the exact MPP with desired target reliability.

4.1 Algorithm

Following are the steps of the proposed method

1. Set the initial values as $k = 0, \mu_{\mathbf{x}}^0$, the standard deviation σ and the target reliability index β_j^t, $\lambda^{(0)} = 10$.

2. Perform the deterministic optimization and generate $\mu_{\mathbf{x}}^*$

$$\begin{aligned} &\text{Find} && \mu_{\mathbf{x}} \\ &\min: && f(\mu_{\mathbf{x}}), \\ &\text{s.t.} && G_i(\mu_{\mathbf{x}}) \geq 0, \qquad i = 1,2,\ldots,nc. \end{aligned} \quad (16)$$

3. Calculate the $\alpha^{(k)}$ of each of the active constraints.

$$\alpha^{(k)} = \left[\frac{\sigma[\mathbf{H}_i]^{-1} \nabla G_i}{\|\sigma[\mathbf{H}_i]^{-1} \nabla G_i\|} \right]_{\mu_{\mathbf{x}}^1 = \mu_{\mathbf{x}}^*}$$

4. Set $k = k+1$ and perform the following optimization

$$\begin{aligned} &\text{Find} && \mu_{\mathbf{x}} \\ &\min: && f(\mu_{\mathbf{x}}), \\ &\text{s.t.} && G_i(\mathbf{X}^{*(k)}) \geq 0, \quad i = 1,2,\ldots,nc, \end{aligned} \quad (17)$$

where $\mathbf{X}^{*(k)} = \mu_{\mathbf{x}}^{(k)} + \sigma\beta_i^t \alpha^{(k-1)}$

$$\alpha^{k-1} = \left[\frac{\sigma[\mathbf{H}_i]^{-1} \nabla G_i}{\|\sigma[\mathbf{H}_i]^{-1} \nabla G_i\|} \right]_{\mathbf{x}^{*(k-1)}}.$$

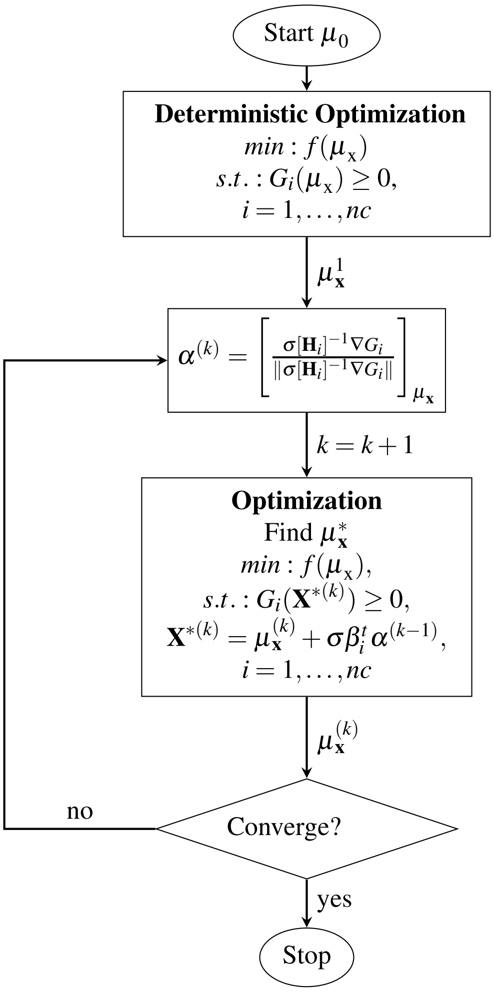

Figure 2: Flowchart of SLSV-MH

5. If the convergence is satisfied, then terminate. Otherwise go to step 3. The convergence criterion is set to be $\|\mu_{\mathbf{x}}^{(k)} - \mu_{\mathbf{x}}^{(k-1)}\| / \|\mu_{\mathbf{x}}^{(k-1)}\| \leq 0.001$.

Note that at Steps 2 and 4, the deterministic optimization and optimization with probabilistic constraints are solved using the sequential quadratic programming (SQP) method by calling fmincon solver of MATLAB. The SQP method gets terminated when the change in the consecutive values of variables is 10^{-6}. The flowchart of the RBDO formulation is also shown Fig. 2.

