Section 6.2 Propositional Calculus

Propositional calculus is the language of *propositions* (statements that are true or false). We represent propositions by formulas called *well-formed formulas* (*wffs*) that are constructed from an alphabet consisting of

Truth symbols:	T (or True) and F (or False)	
Propositional variables:	uppercase letters.	
Connectives (operators):	$\neg$	(not, negation)
	$\wedge$	(and, conjunction)
	$\vee$	(or, disjunction)
	$\rightarrow$	(conditional, implication)
Parentheses symbols:	(and).	

A *wff* is either a truth symbol, a propositional variable, or if V and W are wffs, then so are $\neg V$, $V \wedge W$, $V \vee W$, $V \rightarrow W$, and (W).

Example. The expression $A \neg B$ is not a wff. But each of the following three expressions is a wff: $A \wedge B \rightarrow C$, $(A \wedge B) \rightarrow C$, and $A \wedge (B \rightarrow C)$.

Truth Tables. The connectives are defined by the following truth tables.

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$
T	T	F	T	T	T
T	F	F	F	T	F
F	T	T	F	T	T
F	F	T	F	F	T

Semantics

The meaning of T (or True) is true and the meaning of F (or False) is false. The meaning of any other wff is its truth table, where in the absence of parentheses, we define the hierarchy of evaluation to be $\neg$, $\wedge$, $\vee$, $\rightarrow$, and we assume $\wedge$, $\vee$, $\rightarrow$ are left associative.

Examples.

$\neg A \wedge B$	means	$(\neg A) \wedge B$
$A \vee B \wedge C$	means	$A \vee (B \wedge C)$
$A \wedge B \rightarrow C$	means	$(A \wedge B) \rightarrow C$
$A \rightarrow B \rightarrow C$	means	$(A \rightarrow B) \rightarrow C$.

Three Classes

A *Tautology* is a wff for which all truth table values are T.
A *Contradiction* is a wff for which all truth table values are F.
A *Contingency* is a wff that is neither a tautology nor a contradiction.

Examples. $P \vee \neg P$ is a tautology. $P \wedge \neg P$ is a contradiction. $P \rightarrow Q$ is a contingency.

Equivalence

The wff V is *equivalent* to the wff W (written $V \equiv W$) iff V and W have the same truth value for each assignment of truth values to the propositional variables occurring in V and W.

Example. $\neg A \wedge (B \vee A) \equiv \neg A \wedge B$ and $A \vee \neg A \equiv B \vee \neg B$.

Equivalence and Tautologies

We can express equivalence in terms of tautologies as follows:

$V \equiv W$ iff $(V \rightarrow W)$ and $(W \rightarrow V)$ are tautologies.

Proof: $V \equiv W$ iff V and W have the same truth values iff $(V \rightarrow W)$ and $(W \rightarrow V)$ are tautologies. QED.

Basic Equivalences that Involve True and False

The following equivalences are easily checked with truth tables:

$A \wedge \text{True} \equiv A$	$A \vee \text{True} \equiv \text{True}$	$A \to \text{True} \equiv \text{True}$	$\text{True} \to A \equiv A$
$A \wedge \text{False} \equiv \text{False}$	$A \vee \text{False} \equiv A$	$A \to \text{False} \equiv \neg A$	$\text{False} \to A \equiv \text{True}$
$A \wedge \neg A \equiv \text{False}$		$A \vee \neg A \equiv \text{True}$	$A \to A \equiv \text{True}$

Other Basic Equivalences

The connectives $\wedge$ and $\vee$ are commutative, associative, and distribute over each other. These properties and the following equivalences can be checked with truth tables:

$A \wedge A \equiv A$	$\neg (A \wedge B) \equiv \neg A \vee \neg B$	$A \wedge (A \vee B) \equiv A$	$A \wedge (\neg A \vee B) \equiv A \wedge B$
$A \vee A \equiv A$	$\neg (A \vee B) \equiv \neg A \wedge \neg B$	$A \vee (A \wedge B) \equiv A$	$A \vee (\neg A \wedge B) \equiv A \vee B$
$\neg \neg A \equiv A$	$A \to B \equiv \neg A \vee B$	$\neg (A \to B) \equiv A \wedge \neg B$	

Using Equivalences To Prove Other Equivalences

We can often prove an equivalence without truth tables because of the following two facts:

1. If $U \equiv V$ and $V \equiv W$, then $U \equiv W$.
2. If $U \equiv V$, then any wff W that contains U is equivalent to the wff obtained from W by replacing an occurrence of U by V.

