Section 6.3 Formal Reasoning

A *formal proof* (or *derivation*) is a sequence of wffs, where each wff is either a premise or the result of applying a proof rule to certain previous wffs in the sequence.

Basic Proof Rules

Conjunction (Conj)

$$\frac{A,\ B}{A \wedge B}$$

Simplification (Simp)

$$\frac{A \wedge B}{A} \quad \text{and} \quad \frac{A \wedge B}{B}$$

Addition (Add)

$$\frac{A}{A \vee B} \quad \text{and} \quad \frac{B}{A \vee B}$$

Disjunctive Syllogism (DS)

$$\frac{A \vee B,\ \neg A}{B} \quad \text{and} \quad \frac{A \vee B,\ \neg B}{A}$$

Modus Ponens (MP)

$$\frac{A \to B,\ A}{B}$$

Conditional Proof (CP)

$$\frac{\text{From } A \text{, derive } B}{A \to B}$$

Double Negation (DN)

$$\frac{\neg\neg A}{A} \quad \text{and} \quad \frac{A}{\neg\neg A}$$

Contradiction (Contr)

$$\frac{A,\ \neg A}{\text{False}}$$

Indirect Proof (IP)

$$\frac{\text{From } \neg A \text{, derive False}}{A}$$

Proof Notation

Put each wff on a numbered line along with a reason. Use the letter P for a premise and follow the proof with QED.

Example. We'll prove that the argument with premises $A \vee C \to D$, $\neg B$, $A \vee B$ and conclusion D is valid.

1.	$A \vee C \to D$	P
2.	$\neg B$	P
3.	$A \vee B$	P
4.	A	2, 3, DS
5.	$A \vee C$	4, Add
6.	D	1, 5, MP

QED.

Using CP

If a proof consists of a derivation from a premise A to a conclusion B that does not contain any uses of CP or IP, then we can apply CP to obtain a tautology $A \to B$. The reason we obtain a tautology is that the proof rules used in the derivation are valid arguments. So the truth of A implies the truth of B, which tells us that $A \to B$ is a tautology.

When using CP in this way, instead of writing $A \to B$, we'll write QED along with the line numbers of the derivation followed by CP.

Example. We'll prove that $(A \vee C \rightarrow D) \wedge \neg B \wedge (A \vee B) \rightarrow D$ is a tautology.

1.	$A \vee C \rightarrow D$	P
2.	$\neg B$	P
3.	$A \vee B$	P
4.	A	2, 3, DS
5.	$A \vee C$	4, Add
6.	D	1, 5, MP
	QED	1–6, CP.

Example. We'll prove $(A \vee B \rightarrow C \wedge D) \wedge A \wedge (C \rightarrow E) \rightarrow D \wedge E$ is a tautology.

1.	$A \vee B \rightarrow C \wedge D$	P
2.	A	P
3.	$C \rightarrow E$	P
4.	$A \vee B$	2, Add
5.	$C \wedge D$	1, 4, MP
6.	C	5, Simp
7.	E	3, 6, MP
8.	D	5, Simp
9.	$D \wedge E$	7, 8, Conj
	QED	1–9, CP.

Subproofs

A *subproof* is a proof that is part of another proof. It always starts with a new premise and always ends by applying CP or IP to the derivation from that premise. When this happens, the premise is *discharged* and the wffs of the derivation become inactive.

Indent the statements of the subproof and write down the result of CP or IP without indentation.

Example. We'll prove that $(A \lor B) \to (\neg B \to A)$ is a tautology.

1.	$A \lor B$	P
2.	$\neg B$	P [for $\neg B \to A$]
3.	A	1, 2, DS
4.	$\neg B \to A$	2, 3, CP
	QED	1, 4, CP.

(We'll prove the converse shortly)

Example. We'll prove that $(A \vee B \to C) \to (A \to C) \wedge (B \to C)$ is a tautology.

1.	$A \vee B \to C$	P
2.	$\quad A$	P [for $A \to C$]
3.	$\quad A \vee B$	2, Add
4.	$\quad C$	1, 3, MP
5.	$A \to C$	2–4, CP
6.	$\quad B$	P [for $B \to C$]
7.	$\quad A \vee B$	6, Add
8.	$\quad C$	1, 7, MP
9.	$B \to C$	6–8, CP
10.	$(A \to C) \wedge (B \to C)$	5, 9, Conj
	QED	1, 5, 9–10, CP.

Using IP

If a proof consists of a derivation from a premise $\neg A$ to the conclusion False, then we could apply CP to obtain $\neg A \to$ False. But we also know that $A \equiv \neg A \to$ False. So the result of the derivation is A. This is the IP rule.

Example. We'll prove the tautology $\neg (A \wedge \neg A)$.

