Introduction to Basic Topology

Euler's Theorem

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Outline

1. Introduction
2. Euler's Theorem
3. Topological Equivalence
4. Topological Invariants
5. Bibliography

Introduction

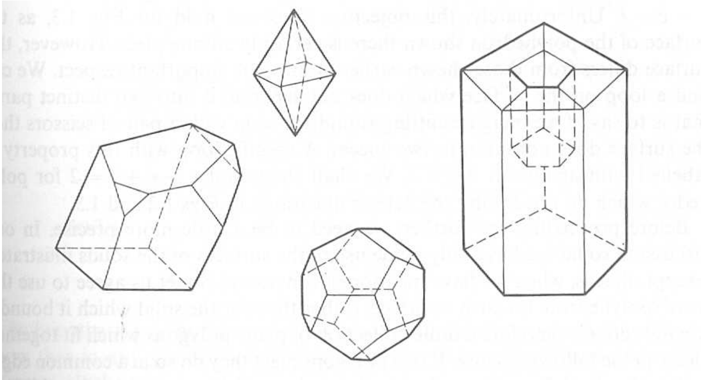

Figure 1 : Polyhedra satisfying Euler characteristic = 2

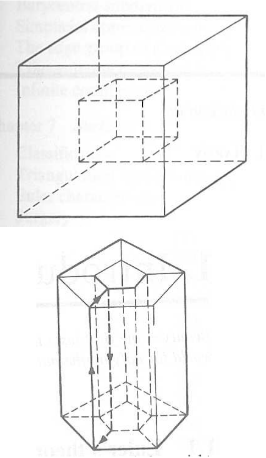

Figure 2 : Polyhedra with Euler characteristic $\neq 2$

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Then

$$v - e + f = 2$$

for P.

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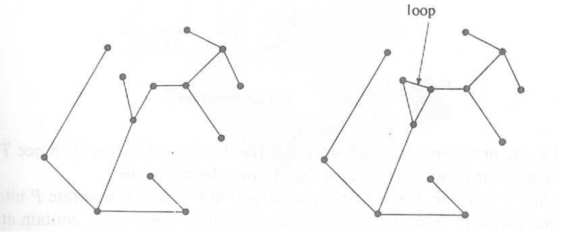
- In 1813 - Lhuilier drew attention to the problems raised by certain polyhedra as we have seen earlier.
- 1847 - von Staudt gave the precise statement as we see it, & a proof of that.

An Outline of von Staudt's Proof



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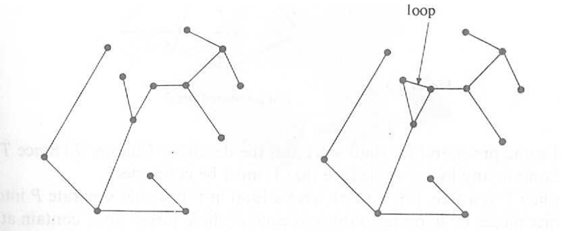

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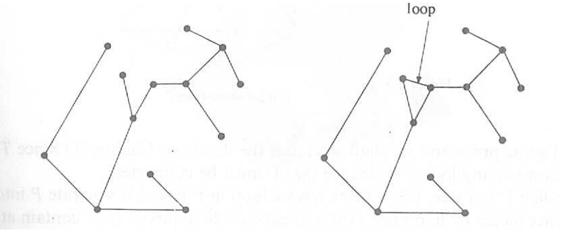

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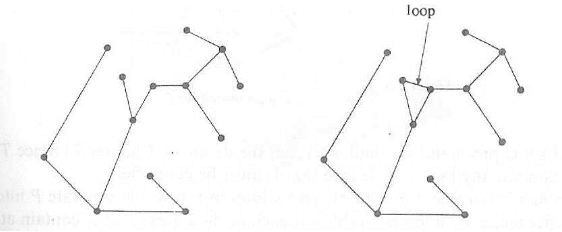

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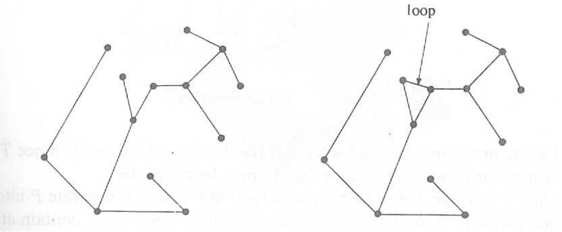

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- Choose a tree T with some edges and all vertices of P.
- Construct a dual T taking faces as vertices and connect two adjacent faces if their common edge $\notin T$.

von Staudt's Proof continues...