5 Examples and Discussion

The proposed SLM-MH method is tested on two mathematical and two engineering problems. The performance of SLM-MH is compared with the double-loop method with PMA (DLM-PMA) and a decoupled-loop method, i.e., sequential optimization and reliability assessment (SORA). These methods are programmed using MATLAB R2016b tool and run on a Intel(R) Core(TM) $i7-7500U$ CPU with 2.70 Ghz processor. The processor has a 12 GB of RAM (internal memory) operating. The four problems are presented and results of RBDO methods are discussed in the following sections.

5.1 *Mathematical problem 1*

The first example is a non-linear mathematical problem[20] with linear objective function and highly non-linear performance functions. The mathematical formulation of the problem is given in equation (18).

$$
\begin{aligned}
\text{min}: \quad & \mu_{\mathbf{x}_1} + \mu_{\mathbf{x}_2} \\
s.t.: \quad & Pr\left[g_1(\mathbf{X}) = 1 - \frac{x_1^2 x_2}{20} > 0\right] \leq \phi(-\beta_1^t), \\
& Pr\left[g_2(\mathbf{X}) = 1 - \frac{(x_1+x_2-5)^2}{30} - \frac{(x_1-x_2-12)^2}{120} > 0\right] \\
& \quad \leq \phi(-\beta_2^t), \\
& Pr\left[g_3(\mathbf{X}) = 1 - \frac{80}{(x_1^2+8x_2+5)} > 0\right] \leq \phi(-\beta_3^t), \\
& 0 \leq \mu_{\mathbf{x}_j} \leq 10, \ \ x_j \sim N(\mu_{x_j}, 0.3^2) \ \text{ for } \ j = 1,2 \\
& \beta_i^t = 3.0, \ \ \mu_x^{(0)} = [5.0, 5.0]^T \ \ i = 1,2,3.
\end{aligned}
\tag{18}
$$

where $\mu_{\mathbf{x}}$ and σ are the mean values and standard deviation respectively. The target reliability of $\beta^t = 3.0$ is taken for each constraint.

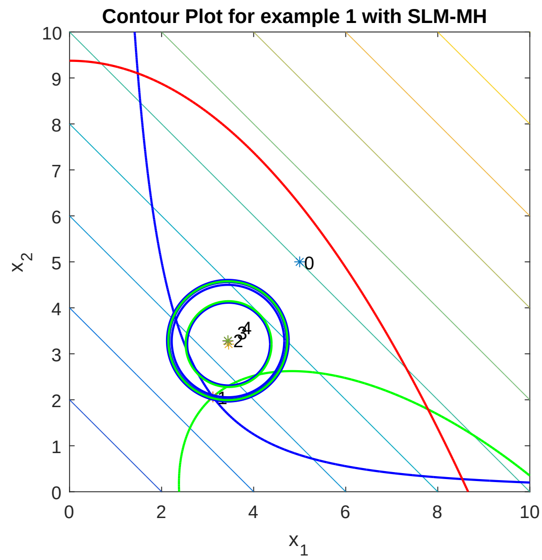

Figure 3: Contour plot for mathematical problem 5.1 along with convergence of SLM-MH

Table 1 presents the results obtained by the three methods. The number of function evaluations of objective function and constraints are represented as f_{FE} and g_{FE} respectively. It can be seen from the table that SLM-MH generated the solution with desired target reliability, which is then verified for each constraints through Monte-Carlo simulation (MCS) with 10^4 sample size. DLM-PMA is also able to achieve the desired reliability, but with an expense

Table 1: RBDO results for mathematical problem 1 with $\beta_t = 3.0$

Methods	f^*	$\mu^*_{\mathbf{x}}$	NFE		β^t_{MCS}			Iter
			f_{FE}	g_{FE}	g_1	g_2	g_3	
DLM-PMA	6.7219	(3.4363, 3.2855)	27	5193	3.0045	3.0307	Inf	3
SORA	6.7226	(3.4369, 3.2857)	76	1137	2.9989	3.0617	Inf	4
SLM-MH	6.7255	(3.4392, 3.2863)	211	757	3.0022	3.0407	Inf	4

of larger function evaluations than SLM-MH. On the other hand, SORA is unable to generate optimal solution with desired reliability for $g_1(\mathbf{X})$. It also takes more number function evaluation than SLM-MH. Although SLM-MH takes one iteration more than DLM-PMA to converge to the solution, but is the most efficient with respect to g_{FE}. Figure 3 also shows that for the proposed method the target reliability is satisfied at the optimum for both the constraints by SLM-MH. For $g_3(\mathbf{X})$, β-circle is not shown in the figure as its MCS value is infinity.

