Section 7.2 Equivalent Formulas

Two wffs A and B are equivalent, written $A \equiv B$, if they have the same truth value for every interpretation.

Property: $A \equiv B$ iff $A \to B$ and $B \to A$ are both valid.

Proof: $A \equiv B$ iff A and B have the same truth value for any interpretation iff $A \to B$ and $B \to A$ are true for any interpretation iff $A \to B$ and $B \to A$ are both valid. QED.

Propositional Equivalence Gives Rise to First-Order Equivalence

In other words, if two propositional wffs are equivalent and each occurrence of a propositional variable is replaced by a first order wff, then the resulting two first order wffs, called instances, are equivalent.

Example. We have $\forall x\, p(x) \to \exists x\, p(x) \equiv \neg \forall x\, p(x) \vee \exists x\, p(x)$ because $A \to B \equiv \neg A \vee B$. So the first-order equivalence is an instance of the propositional equivalence by letting $A = \forall x\, p(x)$ and $B = \exists x\, p(x)$.

Basic Equivalences

(1) $\neg \forall x\, W \equiv \exists x \neg W$ *and* $\neg \exists x\, W \equiv \forall x \neg W$.

Proof (second equivalence). Let I be an arbitrary interpretation with domain D. Then

$\neg \exists x\, W$ is true for I

iff $\exists x\, W$ is false for I

iff $W(x/d)$ is false for I for all $d \in D$

iff $\neg W(x/d)$ is true for I for all $d \in D$

iff $\forall x \neg W$ is true for I.

Since I was arbitrary, the wffs are equivalent. QED.

Example. $\forall x\, p(x) \rightarrow \exists x\, p(x) \equiv \neg \forall x\, p(x) \vee \exists x\, p(x)$ (instance of propositional wff)

$\equiv \exists x \neg p(x) \vee \exists x\, p(x)$ (basic equivalence).

(2) $\forall x\, \forall y\, W \equiv \forall y\, \forall x\, W$ *and* $\exists x\, \exists y\, W \equiv \exists y\, \exists x\, W$.

Proof (first equivalence): Let I be an arbitrary interpretation with domain D. Then

$\forall x\, \forall y\, W$ is true for I

iff $\forall y\, W(x/d)$ is true for I for all $d \in D$

iff $W(x/d)\,(y/e)$ is true for I for all $d, e \in D$

iff $W(y/e)\,(x/d)$ is true for I for all $d, e \in D$

iff $\forall x\, W(y/e)$ is true for I for all $e \in D$

iff $\forall y\, \forall x\, W$ is true for I.

Since I was arbitrary, the wffs are equivalent. QED.

(3) $\exists x\, (A(x) \rightarrow B(x)) \equiv \forall x\, A(x) \rightarrow \exists x\, B(x)$.

Proof: Let I be an arbitrary interpretation with domain D. Assume LHS is true for I. Then $A(c) \rightarrow B(c)$ is true for I for some $c \in D$. Consider the possible values of $A(c)$. If $A(c)$ is true for I, then $B(c)$ is true for I. So $\exists x\, B(x)$ is true for I, which implies RHS is true for I. If $A(c)$ is false for I, then $\forall x\, A(x)$ is false for I, which implies RHS is true for I. So LHS $\rightarrow$ RHS is valid.

Class, prove the converse, RHS $\rightarrow$ LHS. QED.

(4) $\exists x\, (A(x) \vee B(x)) \equiv \exists x\, A(x) \vee \exists x\, B(x)$.

Proof: $\exists x\, (A(x) \vee B(x)) \equiv \exists x\, (\neg A(x) \rightarrow B(x))$ (instance of propositional wff)

$\equiv \forall x \neg A(x) \rightarrow \exists x\, B(x)$ (3)

$\equiv \neg \exists x\, A(x) \rightarrow \exists x\, B(x)$ (1)

$\equiv \exists x\, A(x) \vee \exists x\, B(x)$ (instance of propositional wff). QED.

(5) $\forall x\,(A(x) \wedge B(x)) \equiv \forall x\, A(x) \wedge \forall x\, B(x)$.

Proof: Let I be an arbitrary interpretation with domain D. Then

$\forall x\,(A(x) \wedge B(x))$ is true for I

iff $A(d) \wedge B(d)$ is true for I for all $d \in D$

iff $A(d)$ is true for I for all $d \in D$ and $B(d)$ is true for I for all $d \in D$

iff $\forall x\, A(x)$ is true for I and $\forall x\, B(x)$ is true for I

iff $\forall x\, A(x) \wedge \forall x\, B(x)$ is true for I.

Since I was arbitrary, the wffs are equivalent. QED.

Restricted Equivalences

(6) (Renaming Variables) If y does not occur in $W(x)$, then

$$\exists x\, W(x) \equiv \exists y\, W(y) \;\; \textit{and} \;\; \forall x\, W(x) \equiv \forall y\, W(y).$$

Example. $\forall x\,(\exists y\, p(x, y) \wedge \exists x\, q(x, y)) \equiv \forall z\,(\exists y\, p(z, y) \wedge \exists x\, q(x, y))$.

(7) If x does not occur free in C, then

(a) $\forall x\, C \equiv C$ *and* $\exists x\, C \equiv C$.

(b) $\forall x\,(C \vee A(x)) \equiv C \vee \forall x\, A(x)$ *and* $\exists x\,(C \vee A(x)) \equiv C \vee \exists x\, A(x)$.

(c) $\forall x\,(C \wedge A(x)) \equiv C \wedge \forall x\, A(x)$ *and* $\exists x\,(C \wedge A(x)) \equiv C \wedge \exists x\, A(x)$.

(d) $\forall x\,(C \to A(x)) \equiv C \to \forall x\, A(x)$ *and* $\exists x\,(C \to A(x)) \equiv C \to \exists x\, A(x)$.