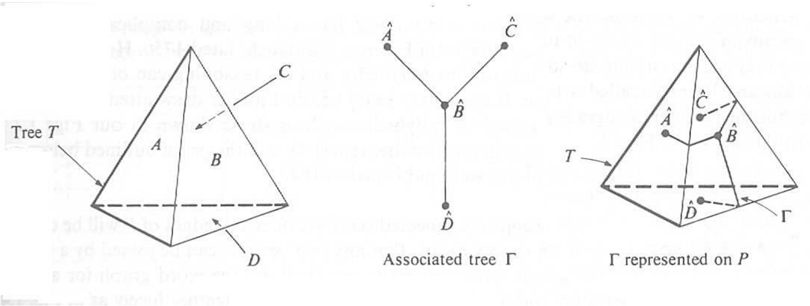

Figure 4 : The Graph T and its dual Γ for tetrahedron

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Warning!!! This proof fails for polyhedrons in Figure 2

Topological Equivalence or Homeomorphism

The above proof gives us some more information other than the formula. Thickening of T and Γ tells that P is made up of two disks. See the figure below.

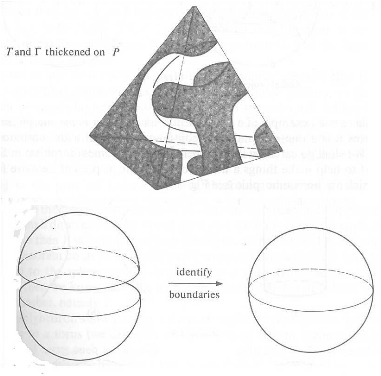

Figure 5 : P is made up of two disks, thickened T & thickened Γ

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So we need a continuously invertible transformation i.e. *homeomorphism* (a bijective bi-continuous map).
Now we can also see that *"homeomorphic"* is an equivalence relation.
Two spaces are topologically equivalent will imply that they are homeomorphic.

Are the following spaces topologically equivalent ?

1. Consider the following four spaces:

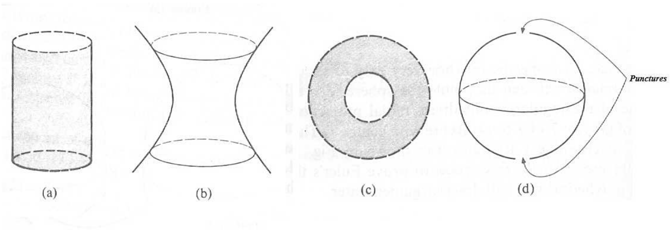

Figure 6 : Example A

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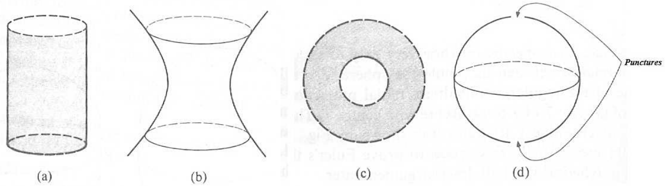
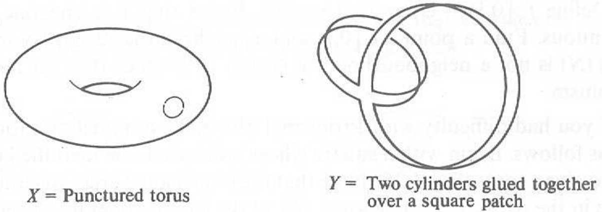


Figure 6 : Example A

2. Consider the following two spaces:



Few More Examples...

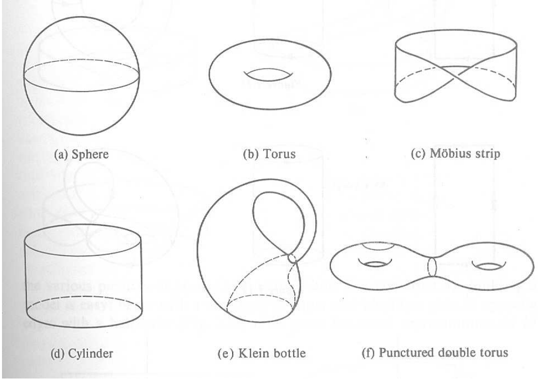


Figure 8 : Are any two of the above spaces homeomorphic?

Theorem

Topologically equivalent polyhedra have the same Euler number.

This remarkable theorem was the starting point of modern topology.
It motivated the search of properties which are preserved by topological equivalences or homeomorphisms.
Such properties are also known as *Topological Invariants*.