5.2 *Mathematical problem 2*

The second mathematical example[21] constitutes of ten random variables and eight probabilistic constraint. The target reliability $\beta^t = 3.0$ is set for all the constraints. The formulation of the problem is given in equation (19).

$$
\begin{aligned}
\min: \quad & x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3-10)^2 \\
& + 4(x_4-5)^2 + (x_5-3)^2 + 2(x_6-1)^2 + 5x_7^2 \\
& + 7(x_8-11)^2 + 2(x_9-10)^2 + (x_{10}-7)^2 + 45 \\
s.t.: \quad & Pr\left[g_i(\mathbf{X}) > 0\right] \leq \phi(-\beta_i^t) \\
where: \quad & \beta_i^t = 3.0, \;\; i = 1,2,\dots,8 \\
& g_1(\mathbf{X}) = \frac{4x_1 + 5x_2 - 3x_7 + 9x_8}{105} - 1 > 0 \\
& g_2(\mathbf{X}) = 10x_1 - 8x_2 - 17x_7 + 2x_8 > 0 \\
& g_3(\mathbf{X}) = \frac{-8x_1 + 2x_2 + 5x_9 - 2x_{10}}{12} - 1 > 0 \\
& g_4(\mathbf{X}) = \frac{3(x_1-2)^2 + 4(x_2-3)^2 + 2x_3^2 - 7x_4}{120} - 1 \\
& g_5(\mathbf{X}) = \frac{5x_1^2 + 8x_2 + (x_3-6)^2 - 2x_4}{40} - 1 > 0 \\
& g_6(\mathbf{X}) = \frac{0.5(x_1-8)^2 + 2(x_2-4)^2 + 3x_5^2 - x_6}{30} - 1 \\
& g_7(\mathbf{X}) = x_1^2 + 2(x_2-2)^2 - x_1x_2 + 14x_5 - 6x_6 > 0 \\
& g_8(\mathbf{X}) = -3x_1 + 6x_2 + 12(x_9-8)^2 - 7x_10 > 0 \\
& 0 \leq \mu_{\mathbf{x}_j} \leq 10, \;\; x_j \sim N(\mu_{x_j}, 0.02^2) \;\; , \;\; j = 1,2,\dots, \\
& \mu_x^{(0)} = [2.17, 2.36, 8.77, 5.10, 0.99, 1.43, \\
& \quad 1.32, 9.83, 8.28, 8.38]^T
\end{aligned} \tag{19}
$$

Table 2 presents the results obtained by the three methods. It can be observed that that none of the methods is able to generate optimal solution which satisfies desired reliability for all the constraints, as verified through MCS. However, SLM-MH is found to be more accurate than the other methods. It can also be observed that DLM-PMA and SORA failed to achieve atrget reliability for $g_1(\mathbf{X})$ and $g_5(\mathbf{X})$. However, with SLM-MH only for constraint $g_1(\mathbf{X})$, reliability is not achieved. Moreover, SLM-MH consumes lesser function evaluations than other methods. It can also be seen that the efficiency is approximately twice compared with SORA.

5.3 *Speed reducer problem*

A speed reducer problem,[22] as illustrated in figure 4 is taken as an engineering RBDO example. The design objective is to minimize the weight of the speed reducer which is subjected to bending stress, contact stress, longitudinal displacement stress, stress of the shaft, transverse deflection and geometric conditions. It has seven independent random variables, such as gear width (x_1), gear module (x_2), the number of pinion teeth (x_3), distance between bearings (x_4,x_5) and shaft diameters (x_6,x_7). All random variables follow normal distribution. The target reliability β^t is fixed to 3.0 for all constraints. The standard deviation is 0.005 for all the random variables and the deterministic solution of the problem is taken as the initial point.