Example. Use equivalences to show that $A \vee B \to A \equiv B \to A$.

Proof:

$$\begin{aligned} A \vee B \to A &\equiv \neg (A \vee B) \vee A \\ &\equiv (\neg A \wedge \neg B) \vee A \\ &\equiv (\neg A \vee A) \wedge (\neg B \vee A) \\ &\equiv \text{True} \wedge (\neg B \vee A) \\ &\equiv \neg B \vee A \\ &\equiv B \to A. \end{aligned}$$

QED.

Quizzes (1 minute each). Use known equivalences in each case.
Prove that $A \vee B \to C \equiv (A \to C) \wedge (B \to C)$.
Prove that $(A \to B) \vee (\neg A \to B)$ is a tautology (i.e., show it is equivalent to true)
Prove that $A \to B \equiv (A \wedge \neg B) \to \text{False}$.
Use absorption to simplify $(P \wedge Q \wedge R) \vee (P \wedge R) \vee R$.
Use absorption to simplify $(S \to T) \wedge (U \vee T \vee \neg S)$.

Is it a tautology, a contradiction, or a contingency?

If P is a variable in a wff W, let $W(P/\text{True})$ denote the wff obtained from W by replacing all occurrences of P by True. $W(P/\text{False})$ is defined similarly. The following properties hold:

W is a tautology iff $W(P/\text{True})$ and $W(P\ \text{False})$ are tautologies.
W is a contradiction iff $W(P/\text{True})$ and $W(P/\text{False})$ are contradictions.

Quine's method uses these properties together with basic equivalences to determine whether a wff is a tautology, a contradiction, or a contingency.

Example. Let $W = (A \wedge B \to C) \wedge (A \to B) \to (A \to C)$. Then we have

$$W(A/\text{False}) = (\text{False} \wedge B \to C) \wedge (\text{False} \to B) \to (\text{False} \to C)$$
$$\equiv (\text{False} \to C) \wedge \text{True} \to \text{True} \equiv \text{True} .$$

So $W(A/\text{False})$ is a tautology. Next look at

$$W(A/\text{True}) = (\text{True} \wedge B \to C) \wedge (\text{True} \to B) \to (\text{True} \to C) \equiv (B \to C) \wedge B \to C.$$

Let $X = (B \to C) \wedge B \to C$. Then we have

$$X(B/\text{True}) = (\text{True} \to C) \wedge \text{True} \to C \equiv C \wedge \text{True} \to C \equiv C \to C \equiv \text{True}.$$
$$X(B/\text{False}) = (\text{False} \to C) \wedge \text{False} \to C \equiv \text{False} \to C \equiv \text{True}.$$

So X is a tautology. Therefore, W is a tautology.

Quizzes (2 minutes each). Use Quine's method in each case.
Show that $(A \vee B \to C) \vee A \to (C \to B)$ is NOT a tautology.
Show that $(A \to B) \to C$ is NOT equivalent to $A \to (B \to C)$.

Normal Forms

A *literal* is either a propositional variable or its negation. e.g., A and $\neg A$ are literals.
A *disjunctive normal form* (DNF) is a wff of the form $C_1 \vee \ldots \vee C_n$, where each C_i is a conjunction of literals, called a *fundamental conjunction*. A *conjunctive normal form* (CNF) is a wff of the form $D_1 \wedge \ldots \wedge D_n$, where each D_i is a disjunction of literals, called a *fundamental disjunction*.

Examples. $(A \wedge B) \vee (\neg A \wedge C \wedge \neg D)$ is a DNF. $(A \vee B) \wedge (\neg A \vee C) \wedge (\neg C \vee \neg D)$ is a CNF. The wffs A, $\neg B$, $A \vee \neg B$, and $A \wedge \neg B$ are both DNF and CNF. Why?

Any wff has a DNF and a CNF.

For any propositional variable A we have True $\equiv A \vee \neg A$ and False $\equiv A \wedge \neg A$. Both forms are DNF and CNF. For other wffs use basic equivalences to: (1) remove conditionals, (2) move negations to the right, and (3) transform into required form. Simplify where desired.