Girard's Theorem

Consider an N-sided spherical polygon on the unit sphere and let A_n denote the n-th interior angle. Then the area of such a polygon is given by (Todhunter)

$$\text{Area of polygon} \equiv E_N = \left(\sum_{n=1}^{N} A_n\right) - (N-2)\pi$$

This theorem leads us to Legendre's proof of Euler's formula.

Proof by Legendre

- Project the polyhedron radially onto a unit sphere.

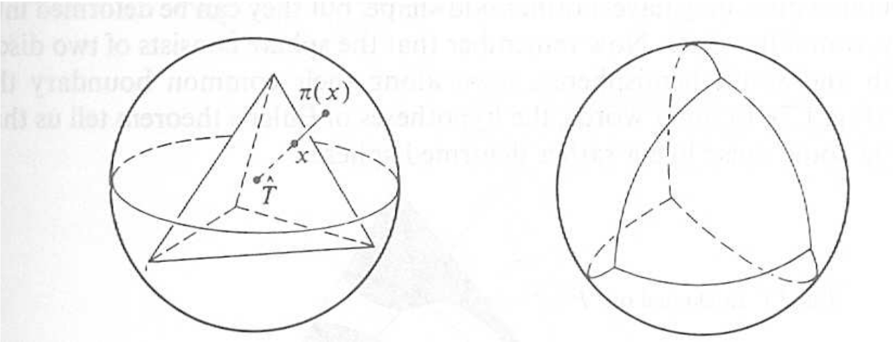


Figure 9 : Radial Projection

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- If Q is a spherical polygon with angles $\alpha_1, \cdots, \alpha_n$ and with n edges, then

$$\text{Area}(Q) = \alpha_1 + \cdots + \alpha_n - (n-2)\pi = \alpha_1 + \cdots + \alpha_n - n\pi + 2\pi.$$

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- Since α's sums up to $2\pi v$ (total angle at each vertex $= 2\pi$), $n\pi'$s add upto $2\pi e$ (as edges are counted twice), and each face contributes a 2π, therefore

$$4\pi = \text{Area(sphere)} = \sum_Q \text{Area}(Q) = 2\pi v - 2\pi e + 2\pi f.$$

Topological Invariants

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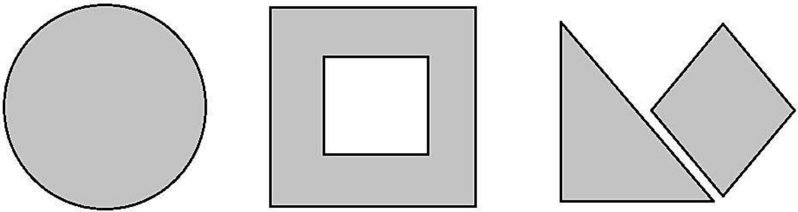
Qn. How to show that two topological spaces X and Y are not homeomorphic ?

Examples of Topological Invariants

- Connectedness, Simply connectedness. [20]
- Compactness [21]
- Fundamental Groups (Poincaré) - The idea is to assign a group to each topological space in such a way that *homeomorphic spaces have isomorphic groups.* [22]

etc.

Examples (Connectedness/Simply Connectedness)



(A) (B) (C)

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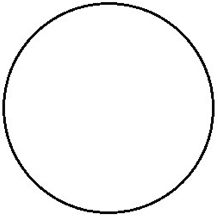
Examples (Compactness)



(A)

(B)

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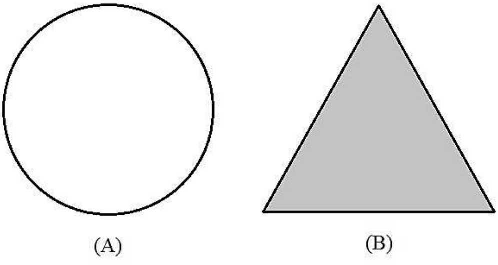
Examples (Fundamental Groups)



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Consequences

Classification Theorem for 1-Manifolds

Every compact, connected, one-dimensional manifold with boundary is diffeomorphic to $[0, 1]$ or S^1.

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Classification Theorem for Closed Surfaces (2-Manifolds)

Any closed surface is homeomorphic either to the sphere or to the sphere with a finite number of handles added, or to the sphere with a finite number of discs removed and replaced by Möbius strips. No two of these surfaces are homeomorphic.

References

[1] M. A. Armstrong, *Basic Topology*, Undergraduate Text in Mathematics, Springer-Verlag, New York, 1983.

[2] Victor Guillemin Allan Pollack, *Differential Topology*, Prentice-Hall, Inc., New Jersey, 1974.

Thank You