# *Introduction to Frame Theory*

**RSF Lecture Series**

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# Outline

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- 30 years later in 1980, Young in "an introduction to non-harmonic Fourier series" and in 1986, Daubechies, Grossmann and Meyer in "Painless non-orthonormal expansions" reintroduce frame and used them as bases in Hilbert spaces (especially $L^2(\mathbb{R})$).

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- Recent research has shown that frame theory has broad application in mathematics and engineering to a variety of areas including pure mathematics ([2]), applied mathematics ([5]), sampling theory ([1]), operator theory ([3]), harmonic analysis, Wavelet theory, etc.

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$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2 \quad \text{for all } f \in \mathbb{H}. \tag{1}$$

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- The frame is call exact frame if it ceases to be a frame whenever any single element is deleted from the sequence.
- The numbers $\{\langle f, f_i\rangle\}_{i\in I}$ are call the frame coefficients of $f$.

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- $\{f_i\}_{i\in I}$ is a Riesz basis for $\mathbb{H}$ if and only if there are constants $0 \le A \le B$ such that for all scalars $\{a_i\}_{i\in I}$ we have:

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- $\{f_i\}_{i\in I}$ is a Riesz basis iff it is frame and $w$-independent.

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- $$f = S^{-1} Sf = \sum_{i \in I} \langle f, f_i \rangle S^{-1} f_i = \sum_{i \in I} \langle f, S^{-1} f_i \rangle f_i$$

  The family $\{S^{-1} f_i\}_{i \in I}$ is also frame for $\mathbb{H}$, called the canonical dual frame of $\{f_i\}_{i \in I}$.

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**The Role of eigenvalues and eigenvectors:**

Let $\mathbb{H}_N$ be an $N$-dimensional Hilbert spaces and let $\{f_i\}_{i=1}^M$ be a frame for $\mathbb{H}_N$. Let $S$ be frame operator and $\{\lambda_j\}_{j=1}^N$ be the eigenvalues for $S$ with respective eigenvectors $\{g_j\}_{j=1}^N$. Then following properties holds:

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- **General frame:** $\sum_{j=1}^N \lambda_j = \sum_{i=1}^M \|f_i\|^2$.
- **Equal norm frame:** $\sum_{j=1}^N \lambda_j = M\|f_1\|^2$.
- **Tight frame:** $N \cdot A = \sum_{j=1}^N \lambda_j = \sum_{i=1}^M \|f_i\|^2$.
- **Parseval frame:** $N = \sum_{j=1}^N \lambda_j = \sum_{i=1}^M \|f_i\|^2$.

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$$(F_g f)(t,\omega) = \langle f, \tau(t,\omega)g\rangle = \int_{\mathbb{R}^d} f(x)\overline{g(x-t)}e^{-2\pi i\langle x,\omega\rangle}\,dx. \qquad (2)$$

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- Let $a, b > 0$ and $g \in L^2(\mathbb{R}^d)$. Then the system $\{\tau(na, mb)g : m, n \in \mathbb{Z}^d\}$ is called **Gabor system**, denoted by $(g, a, b)$ and $g$ is called the window function.

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*Given a window function $g \in L^2(\mathbb{R}^d)$ and $a, b > 0$. If there exist constants $A, B > 0$ such that*
$$A\|f\|_2^2 \le \sum_{n,m\in\mathbb{Z}^d} |\langle f, \tau(na, mb)g\rangle|^2 \le B\|f\|_2^2, \quad \forall f \in L^2(\mathbb{R}^d)$$
*then the collection $(g, a, b)$ is called* **Gabor frame** *or* **Weyl-Heisenberg frame** *for $L^2(\mathbb{R}^d)$.*

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Sufficient condition for Gabor frame: Given, $g \in L^2(\mathbb{R})$ and $a, b > 0$. Suppose that (i) there are real constant's $A, B$ such that

$$0 < A \leq \sum_{n\in\mathbb{Z}} |g(t - na)|^2 \leq B, \quad \text{a.e.} t \in \mathbb{R}.$$

(ii) f has compact support with $suppg \subset I$ where $I$ is an interval of length $1/b$. Then $(g, a, b)$ is a Gabor frame for $L^2(\mathbb{R})$ with frame bound $b^{-1}A$ and $b^{-1}B$.

