Section 7.3 Formal Proofs in Predicate Calculus

All proof rules for propositional calculus extend to predicate calculus.

Example. …

k.	$\forall x\, p(x)$	P
k+1.	$\forall x\, p(x) \to \exists x\, p(x)$	P
k+2.	$\exists x\, p(x)$	1, 2, MP

…

But we need additional proof rules to reason with most quantified wffs. For example, suppose we want to prove that the following wff is valid.

$$\exists x\, \forall y\, p(x, y) \to \forall y\, \exists x\, p(x, y).$$

We might start with

Proof: 1. $\exists x\, \forall y\, p(x, y)$ $\qquad$ P

But what do we do for the next line of the proof? We're stuck if we want to use proof rules. We need more proof rules.

Free to Replace

For a wff $W(x)$ and a term t we say t is *free to replace* x in $W(x)$ if $W(t)$ has the same bound occurrences of variables as $W(x)$.

Example. Let $W(x) = \exists y\, p(x, y)$. Then

$W(y) = \exists y\, p(y, y)$, so y is not free to replace x in $W(x)$.
$W(f(x)) = \exists y\, p(f(x), y)$, so $f(x)$ is free to replace x in $W(x)$.
$W(c) = \exists y\, p(c, y)$, so c is free to replace x in $W(x)$.
$W(x) = \exists y\, p(x, y)$, so x is free to replace x in $W(x)$.

Universal Instantiation (UI)

$$\frac{\forall x W(x)}{W(t)}$$ if t is free to replace x in $W(x)$.

Special cases that satisfy the restriction: $\frac{\forall x W(x)}{W(x)}$ $\frac{\forall x W(x)}{W(c)}$

Existential Generalization (EG)

$$\frac{W(t)}{\exists x W(x)}$$ if t is free to replace x in $W(x)$.

Special cases that satisfy the restriction: $\frac{W(x)}{\exists x W(x)}$ $\frac{W(c)}{\exists x W(x)}$

Existential Instantiation (EI)

If $\exists x\ W(x)$ occurs on some line of a proof, then $W(c)$ may be placed on any subsequent line of the proof subject to the following restrictions:

Choose c to be a new constant in the proof and such that c does not occur in the statement to be proven.

Universal Generalization (UG)

If $W(x)$ occurs on some line of a proof, then $\forall x\ W(x)$ may be placed on any subsequent line of the proof subject to the following restrictions:

Among the wffs used to obtain $W(x)$, x is not free in any premise and x is not free in any wff obtained by EI.

Restrictions on quantifier inference rules are necessary

Example. $\forall x\, \exists y\, p(x, y) \to \exists y\, \forall x\, p(x, y)$ is invalid. Here is an *attempted* proof.

1.	$\forall x\, \exists y\, p(x, y)$	P
2.	$\exists y\, p(x, y)$	1, UI
3.	$p(x, c)$	2, EI
4.	$\forall x\, p(x, c)$	3, UG (NO, x on line 3 is free in wff obtained by EI)
5.	$\exists y\, \forall x\, p(x, y)$	4, EG
	NOT QED	1–5, CP

Example. $\exists x\, p(x) \to \forall x\, p(x)$ is invalid. Here is an *attempted* proof.

1.	$\exists x\, p(x)$	P
2.	$p(x)$	1, EI (NO, x is not a new constant in the proof)
3.	$\forall x\, p(x)$	2, UG (NO, x on line 2 is free in wff obtained by EI)
	NOT QED	1–3, CP

Example. $\exists x\, p(x) \wedge \exists x\, q(x) \to \exists x\, (p(x) \wedge q(x))$ is invalid. Here is an *attempted* proof.

1.	$\exists x\, p(x)$	P
2.	$\exists x\, q(x)$	P
3.	$p(c)$	1, EI
4.	$q(c)$	2, EI (NO, c is not a new constant in the proof)
5.	$p(c) \wedge q(c)$	3, 4, Conj
6.	$\exists x\, (p(x) \wedge q(x))$	5, EG
	NOT QED	1–6, CP

Example. $p(x) \to \forall x\, p(x)$ is invalid. Here is an *attempted* proof.

1.	$p(x)$	P
2.	$\forall x\, p(x)$	1, UG (NO, x is free in a premise)
	NOT QED	1, 2, CP

Example. $\forall x\, \exists y\, p(x, y) \to \exists y\, p(y, y)$ is invalid. Here is an *attempted* proof.