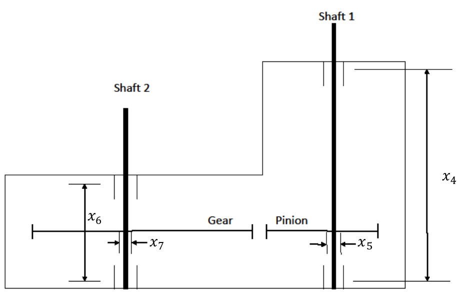

Figure 4: A speed reducer

Table 2: RBDO results for mathematical problem 2 with $\beta_t = 3.0$

Methods	f^*	$\mu_{\mathbf{x}}^*$	NFE		β_{MCS}^t	Iter
			f_{FE}	g_{FE}	g_i	
DLM-PMA	27.5435	$(2.1322, 2.3378, 8.7106, 5.1026, 0.9238, 1.4449, 1.3847, 9.8185, 8.1501, 8.4799$	249	896856	$2.5116, 3.0258, 3.0162,$ $3.0091, 2.9576,$inf, 3.0045, inf	2
SORA	27.5435	$(2.1322, 2.3378, 8.7106, 5.1026, 0.9238, 1.4449, 1.3847, 9.8186, 8.1501, 8.4799)$	487	16959	$2.5116, 3.0258, 3.0162,$ $3.0091, 2.9576$, inf, 3.0045, inf	3
SLM-MH	27.7465	$(2.1350, 2.3309, 8.7094, 5.1021, 0.9225, 1.4452, 1.3885, 9.8094, 8.1556, 8.4755)$	555	8653	$2.9957, 3.0332, 3.0013,,$ $3.0307, 3.0185$, inf, 3.0407, inf	2

$$
\begin{aligned}
\min: \quad & 0.7854x_1x_2^2(3.3333x_3^2 + 14.9334x_3 - 43.0934) - \\
& 1.508x_1(x_6^2 + x_7^2) + 7.477(x_6^3 + x_7^3) \\
& + 0.7854(x_4x_6^2 + x_5x_7^2) \\
s.t.: \quad & Pr\left[g_i(\mathbf{X}) > 0\right] \leq \phi(-\beta_i^t), \\
& g_1(\mathbf{X}) = \frac{27}{x_1x_2^2x_3} - 1 > 0;\ g_2(\mathbf{X}) = \frac{397.5}{x_1x_2^2x_3^2} - 1 > 0, \\
& g_3(\mathbf{X}) = \frac{1.93x_4^3}{x_2x_3x_6^4} - 1 > 0;\ g_4(\mathbf{X}) = \frac{1.93x_5^3}{x_2x_3x_7^4} - 1 > 0, \\
& g_5(\mathbf{X}) = \frac{\sqrt{(\frac{745x_4}{x_2x_3})^2 + 16.9 \times 10^6}}{0.1x_6^3} - 1100 > 0, \\
& g_6(\mathbf{X}) = \frac{\sqrt{(\frac{745x_5}{x_2x_3})^2 + 157.5 \times 10^6}}{0.1x_7^3} - 850 > 0, \\
& g_7(\mathbf{X}) = x_2x_3 - 40 > 0;\ g_8(\mathbf{X}) = 5 - \frac{x_1}{x_2} > 0, \\
& g_9(\mathbf{X}) = \frac{x_1}{x_2} - 12 > 0;\ g_{10}(\mathbf{X}) = \frac{1.5x_6 + 1.9}{x_4} - 1 > 0, \\
& g_{11}(\mathbf{X}) = \frac{1.1x_7 + 1.9}{x_5} - 1 > 0, \\
& 2.6 \leq x_1 \leq 3.6,\ 0.7 \leq x_2 \leq 0.8,\ 17 \leq x_3 \leq 28, \\
& 7.3 \leq x_4 \leq 8.3,\ 7.3 \leq x_5 \leq 8.3,\ 2.9 \leq x_6 \leq 3.9, \\
& 5 \leq x_7 \leq 5.5, \\
& x_j \sim N(\mu_{x_j}, 0.005^2) \text{ for } j = 1, 2, \ldots, 7, \\
& \beta_i^t = 3.0,\ i = 1, 2, \ldots, 11, \\
& \mu_x^{(0)} = [3.5, 0.7, 17, 7.3, 7.72, 3.35, 5.29]^T.
\end{aligned}
\tag{20}
$$