Necessary condition for Gabor frame: Given, $g \in L^2(\mathbb{R})$, $a, b > 0$ and $(g, a, b)$ be a Gabor frame with frame bound $A, B$. Then $bA \leq \sum_{n\in\mathbb{Z}} |g(t - na)|^2 \leq bB$ a.e.

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- If $(g, a, b)$ is Gabor frame then there exists a dual window $\tilde{g} \in L^2(\mathbb{R}^d)$ such that $\{\tau(na, mb)\tilde{g} : m, n \in \mathbb{Z}^d\}$ is also a Gabor frame for $L^2(\mathbb{R}^d)$ and $f$ can be represented as

$$f = \sum_{n,m\in\mathbb{Z}^d} \langle f, \tau(na, mb)g\rangle \tau(na, mb)\tilde{g} = \sum_{n,m\in\mathbb{Z}^d} \langle f, \tau(na, mb)\tilde{g}\rangle \tau(na, mb)g, \ \forall f \in L^2(\mathbb{R}^d).$$

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- Define

$$S_{a,b;g,\tilde{g}}f = \frac{(ab)^d}{\langle \tilde{g}, g\rangle} \sum_{n,m\in\mathbb{Z}^d} \langle f, \tau(na, mb)g\rangle \tau(na, mb)\tilde{g}.$$

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- **Feichtinger's algebra**: $S_0(\mathbb{R}^d) := \{g : F_g g \in L^1(\mathbb{R}^{2d})\}$
- **Wiener space**: A measurable function $f$ belongs to the Wiener space $W(\mathbb{R}^d)$ if

$$\|f\|_{W(\mathbb{R}^d)} := \sum_{k\in\mathbb{Z}^d} \|f \cdot T_k\chi_Q\|_\infty < \infty.$$

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- $Q = [0, 1)^d$. A measurable function $f$ belongs to the Wiener amalgam space $W(L^p, \ell^q)(\mathbb{R}^d)$ $(1 \leqslant p, q \leqslant \infty)$ if

$$\|f\|_{W(L^p,\ell^q)(\mathbb{R}^d)} := \left( \sum_{k\in\mathbb{Z}^d} \|f \cdot T_k \chi_Q\|_p^q \right)^{\frac{1}{q}} < \infty.$$

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- $Q = [0,1)^d$. A measurable function $f$ belongs to the Wiener amalgam space $W(L^p, \ell^q)(\mathbb{R}^d)$ $(1 \leqslant p, q \leqslant \infty)$ if

$$\|f\|_{W(L^p,\ell^q)(\mathbb{R}^d)} := \left( \sum_{k \in \mathbb{Z}^d} \|f \cdot T_k \chi_Q\|_p^q \right)^{\frac{1}{q}} < \infty.$$

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$$W(L^\infty, \ell^1)(\mathbb{R}^d) \subset L^p(\mathbb{R}^d) \subset W(L^1, \ell^\infty)(\mathbb{R}^d) \quad (1 \leq p \leq \infty).$$

# OUTLINE

# Weight Functions

- A function $w : \mathbb{R}^d \to (0, +\infty)$ is called a weight if it is continuous and symmetric (i.e. $w(x) = w(-x)$). A weight $w$ is submultiplicative if

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- For $s > 0$, $w(x) = (1 + |x|)^s$ is a submultiplicative weight.
- Consider $v(x) = (1 + |x|)^t$, $w(x) = (1 + |x|)^s$, then $v$ is $w$-moderate if $|t| \leq s$.