(e) Be careful with these:

$$\forall x\,(A(x) \to C) \equiv \exists x\, A(x) \to C.$$

$$\exists x\,(A(x) \to C) \equiv \forall x\, A(x) \to C.$$

Proof: There are various ways to prove the equivalences. For example, prove (a) and (b) with models and then use known equivalences to prove (c), (d), and (e). QED.

Example. We know that $\exists x\, \forall y\, p(x, y) \to \forall y\, \exists x\, p(x, y)$ is valid, but the converse is not valid. So we can't interchange $\exists x$ and $\forall y$. But for predicates that take single arguments, the two quantifiers can be interchanged. For example, we have the following equivalence:

$$\exists x\, \forall y\, (p(x) \to q(y)) \equiv \forall y\, \exists x\, (p(x) \to q(y)).$$

Proof:
$$\begin{aligned} \exists x\, \forall y\, (p(x) \to q(y)) &\equiv \exists x\, (p(x) \to \forall y\, q(y)) && \text{(7d)} \\ &\equiv \forall x\, p(x) \to \forall y\, q(y) && \text{(7e)} \\ &\equiv \forall y\, (\forall x\, p(x) \to q(y)) && \text{(7d)} \\ &\equiv \forall y\, \exists x\, (p(x) \to q(y)) && \text{(7e)} \quad \text{QED.} \end{aligned}$$

Normal Forms

A wff in *prenex normal form* has all quantifiers at the left end. *e.g.*, $\exists x\, \forall y\, (p(x) \to q(y))$.

Prenex Normal Form Algorithm

(a) Rename variables to get distinct quantifier names and distinct free variable names.

(b) Move quantifiers left using (1), (7b), (7c), (7d), and (7e).

Example. $q(x) \land \exists x\, (r(x) \to \neg\, \exists y\, p(x, y))$
$$\begin{aligned} &\equiv q(x) \land \exists z\, (r(z) \to \neg\, \exists y\, p(z, y)) && \text{(rename)} \\ &\equiv \exists z\, (q(x) \land (r(z) \to \neg\, \exists y\, p(z, y))) && \text{(7b)} \\ &\equiv \exists z\, (q(x) \land (r(z) \to \forall y\, \neg\, p(z, y))) && \text{(1)} \\ &\equiv \exists z\, (q(x) \land \forall y\, (r(z) \to \neg\, p(z, y))) && \text{(7d)} \\ &\equiv \exists z\, \forall y\, (q(x) \land (r(z) \to \neg\, p(z, y))) && \text{(7b).} \end{aligned}$$

Prenex CNF/DNF

A wff in prenex normal form is in *prenex CNF (or prenex DNF)* if the wff to the right of the quantifiers is in CNF (or DNF), where a literal is now either an atom or its negation.

Example. $p(x)$ and $\neg p(x)$ are literals.

Example. $\exists z\ \forall y\ (q(x) \wedge (\neg r(z) \vee \neg p(z, y)))$ is a prenex CNF.

Example. $\exists z\ \forall y\ ((q(x) \wedge \neg r(z)) \vee (q(x) \wedge \neg p(z, y)))$ is a prenex DNF.

Prenex CNF/DNF Algorithm

(a) Put wff in prenex normal form.
(b) Remove $\to$.
(c) Move $\neg$ to the right to form literals.
(d) Distribute $\vee$ over $\wedge$ and/or $\wedge$ over $\vee$ for desired form.

Example. $\forall x\ \forall y\ \exists z\ (q(x) \vee r(z, x) \to p(z, y))$ (prenex normal form)

$\equiv \forall x\ \forall y\ \exists z\ (\neg (q(x) \vee r(z, x)) \vee p(z, y))$ (remove $\to$)

$\equiv \forall x\ \forall y\ \exists z\ ((\neg q(x) \wedge \neg r(z, x)) \vee p(z, y))$ (move $\neg$ right) (prenex DNF)

$\equiv \forall x\ \forall y\ \exists z\ ((\neg q(x) \vee p(z, y)) \wedge (\neg r(z, x)) \vee p(z, y)))$ (distribute) (prenex CNF).

Formalizing English Sentences. Some rules that usually work for English sentences are:

- $\forall x$ quantifies a conditional.
- $\exists x$ quantifies a conjunction.
- Use $\forall x$ with conditional for "all," "every," and "only."
- Use $\exists x$ with conjunction for "some," "there is," and "not all."
- Use $\forall x$ with conditional or $\neg \exists x$ with conjunction for "no *A* is *B*."
- Use $\exists x$ with conjunction or $\neg \forall x$ with conditional for "not all *A's* are *B*."

Example/Quiz. For a person x let $c(x)$ mean x is a criminal and let $s(x)$ mean x is sane. Then formalize each of the following sentences.

1. All criminals are sane.

Solution: $\forall x\ (c(\mathrm{x}) \to s(x))$.

2. Every criminal is sane.

Solution: $\forall x\ (c(\mathrm{x}) \to s(x))$.

3. Only criminals are sane.

Solution: $\forall x\ (s(\mathrm{x}) \to c(x))$.

4. Some criminals are sane.

Solution: $\exists x\ (c(x) \wedge s(x))$.

5. There is a criminal that is sane.

Solution: $\exists x\ (c(x) \wedge s(x))$.

6. No criminal is sane.

Solution: $\forall x\ (c(\mathrm{x}) \to \neg\, s(x)) \equiv \neg\, \exists x\ (c(x) \wedge s(x))$.

7. Not all criminals are sane.

Solution: $\exists x\ (c(x) \wedge \neg\, s(x)) \equiv \neg\, \forall x\ (c(x) \to s(x))$.