Section 7.1 First-Order Predicate Calculus

Predicate calculus studies the internal structure of sentences where subjects are applied to predicates existentially or universally.

A **predicate** describes a property of the subject or subjects of a sentence. If p is a predicate that describes a property of x, then we write $p(x)$.

Example. If p is the "is prime" predicate and x is prime, then we write $p(x)$ for "x is prime".

Example. If q is the "is a parent of" predicate, and x is a parent of y, then we write $q(x, y)$ for "x is a parent of y."

Existential Quantifier. The phrase, "there exists an x such that $p(x)$" is denoted by $\exists x\ p(x)$. The symbol $\exists x$ is an existential quantifier and it indicates disjunction.

Example. If $x \in \{1, 2, 3\}$, then $\exists x\ p(x) = p(1) \vee p(2) \vee p(3)$.

Quiz (1 minute). $p(y, a) \vee p(y, b) \vee p(y, c) = ?$

Answer: $\exists x\ p(y, x)$, where $x \in \{a, b, c\}$.

Universal Quantifier. The phrase, "for every x, $p(x)$" is denoted by $\forall x\ p(x)$. The symbol $\forall x$ is a universal quantifier and it indicates conjunction.

Example. If $x \in \{1, 2, 3\}$, then $\forall x\ p(x) = p(1) \wedge p(2) \wedge p(3)$.

Quiz (1 minute). $p(a, x) \wedge p(b, x) \wedge p(c, x) = ?$
Answer: $\forall y\ p(y, x)$ where $y \in \{a, b, c\}$.

Quiz (2 minutes). If $x, y \in \{0, 1\}$, then $\exists x\ \forall y\ p(x, y, z) = ?$
Answer 1: $\forall y\ p(0, y, z) \vee \forall y\ p(1, y, z) = (p(0, 0, z) \wedge p(0, 1, z)) \vee (p(1, 0, z) \wedge p(1, 1, z))$.
Answer 2: $\exists x\ (p(x, 0, z) \wedge p(x, 1, z)) = (p(0, 0, z) \wedge p(0, 1, z)) \vee (p(1, 0, z) \wedge p(1, 1, z))$.

Syntax of wffs in first-order predicate calculus

Terms are nonlogical things: *constants* $a, b, c, \ldots$, *variables* $x, y, z, \ldots$. and function symbols applied to terms $f(a), g(x, f(y)), h(c, z), \ldots$.

Atoms are predicate symbols applied to terms $p(x), q(a, f(x)), \ldots$.

Wffs are either atoms or if U and V are wffs then the following expressions are also wffs:

$\neg U, U \wedge V, U \vee V, U \to V, \exists x\, U, \forall x\, U,$ and (U).

Hierarchy in the absence of parentheses

$\neg, \exists x, \forall x$ (highest, group rightmost operator with smallest wff to its right)

$\wedge$

$\vee$

$\to$ (lowest, and it is left associative)

Example. $\forall x \neg \exists y\, p(x, y) \to \forall x\, q(x) = (\forall x\, (\neg\, (\exists y\, p(x, y)))) \to (\forall x\, q(x))$.

Scope, Bound, and Free

The *scope* of $\exists x$ in $\exists x\, W$ is W. The *scope* of $\forall x$ in $\forall x\, W$ is W. An occurrence of x is *bound* if it occurs in either $\exists x$ or $\forall x$ or in their scope. Otherwise the occurrence of x is *free*.

Example.

$\exists x$	$p(x)$	$\to$	$\forall y$	$q(x,$	$y)$
⋮	⋮		⋮	⋮	⋮
B	B		B	F	B

Interpretations

An *interpretation* for a first-order wff consists of a nonempty set D, called the *domain*, together with an assignment of the symbols of the wff as follows:

1. Predicate letters are assigned to relations over D.
2. Function letters are assigned to functions over D.
3. Constants are assigned to elements of D.
4. Free occurrences of variables are assigned to elements of D.

Notation: If x is free in W and $d \in D$, then $W(x/d)$ denotes the wff obtained from W by replacing all free occurrences of x by d. We also write $W(d) = W(x/d)$.

Example. Let $W = \forall y\ (p(x, y) \to q(x))$. Then $W(x/d) = \forall y\ (p(d, y) \to q(d))$.

Truth Value of a Wff

The truth value of a wff with respect to an interpretation with domain D is obtained by recursively applying the following rules:

1. An atom has the truth value of the proposition obtained from its interpretation.
2. Truth values for $\neg U$, $U \wedge V$, $U \vee V$, $U \to V$ are obtained by applying truth tables for $\neg, \wedge, \vee, \to$ to the truth values for U and V.
3. $\forall x\ W$ is true iff $W(x/d)$ is true for all $d \in D$.
4. $\exists x\ W$ is true iff $W(x/d)$ is true for some $d \in D$.