# OUTLINE

# Modulation spaces

- **Weightd mixed-norm spaces:** Let $m$ be a weight function on $\mathbb{R}^{2d}$ and let $1 \leq p, q < \infty$. Then the weighted mixed-norm space $L_m^{p,q}(\mathbb{R}^{2d})$ consists of all (Lebesgue) measurable functions on $\mathbb{R}^{2d}$, such that the norm

$$\|F\|_{L_m^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |F(x,w)|^p m(x,w)^p dx \right)^{q/p} dw \right)^{1/q}$$

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- **Modulation spaces**: Given $g \in S(\mathbb{R}^d)$ and $1 \leq p, q \leq \infty$, the Modulation space $M_{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in S'(\mathbb{R}^d)$ such that $F_g f \in L_m^{p,q}(\mathbb{R}^{2d})$ i.e.,

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- If $p = q$, we write $M_p$ instead $M_{p,q}$. It is known that $M_2(\mathbb{R}^d) = L_2(\mathbb{R}^d)$.
- For $1 \le p_1 \le p_2 \le \infty$ and $1 \le q_1 \le q_2 \le \infty$ we have:

$$M_1(\mathbb{R}^d) \hookrightarrow M_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow M_{p_2,q_2}(\mathbb{R}^d) \hookrightarrow M_\infty(\mathbb{R}^d)$$

# Outline

# Gabor frame operators on $W(L^p, \ell^q)$

For $g, \tilde{g} \in W(\mathbb{R}^d)$ and $a, b > 0$, define

$$G_{a,b;n}(x) = \sum_{k \in \mathbb{Z}^d} \overline{g}\left(x - \frac{n}{b} - ak\right) \tilde{g}(x - ak), \quad n \in \mathbb{Z}^d.$$

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**Lemma**

*For any $g, \tilde{g} \in W(\mathbb{R}^d)$, we have*

$$\sum_{n\in\mathbb{Z}^d} \|G_{a,b;n}\|_\infty \leqslant \left(1 + \frac{1}{a}\right)^d (2 + 2b)^d \|g\|_{W(\mathbb{R}^d)} \|\tilde{g}\|_{W(\mathbb{R}^d)}, \quad \forall a, b > 0, \tag{4}$$

*and*

$$\lim_{(a,b)\to(0,0)} \sum_{n\in\mathbb{Z}^d\setminus\{0\}} a^d \|G_{a,b;n}\|_\infty = 0. \tag{5}$$

# Gabor frame operators on $W(L^p, \ell^q)$

**Proposition (Walnut's representation)**

*Let $g, \tilde{g} \in W(\mathbb{R}^d)$ and let $a, b > 0$. Then the operator $S_{a,b;g,\tilde{g}}$ can be written as*

$$(S_{a,b;g,\tilde{g}}f)(x) = \frac{1}{\langle \tilde{g}, g \rangle} \sum_{n \in \mathbb{Z}^d} a^d G_{a,b;n}(x) f\left(x - \frac{n}{b}\right)$$

*is bounded from $W(L^p, \ell^q)(\mathbb{R}^d)$ to $W(L^p, \ell^q)(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$ with operator norm*

$$\|S_{a,b;g,\tilde{g}}\|_{W(L^p,\ell^q)(\mathbb{R}^d) \to W(L^p,\ell^q)(\mathbb{R}^d)} \leq \frac{a^d}{|\langle \tilde{g}, g \rangle|} \left(1 + \frac{1}{a}\right)^d (2 + 2b)^d \|g\|_{W(\mathbb{R}^d)} \|\tilde{g}\|_{W(\mathbb{R}^d)}.$$

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$$\|S_{a,b;g,\tilde{g}}\|_{W(L^p,\ell^q)(\mathbb{R}^d) \to W(L^p,\ell^q)(\mathbb{R}^d)} \leq \frac{a^d}{|\langle \tilde{g}, g \rangle|} \left(1 + \frac{1}{a}\right)^d (2 + 2b)^d \|g\|_{W(\mathbb{R}^d)} \|\tilde{g}\|_{W(\mathbb{R}^d)}.$$