1.	$\forall x\, \exists y\, p(x, y)$	P
2.	$\exists y\, p(y, y)$	1, UI (NO, y is not free to replace x in $\exists y\, p(x, y)$)
	NOT QED	1, 2, CP

Example. $\forall x\, p(x, f(x)) \to \exists x\, p(x, x)$ is invalid. Here is an *attempted* proof.

1.	$\forall x\, p(x, f(x))$	P
2.	$p(x, f(x))$	1, UI
3.	$\exists x\, p(x, x)$	2, EG (NO, $p(x, f(x)) \neq p(x, x)(x/t)$ for any term t)
	NOT QED	1–3, CP

Example. $\forall x\, p(x, f(x)) \to \exists y\, \forall x\, p(x, y)$ is invalid. Here is an *attempted* proof.

1.	$\forall x\, p(x, f(x))$	P
2.	$\exists y\, \forall x\, p(x, y)$	1, EG (NO, $f(x)$ is not free to replace y in $\forall x\, p(x, y)$)
	NOT QED	1, 2, CP

Example. $\exists x\, p(x) \to p(c)$ is invalid. Here is an *attempted* proof.

1.	$\exists x\, p(x)$	P
2.	$p(c)$	1, EI (NO, c occurs in statement to be proved)
	NOT QED	1, 2, CP

Now Some Valid Wffs

Example. $\forall x\, \forall y\, p(x, y) \to \forall y\, p(y, y)$ is valid. Here is an *attempted* proof.

1.	$\forall x\, \forall y\, p(x, y)$	P
2.	$\forall y\, p(y, y)$	1, UI (NO, y is not free to replace x in $\forall y\, p(x, y)$)
	NOT QED	1, 2, CP

But here is a *correct* proof.

1.	$\forall x\, \forall y\, p(x, y)$	P
2.	$\forall y\, p(x, y)$	1, UI
3.	$p(x, x)$	2, UI
4.	$\forall x\, p(x, x)$	3, UG
5.	$p(y, y)$	4, UI
6.	$\forall y\, p(y, y)$	5, UG
	QED	1–6, CP.

Quiz. Find a proof of the statement that uses IP.

Example/Quiz. $\forall x\, (A(x) \to B(x)) \to (\forall x\, A(x) \to \forall x\, B(x))$ is valid. Find a proof.

1.	$\forall x\, (A(x) \to B(x))$	P
2.	$\quad\quad \forall x\, A(x)$	P [for $\forall x\, A(x) \to \forall x\, B(x)$]
3.	$\quad\quad A(x)$	2, UI
4.	$\quad\quad A(x) \to B(x)$	1, UI
5.	$\quad\quad B(x)$	3, 4, MP
6.	$\quad\quad \forall x\, B(x)$	5, UG
7.	$\forall x\, A(x) \to \forall x\, B(x)$	2–6, CP
	QED	1, 7, CP.

Example/Quiz. Prove that the following wff is valid using IP.

$$\forall x\, \neg p(x, x) \wedge \forall x\, \forall y\, \forall z\; (p(x, y) \wedge p(y, z) \rightarrow p(x, z)) \rightarrow \forall x\, \forall y\, \neg (p(x, y) \wedge p(y, x)).$$

1.	$\forall x\, \neg p(x, x)$	P
2.	$\forall x\, \forall y\, \forall z\; (p(x, y) \wedge p(y, z) \rightarrow p(x, z))$	P
3.	$\exists x\, \exists y\, (p(x, y) \wedge p(y, x))$	P [for $\forall x\, \forall y\, \neg (p(x, y) \wedge p(y, x)$], T
4.	$p(a, b) \wedge p(b, a)$	3, EI, EI
5.	$p(a, b) \wedge p(b, a) \rightarrow p(a, a)$	2, UI, UI, UI
6.	$p(a, a)$	4, 5, MP
7.	$\neg p(a, a)$	1, UI
8.	False	6, 7, Contr
9.	$\forall x\, \forall y\, \neg (p(x, y) \wedge p(y, x)$	3–8, IP
	QED	1, 2, 9, CP.

Quiz. Find a CP proof of the statement.

1.	$\forall x\, \neg p(x, x)$	P
2.	$\forall x\, \forall y\, \forall z\; (p(x, y) \wedge p(y, z) \rightarrow p(x, z))$	P
3.	$\neg p(x, x)$	1, UI
4.	$p(x, y) \wedge p(y, x) \rightarrow p(x, x)$	2, UI, UI, UI
5.	$\neg (p(x, y) \wedge p(y, x))$	3, 4, MT
6.	$\forall x\, \forall y\, \neg (p(x, y) \wedge p(y, x))$	5, UG, UG
	QED	1– 6, CP.