If a wff is true with respect to an interpretation I, we say the wff *is true for I*.

Example. Let $W = p(x)$. We can define an interpretation I by letting $D = \mathbf{N}$, $p(x)$ means x is odd, and $x = 4$. Then W is false for I because $W(x/4) = p(x)(x/4) = p(4) =$ "4 is odd," which is false. If we let J be the same as I except that we assign $x = 3$, then W is true for J because $W(x/3) = p(x)(x/3) = p(3) =$ "3 is odd," which is true.

Example. Let $W = \forall x\ (p(x) \to q(x, y))$. Here are two interpretations of W:

1. Let I be defined by $D = \{0, 1\}$, $p(0) = \text{True}$, $p(1) = \text{False}$, $q(0, 1) = \text{True}$, otherwise $q(x, y)$ is false, and $y = 1$. Then W is true for I because it becomes
$$\forall x\ (p(x) \to q(x, 1)) = (p(0) \to q(0, 1)) \wedge (p(1) \to q(1, 1))$$
$$= (\text{True} \to \text{True}) \wedge (\text{False} \to \text{False}) \equiv \text{True} \wedge \text{True} \equiv \text{True}.$$

2. If J has any domain D, p is true, q is false, and y any value of D, then W is false for J.

Example. Let $W = \exists x\ (p(x) \wedge q(x))$. Here are two interpretations of W:

1. Let I be defined by $D = \mathbf{N}$, $p(x)$ means x is prime, and $q(x)$ means x is odd. Then W is true for I because, for example, $p(3) \wedge q(3) = \text{True} \wedge \text{True} \equiv \text{True}$.

2. If J consists of $D = \mathbf{N}$, $p(x)$ means x is even, and $q(x)$ means x is odd, then W is false for I because, for example, $p(3) \wedge q(3) = \text{False} \wedge \text{True} \equiv \text{False}$.

Example. Let $W = \forall x\ (g(x, c) \to \exists y\ (p(y) \wedge d(y, x)))$. Here are two interpretations of W:

1. Let I be defined by $D = \{a\}$, $p(a) = \text{False}$, $d(a, a) = \text{True}$, $g(a, a) = \text{True}$, and $c = a$. Then W is false for I because $W = g(a, a) \to p(a) \wedge d(a, a) \equiv \text{True} \to \text{False} \wedge \text{True} \equiv \text{False}$.

2. Let I be defined by $D = \mathbf{N}$, $g(x, c)$ means $x > c$, $p(x) =$ means x is prime, $d(x, y)$ means x divides y, and $c = 1$. This gives the sentence, "every natural number greater than 1 has a prime divisor," which is known to be true.

Example. Let $W = \forall x\ \forall y\ (\neg\ (p(x) \wedge p(y)) \to \exists z\ q(z, x, y))$. Let I be defined by $D = \mathbf{N}$, $p(x)$ means $x = 0$ and $q(z, x, y)$ means $z = \gcd(x, y)$. Then the meaning of W wrt I is

"Every pair of natural numbers that are not both zero has a greatest common divisor,"

which is known to be true.

Models and Countermodels

An interpretation that makes a wff true is called a *model*. An interpretation that makes a wff false is called a *countermodel*. See the previous examples.

Example. Let W be the following wff.

$$\forall x\, (r(x, a) \wedge \neg\, p(x) \to \exists y\, (r(x, y) \wedge r(y, a) \wedge d(y, x))).$$

1. Any interpretation for which r is always false is a model for W.
2. Any interpretation for which r is always true, p is always false, and d is always false is a countermodel for W.

Quiz (1 minute). For the preceding example, let I be the interpretation defined by $D = \mathbf{N}$, $r(x, y)$ means $x > y$, $p(x)$ means x is prime, $d(y, x)$ means y divides x, and $a = 1$. Is I a model or a countermodel for W?

Answer: We get the statement,

"every natural number $x > 1$ that is not a prime has a divisor y between 1 and x,"

which is known to be true. So I is a model of W.

Validity

A wff is *valid* if every interpretation is a model. Otherwise the wff is *invalid*. A wff is *unsatisfiable* if every interpretation is a countermodel. Otherwise the wff is *satisfiable*.

Quiz (1 minute). Every wff has exactly two of these four properties. What are the possible pairs of properties that a wff can have?

Answer: {valid, satisfiable}, {unsatisfiable, invalid}, and {satisfiable, invalid}

Proofs of Validity or Unsatisfiability

We can't check every interpretation (there are too many). So we need to reason informally with interpretations.

Example: $\forall x\,(p(x) \to p(x))$ is valid because $p(x) \to p(x)$ is true for all interpretations.

Example: $\forall x\,(p(x) \wedge \neg\, p(x))$ is unsatisfiable because $p(x) \wedge \neg\, p(x)$ is always false.