Define $G_a(x) := \frac{a^d}{\langle \tilde{g}, g \rangle} G_{a,b;0}(x) = \frac{a^d}{\langle \tilde{g}, g \rangle} \sum_{k \in \mathbb{Z}^d} \overline{g}(x - ak) \tilde{g}(x - ak), \quad x \in \mathbb{R}^d.$

# GABOR FRAME OPERATORS ON $W(L^p, \ell^q)$

## LEMMA

*Let $g, \tilde{g} \in W(\mathbb{R}^d)$ and $1 \leqslant p, q \leqslant \infty$. Then we have:*
*(i) For any $f \in W(L^p, \ell^q)(\mathbb{R}^d)$,*

$$\lim_{(a,b)\to(0,0)} (\|S_{a,b;g,\tilde{g}} f - f\|_{W(L^p,\ell^q)(\mathbb{R}^d)} - \|(G_a - 1)f\|_{W(L^p,\ell^q)(\mathbb{R}^d)}) = 0;$$

*(ii)*

$$\lim_{(a,b)\to(0,0)} (\|S_{a,b;g,\tilde{g}} - I\|_{W(L^p,\ell^q)(\mathbb{R}^d)\to W(L^p,\ell^q)(\mathbb{R}^d)} - \|G_a - 1\|_{\infty}) = 0.$$

# Gabor frame operators on $W(L^p, \ell^q)$

## Lemma

*Let $g, \tilde{g} \in W(\mathbb{R}^d)$ and $1 \leqslant p, q \leqslant \infty$. Then we have:*
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## Theorem

*Let $g, \tilde{g} \in W(\mathbb{R}^d)$. Then we have:*

1. *For any $f \in W(L^p, \ell^q)(\mathbb{R}^d)$, $1 \leqslant p, q < \infty$,*

$$\lim_{(a,b)\to(0,0)} \|S_{a,b;g,\tilde{g}} f - f\|_{W(L^p,\ell^q)(\mathbb{R}^d)} = 0, \tag{6}$$

   *and conclusion true for $q = \infty$ but fails if $p = \infty$.*

2. *Moreover, if $\overline{g} \cdot \tilde{g}$ is locally Riemann integrable,then for any $1 \leqslant p, q \leqslant \infty$,*

$$\lim_{(a,b)\to(0,0)} \|S_{a,b;g,\tilde{g}} - I\|_{W(L^p,\ell^q)(\mathbb{R}^d)\to W(L^p,\ell^q)(\mathbb{R}^d)} = 0. \tag{7}$$

# GABOR FRAME OPERATORS ON $W(L^p, \ell^q)$

**Example:**
Take some $E \subset [0,1]$ such that $E$ is nowhere dense and is of positive measure. Let $g = \tilde{g} = \chi_E$. Then for any $a > 0$,

$$\{x \in [0,1] : G_a(x) > 0\} = \bigcup_{n\in\mathbb{Z}} (na + E) \cap [0,1] = \bigcup_{\|n\|_\infty \leq \frac{1}{a}} (na + E) \cap [0,1].$$

$$|\{x \in [0,1] : |G_a(x) - 1| = 1\}| \geq |\{x \in [0,1] : G_a(x) = 0\}| > 0.$$

Let $f_0 = \chi_{[0,1]}$. Then we have $\|(G_a - 1)f_0\|_\infty \geq 1, \quad \forall a > 0$.
Now

$$\|(G_a - 1)f_0\|^q_{W(L^\infty, l^q)(\mathbb{R})} = \sum_{k\in\mathbb{Z}} \|(G_a - 1)\chi_{[0,1]} \cdot \chi_{[k,k+1]}\|^q_\infty \geq \|(G_a - 1)\chi_{[0,1]}\|^q_\infty$$

That is,

$$\|(G_a - 1)f_0\|_{W(L^\infty, l^q)(\mathbb{R})} \geq 1, \quad \forall a > 0.$$

So $\lim\limits_{(a,b)\to(0,0)} \|S_{a,b;g,\tilde{g}} f_0 - f_0\|_{W(L^\infty, l^q)(\mathbb{R})} \geq 1.$

# Outline

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# THANK YOU