Example/Quiz. Use IP to prove that the $\forall x\, \exists y\, (p(x) \rightarrow p(y))$ is valid.

1.	$\exists x\, \forall y\, (p(x) \wedge \neg p(y))$	P [for IP]
2.	$\forall y\, (p(c) \wedge \neg p(y))$	1, EI
3.	$p(c) \wedge \neg p(c)$	2, UI
4.	$p(c)$	3, Simp
5.	$\neg p(c)$	4, Simp
6.	False	4, 5, Contr
	QED	1–6, IP.

Group Quiz. Divide the class into six subgroups and assign each group one of the following six wffs to prove using CP (no IP and no T's). Assume that x does not occur free in C.

1. $\forall x\,(A(x) \to C) \to (\exists x\,A(x) \to C)$.
2. $(\exists x\,A(x) \to C) \to \forall x\,(A(x) \to C)$.
3. $(C \to \forall x\,A(x)) \to \forall x\,(C \to A(x))$.
4. $(C \to \exists x\,A(x)) \to \exists x\,(C \to A(x))$.
5. $\exists x\,(C \to A(x)) \to (C \to \exists x\,A(x))$.
6. $\exists x\,(A(x) \to C) \to (\forall x\,A(x) \to C)$.

Solutions.

1. $\forall x\,(A(x) \to C) \to (\exists x\,A(x) \to C)$.

1.	$\forall x\,(A(x) \to C)$	P
2.	$\quad\exists x\,A(x)$	P[for $\exists x\,A(x) \to C$]
3.	$\quad A(d)$	2, EI
4.	$\quad A(d) \to C$	1, UI
5.	$\quad C$	3, 4, MP
6.	$\exists x\,A(x) \to C$	2–5, CP
	QED	1, 6, CP.

2. $(\exists x\,A(x) \to C) \to \forall x\,(A(x) \to C)$.

1.	$\exists x\,A(x) \to C$	P
2.	$\quad A(x)$	P [for $A(x) \to C$]
3.	$\quad\exists x\,A(x)$	2, EG
4.	$\quad C$	1, 3, MP
5.	$A(x) \to C$	2–4, CP
6.	$\forall x\,(A(x) \to C)$	5, UG
	QED	1, 5–6, CP.

3. $(C \to \forall x\, A(x)) \to \forall x\, (C \to A(x))$.

1.	$C \to \forall x\, A(x)$	P
2.	$\quad C$	P [for $C \to A(x)$]
3.	$\quad \forall x\, A(x)$	1, 2, MP
4.	$\quad A(x)$	3, UI
5.	$C \to A(x)$	2–4, CP
6.	$\forall x\, (C \to A(x))$	5, UG
	QED	1, 5–6, CP.

4. $(C \to \exists x\, A(x)) \to \exists x\, (C \to A(x))$.

1.	$C \to \exists x\, A(x)$	P
2.	$\quad C$	P [for $C \to A(?)$]
3.	$\quad \exists x\, A(x)$	1, 2, MP
4.	$\quad A(d)$	3, EI
5.	$C \to A(d)$	2–4, CP
6.	$\exists x\, (C \to A(x))$	5, EG
	QED	1, 5–6, CP.

5. $\exists x\, (C \to A(x)) \to (C \to \exists x\, A(x))$.

1.	$\exists x\, (C \to A(x))$	P
2.	$\quad C$	P [for $C \to \exists x\, A(x)$]
3.	$\quad C \to A(d)$	1, EI
4.	$\quad A(d)$	2, 3, MP
5.	$\quad \exists x\, A(x)$	4, EG
6.	$C \to \exists x\, A(x)$	2–5, CP
	QED	1, 6, CP.

6. $\exists x\, (A(x) \to C) \to (\forall x\, A(x) \to C)$.

1.	$\exists x\, (A(x) \to C)$	P
2.	$\quad \forall x\, A(x)$	P [for $\forall x\, A(x) \to C$]
3.	$\quad A(d) \to C$	1, EI
4.	$\quad A(d)$	2, UI
5.	$\quad C$	3, 4, MP
6.	$\forall x\, A(x) \to C$	2–5, CP
	QED	1, 6, CP.