1.	$\neg \neg (A \wedge \neg A)$	P [for $\neg (A \wedge \neg A)$]
2.	$A \wedge \neg A$	1, DN
3.	A	2, Simp
4.	$\neg A$	2, Simp
5.	False	3, 4, Contr
	QED	1–5, IP.

IP is most often used in a subproof setting when proving a conditional of the form $V \to W$. Start with V as a premise for a CP proof. Then start an IP subproof with premise $\neg W$. When a contradiction is reached, we obtain W by IP. Then CP gives the result $V \to W$.

As with CP subproofs, the result of IP is written with no indentation.

Example/Quiz. Prove the tautology $(A \to B) \wedge (A \vee B) \to B$.

1.	$A \to B$	P
2.	$A \vee B$	P
3.	$\neg B$	P [for B]
4.	A	2, 3, DS
5.	B	1, 4, MP
6.	False	3, 5, Contr
7.	B	3–6, IP
	QED	1–2, 7, CP.

Example. We'll prove that the converse of $(A \vee B) \to (\neg B \to A)$.
Proof of $(\neg B \to A) \to (A \vee B)$:

1.	$\neg B \to A$	P
2.	$\neg (A \vee B)$	P [for $A \vee B$]
3.	$\neg B$	P [for B]
4.	A	1, 3, MP
5.	$A \vee B$	4, Add
6.	False	2, 5, Contr
7.	B	3–6, IP
8.	$A \vee B$	7, Add
9.	False	2, 8, Contr
10.	$A \vee B$	2, 7–9, IP
	QED	1, 4, CP.

Derived Rules (they follow from the original rules)

Modus Tollens (MT)

$$\frac{A \to B,\ \neg B}{\neg A}$$

Proof by Cases (Cases)

$$\frac{A \vee B,\ A \to C,\ B \to C}{C}$$

Hypothetical Syllogism (HS)

$$\frac{A \to B,\ B \to C}{A \to C}$$

Constructive Dilemma (CD)

$$\frac{A \vee B,\ A \to C,\ B \to D}{C \vee D}$$

Example. We'll give two proofs of the tautology

$$(A \to C) \wedge (B \to C) \to (A \vee B \to C).$$

First proof:

1.	$A \to C$	P
2.	$B \to C$	P
3.	$\quad A \vee B$	P [for $A \vee B \to C$]
4.	$\quad C$	1, 2, 3, Cases
5.	$A \vee B \to C$	3–4, CP
	QED	1–2, 5, CP.

Second proof:

1.	$A \to C$	P
2.	$B \to C$	P
3.	$\quad A \vee B$	P [for $A \vee B \to C$]
4.	$\quad\quad \neg C$	P [for C]
5.	$\quad\quad \neg A$	1, 3, MT
6.	$\quad\quad B$	3, 5, DS
7.	$\quad\quad \neg B$	2, 4, MP
8.	$\quad\quad$ False	6, 7, Contr
9.	$\quad C$	4–8, IP
10.	$A \vee B \to C$	3, 9, CP
	QED	1–2, 10, CP.

Example/Quiz. Prove $(A \to C) \wedge \neg (A \to B) \to \neg (C \to B)$ with IP somewhere.

1.	$A \to C$	P
2.	$\neg (A \to B)$	P
3.	$\quad \neg \neg (C \to B)$	P [for $\neg (C \to B)$]
4.	$\quad C \to B$	3, DN
5.	$\quad A \to B$	1, 4, HS
6.	$\quad$ False	2, 5, Contr
7.	$\neg (C \to B)$	3–6, IP
	QED	1–2, 7, CP.

Example/Quiz. Consider the following argument:

I eat spinach (S) or ice cream (I). If I study logic (L) then I will pass the exam (P). If I eat ice cream then I will study logic. If I eat spinach then I will play golf (G). I failed the exam. Therefore, I played golf.

The argument has five premises $\{S \vee I, L \to P, I \to L, S \to G, \neg P\}$ and conclusion G.

Prove that the argument is valid.

1.	$S \vee I$	P
2.	$L \to P$	P
3.	$I \to L$	P
4.	$S \to G$	P
5.	$\neg P$	P
6.	$\neg L$	2, 5, MT
7.	$\neg I$	3, 6, MT
8.	S	1, 7, DS
9.	G	4, 8, MP
	QED.	

Alternative Proof:

6.	$I \to P$	2, 3, HS
7.	$\neg I$	5, 6, MT
8.	S	1, 7, DS
9.	G	4, 8, MP
	QED.	

Alternative Proof:

6.	$G \vee L$	1, 3, 4, CD
7.	$\neg L$	2, 5, MT
8.	G	6, 7, DS
	QED.	

Alternative Proof:

6.	$I \to P$	2, 3, HS
7.	$G \vee L$	1, 4, 6, CD
8.	G	5, 7, DS
	QED.	