The RBDO results are shown in table 3. It can be concluded that all RBDO methods converge to the same optimal solution, 3038.612. However, the function evaluations required by SLM-MH are lesser than other methods. Also, SLM-MH converges to the optima with only two iterations.

5.4 *Welded beam problem*

A welded beam problem[22] is taken as another RBDO example. There are four independent random variables with normal distribution and the objective function is to minimize the welding cost. Five probabilistic constraints related to shear stress, bending stress, buckling and displacement are used. The target reliability β^t is 3.0 for all the probabilistic constraints and the initial point is taken as the deterministic optima. The formulation of the problem is given in equation (21). The system parameters are given in the table 4.

From table 5, it can be seen that all methods generate the optimal solution and achieve the target reliability for all constraints. SORA seems to be slightly better than SLM-MH in acquiring lesser number of function evaluations . However, the range of function evaluations is same.

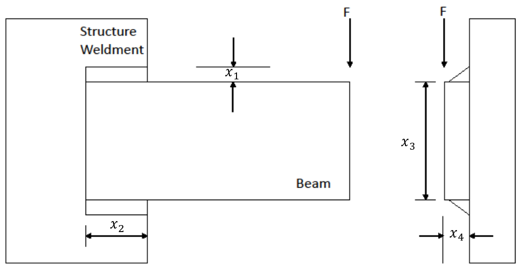

Figure 5: A welded beam

Table 3: RBDO results for speed reducer problem with $\beta_t = 3.0$

Methods	f^*	$\mu_{\mathbf{x}}^*$	NFE		β_{MCS}^t	Iter
			f_{FE}	g_{FE}	g_i	
DLM-PMA	3038.612	$(3.5765, 0.7000, 17.0000, 7.3000, 7.7541, 3.3652, 5.3017)$	56	229680	Inf,Inf,Inf,Inf,3.0307,3.3082, Inf,3.0000,Inf,Inf,3.0068	2
SORA	3038.612	$(3.5765, 0.7000, 17.0000, 7.3000, 7.7541, 3.3652, 5.3017)$	77	14874	Inf,Inf,Inf,Inf,3.0307,3.3082, Inf,3.0000,Inf,Inf,3.0068	3
SLM-MH	3038.612	$(3.5764, 0.7000, 17.0000, 7.3000, 7.7541, 3.3652, 5.3017)$	213	5439	Inf,Inf,Inf,Inf,3.0307,3.3082, Inf,3.0000,Inf,Inf,3.0068	2