The previous two examples were quite simple. Here is a more realistic example.

Example. Prove the following wff is valid.

$$\exists x\,(A(x) \wedge B(x)) \to \exists x\, A(x) \wedge \exists x\, B(x).$$

Direct Proof: Let I be an interpretation with domain D for the wff and assume that the antecedent is true for I. Then $A(d) \wedge B(d)$ is true for I for some $d \in D$. So both $A(d)$ and $B(d)$ are true for I. Therefore both $\exists x\, A(x)$ and $\exists x\, B(x)$ are true for I. So the consequent is true for I. Thus I is a model for the wff. Since I was an arbitrary interpretation for the wff, every interpretation for the wff is a model. Therefore, the wff is valid. QED.

Indirect Proof: Suppose, BWOC, that the wff is invalid. Then there is a countermodel I with domain D for the wff. So the antecedent is true for I and the consequent is false for I. Since the antecedent is true for I, it follows that $A(d) \wedge B(d)$ is true for I for some $d \in D$. Since the consequent is false for I, either $\exists x\, A(x)$ or $\exists x\, B(x)$ is false for I. So either $A(x)$ is false for all $x \in D$ or $B(x)$ is false for all $x \in D$. These cases contradict the fact that both $A(d)$ and $B(d)$ are true for I for some $d \in D$. So the wff is valid. QED.

Quiz (1 minute). Is $\exists x\, A(x) \wedge \exists x\, B(x) \to \exists x\,(A(x) \wedge B(x))$ valid?

Answer: No. e.g., let $D = \mathbf{N}$, $A(x)$ mean x is even, and $B(x)$ mean x is odd.

Some Valid Conditionals Whose Converses Are Invalid

(a) $\forall x\, A(x) \to \exists x\, A(x)$.

(b) $\exists x\, (A(x) \wedge B(x)) \to \exists x\, A(x) \wedge \exists x\, B(x)$.

(c) $\forall x\, A(x) \vee \forall x\, B(x) \to \forall x\, (A(x) \vee B(x))$.

(d) $\forall x\, (A(x) \to B(x)) \to (\forall x\, A(x) \to \forall x\, B(x))$.

(e) $\exists x\, \forall y\, P(x, y) \to \forall y\, \exists x\, P(x, y)$.

Quiz (5 minutes). Find a countermodel for each converse of the above wffs.

Closures

Let W be a wff with free variables $x_1, \ldots, x_n$. Then we have the following definitions:

$\forall x_1 \ldots \forall x_n\, W$ is the *universal closure* of W.
$\exists x_1 \ldots \exists x_n\, W$ is the *existential closure* of W.

Sometimes a wff and one of its closures have the same properties and sometimes they don't, as we can see in the following examples.

Example. The wff $p(x) \vee \neg\, p(y)$ is satisfiable and invalid.

The universal closure $\forall x\, \forall y\, (p(x) \vee \neg\, p(y))$ is satisfiable and invalid.
The existential closure $\exists x\, \exists y\, (p(x) \vee \neg\, p(y))$ is valid.

Example. The wff $p(x) \wedge \neg\, p(y)$ is satisfiable and invalid.

The universal closure $\forall x\, \forall y\, (p(x) \wedge \neg\, p(y))$ is unsatisfiable.
The existential closure $\exists x\, \exists y\, (p(x) \wedge \neg\, p(y))$ is satisfiable and invalid.

Closure Properties (proofs in text)

1. A wff is valid if and only if its universal closure is valid.
2. A wff is unsatisfiable if and only if its existential closure is unsatisfiable .

Example. The wff $p(x) \to \exists y\, p(y)$ is valid and its universal closure $\forall x\,(p(x) \to \exists y\, p(y))$ is also valid.

Example. The wff $p(x) \wedge \forall y\, \neg\, p(y)$ is unsatisfiable and its existential closure $\exists x\,(p(x) \wedge \forall y\, \neg\, p(y)\,))$ is also unsatisfiable.

Decidability (Solvability)

A problem in the form of a yes/no question is *decidable* if there is an algorithm that takes as input any instance of the problem and halts with the answer. Otherwise, the problem is *undecidable*. A problem is *partially decidable* if there is an algorithm that takes as input any instance of the problem and halts if the answer is yes, but might not halt if the answer is no.

The Validity Problem for Propositional Calculus

The problem of determining whether a propositional wff is a tautology *is decidable*. An algorithm can build a truth table for the wff and then check it.

The Validity Problem for First-Order Predicate Calculus

The problem of determining whether a first-order wff is valid *is undecidable*, but it *is partially decidable*. Two partial decision procedures are *natural deduction* (due to Gentzen in 1935) and *resolution* (due to Robinson in 1965). We'll study natural deduction in Section 7.3 and resolution in Chapter 9.