Example. $(A \to B \vee C) \to (A \wedge D)$

$\equiv \neg (A \to B \vee C) \vee (A \wedge D)$	$(X \to Y \equiv \neg X \vee Y)$
$\equiv (A \wedge \neg (B \vee C)) \vee (A \wedge D)$	$(\neg (X \to Y) \equiv X \wedge \neg Y)$
$\equiv (A \wedge \neg B \wedge \neg C) \vee (A \wedge D)$	$(\neg (X \vee Y) \equiv \neg X \wedge \neg Y)$ (DNF)
$\equiv ((A \wedge \neg B \wedge \neg C) \vee A) \wedge ((A \wedge \neg B \wedge \neg C) \vee D)$	(distribute $\vee$ over $\wedge$)
$\equiv A \wedge ((A \wedge \neg B \wedge \neg C) \vee D)$	(absorption)
$\equiv A \wedge (A \vee D) \wedge (\neg B \vee D) \wedge (\neg C \vee D)$	(distribute $\vee$ over $\wedge$) (CNF)
$\equiv A \wedge (\neg B \vee D) \wedge (\neg C \vee D)$	(absorption) (CNF).

Quiz (2 minutes). Transform $(A \wedge B) \vee \neg (C \to D)$ into DNF and into CNF.

Every Truth Function Is a Wff

A *truth function* is a function whose arguments and results take values in {true, false}. So a truth function can be represented by a truth table. The task is to find a wff with the same truth table. We can construct both a DNF and a CNF.

Technique. To construct a DNF, take each line of the table with a true value and construct a fundamental conjunction that is true only on that line. To construct a CNF, take each line with a false value and construct a fundamental disjunction that is false only on that line.

Example. Let f be defined by

$f(A, B)$ = if $A = B$ then True else False.

The picture shows the truth table for f together with the fundamental conjunctions for the DNF and the fundamental disjunctions for the CNF.

A	B	$f(A,B)$	(DNF Parts)	(CNF Parts)
T	T	T	$A \wedge B$	
T	F	F		$\neg A \vee B$
F	T	F		$A \vee \neg B$
F	F	T	$\neg A \wedge \neg B$	

So $f(A, B)$ can be written as follows:

$$f(A, B) \equiv (A \wedge B) \vee (\neg A \wedge \neg B) \quad \text{(DNF)}$$
$$f(A, B) \equiv (\neg A \vee B) \wedge (A \vee \neg B) \quad \text{(CNF)}$$

Full CNF and Full DNF. A DNF for a wff W is a *Full DNF* if each fundamental conjunction contains the same number of literals, one for each propositional variable of W. A CNF for a wff W is a *Full CNF* if each fundamental disjunction contains the same number of literals, one for each propositional variable of W.

Example. The wffs in the previous example are full DNF and full CNF.

Constructing Full DNF and Full CNF

We can use the technique for truth functions to find a full DNF or full CNF for any wff with the restriction that a tautology does not have a full CNF and a contradiction does not have a full DNF. For example,

True $\equiv A \vee \neg A$, which is a full DNF and a CNF, but it is not a full CNF.
False $\equiv A \wedge \neg A$, which is a full CNF and a DNF, but it is not a full DNF.

Alternative Constructions for Full DNF and Full CNF. Use basic equivalences together with the following tricks to add a propositional variable A to a wff W:

$W \equiv W \wedge \text{True} \equiv W \wedge (A \vee \neg A) \equiv (W \wedge A) \vee (W \wedge \neg A)$.
$W \equiv W \vee \text{False} \equiv W \vee (A \wedge \neg A) \equiv (W \vee A) \wedge (W \vee \neg A)$.

Example. Find a full DNF for $(A \wedge \neg B) \vee (A \wedge C)$.

Answer. $(A \wedge \neg B \wedge C) \vee (A \wedge \neg B \wedge \neg C) \vee (A \wedge C \wedge \neg B) \vee (A \wedge C \wedge B)$, which can be simplified to: $(A \wedge \neg B \wedge C) \vee (A \wedge \neg B \wedge \neg C) \vee (A \wedge C \wedge B)$,

Quiz (1 minute). Find a full CNF for $\neg A \wedge B$.

Ans. $(\neg A \vee B) \wedge (\neg A \vee \neg B) \wedge (B \vee A) \wedge (B \vee \neg A) \equiv (\neg A \vee B) \wedge (\neg A \vee \neg B) \wedge (B \vee A)$.

Complete Sets of Connectives

A set S of connectives is *complete* if every wff is equivalent to a wff constructed from S. So $\{\neg, \wedge, \vee, \rightarrow\}$ is complete by definition.

Examples. Each of the following sets is a complete set of connectives.

$\{\neg, \wedge, \vee\}$, $\{\neg, \wedge\}$, $\{\neg, \vee\}$, $\{\neg, \rightarrow\}$, {False, $\rightarrow$}, {NAND}, {NOR}.

Quiz (2 minutes). Show that $\{\neg, \rightarrow\}$ is a complete.
Quiz (2 minutes). Show that {*if-then-else*, True, False} is a complete.