$$
\begin{aligned}
\min: \quad & c_1 x_1^2 x_2 + c_2 x_3 x_4 (z_2 + x_2) \\
s.t.: \quad & Pr\left[g_1(\mathbf{X}) = \frac{\tau(\mathbf{X},\mathbf{z})}{z_6} - 1 > 0\right] \le \phi(-\beta_1^t), \\
& Pr\left[g_2(\mathbf{X}) = \frac{\sigma(\mathbf{X},\mathbf{z})}{z_7} - 1 > 0\right] \le \phi(-\beta_2^t), \\
& Pr\left[g_3(\mathbf{X}) = \frac{x_1}{x_4} - 1 > 0\right] \le \phi(-\beta_3^t), \\
& Pr\left[g_4(\mathbf{X}) = \frac{\delta(\mathbf{X},\mathbf{z})}{z_5} - 1 > 0\right] \le \phi(-\beta_4^t), \\
& Pr\left[g_5(\mathbf{X}) = 1 - \frac{P_c(\mathbf{X},\mathbf{z})}{z_1} > 0\right] \le \phi(-\beta_5^t), \\
& 3.175 \le x_1 \le 50.8,\ \ 0 \le x_2 \le 254,\ \ 0 \le x_3 \le 254, \\
& 0 \le x_4 \le 50.8, \\
& x_{1,2} \sim N(\mu_{x_{1,2}}, 0.1693^2),\ \ x_{3,4} \sim N(\mu_{x_{3,4}}, 0.0107^2,) \\
& \beta_i^t = 3.0,\ \ i = 1, 2, \ldots, 5, \\
& \mu_x^{(0)} = [6.208, 157.82, 210.62, 6.208]^T, \\
& t(\mathbf{X},\mathbf{z}) = \frac{z_1}{\sqrt{2} x_1 x_2},\ \ tt(\mathbf{X},\mathbf{z}) = M(\mathbf{X},\mathbf{z}) \frac{R(\mathbf{X},\mathbf{z})}{J(\mathbf{X},\mathbf{z})}, \\
& M(\mathbf{X},\mathbf{z}) = z_1 \left(z_2 + \frac{x_2}{2}\right), \\
& R(\mathbf{X},\mathbf{z}) = \frac{\sqrt{x_2^2 + (x_1 + x_3)^2}}{2}, \\
& J(\mathbf{X},\mathbf{z}) = \sqrt{2} x_1 x_2 \left\{\frac{x_2^2}{12} + \frac{(x_1 + x_3)^2}{4}\right\},
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
& \sigma(\mathbf{X},\mathbf{z}) = \frac{6 z_1 z_2}{x_3^2 x_4},\ \ \delta(\mathbf{X},\mathbf{z}) = \frac{4 z_1 z_2^3}{z_3 x_3^3 x_4}, \\
& P_c(\mathbf{X},\mathbf{z}) = \frac{4.013 x_3 x_4^3 \sqrt{z_3 z_4}}{6 z_2^2} \left(1 - \frac{x_3}{4 z_2} \sqrt{\frac{z_3}{z_4}}\right), \\
& \tau(\mathbf{X},\mathbf{z}) = \left\{ t(\mathbf{X},\mathbf{z})^2 + 2 t(\mathbf{X},\mathbf{z}) tt(\mathbf{X},\mathbf{z}) \left(\frac{x_2}{2R(\mathbf{X},\mathbf{z})}\right) \right. \\
& \left. + tt(\mathbf{X},\mathbf{z})^2 \right\}^{1/2}.
\end{aligned}
$$

Table 4: Fixed parameters for the welded beam problem

z_1:	2.6688×10^4 (N)
z_2:	3.556×10^2 (mm)
z_3:	2.0685×10^5 (MPa)
z_4:	8.274×10^4 (MPa)
z_5:	6.35 (mm)
z_6:	9.377×10 (MPa)
z_7:	2.0685×10^2 (MPa)
c_1:	6.74135×10^{-5} ($/mm^3)
c_2:	2.93585×10^{-6} ($/mm^3)

6 Conclusion

In this paper, SLM-MH was proposed which was developed on the single-loop method. The primary purpose of this method was to calculate approximate MPP using the direction which was designed on iteratively modifying the Hessian. The method was tested on two mathematical problems and two engineering RBDO problems. Results showed that SLM-MH was able to generate the optimal solution with the desired target reliability on three problems. Although, SLM-MH was unable to achieve the target reliability in one problem, it emerged as the most accurate among DLM-PMA and SORA. Moreever, SLM-MH requires lesser function evaluations to generate the optimal solutions. As a future work, this method can be tested on other RBDO problems.

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Table 5: RBDO results for welded beam problem with $\beta_t = 3.0$

Methods	f^*	$\mu_{\mathbf{x}}^*$	NFE		β_{MCS}^t	Iter
			f_{FE}	g_{FE}	g_i	
DLM-PMA	2.5913	(5.7300, 200.8982, 210.5977, 6.2389)	115	57750	3.0233, 3.0233 3.0091, Inf, 3.0091	2
SORA	2.5913	(5.7300, 200.8982, 210.5977, 6.2389)	160	2155	3.0233, 3.0233 3.0091, Inf, 3.0091	3
SLM-MH	2.5913	(5.7300, 200.8982, 210.5977, 6.2389)	439	2657	3.0233, 3.0233 3.0091, Inf, 3.0091	3

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