Design of Optimal Sliding Mode Controller for Uncertain Systems

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Design of Optimal Sliding Mode Controller for Uncertain Systems

A

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By

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Certificate

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This is dedicated to my parents

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and

Ranja Das

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(Madhulika Das)

Abstract

Generally, a conventional optimal controller designed for a system cannot guarantee its performance when the system is affected by uncertainties caused by modeling error, parameter change or external disturbance. In order to address this problem, this thesis attempts to design a robust optimal controller for uncertain systems. The focus of the research work is to design an optimal sliding mode controller which can minimize the control input and ensures that its performance does not degrade even when the system is affected by parametric uncertainty and external noise. The optimal controller is designed by using classical optimal control algorithm. A sliding mode controller (SMC) is integrated with the optimal controller to impart robustness. This thesis develops an optimal second order sliding mode controller (OSOSMC) which can mitigate high frequency chattering present in conventional first order sliding mode controllers. The optimal control law for linear systems is based on simple linear quadratic regulator (LQR) technique. An optimal adaptive sliding mode controller is designed for linear uncertain systems where the upper bound of uncertainty is unknown. For nonlinear systems, extended linearization is used to represent it in a linear like structure having state dependent coefficient (SDC) matrices and a state dependent Riccati equation (SDRE) based optimal controller is designed for the nominal part. For nonlinear systems which cannot be represented as linear like structures, the optimal control strategy is developed by using the control Lyapunov function (CLF). The second order sliding mode strategy is realized by designing a non-singular terminal sliding mode control based on the integral sliding surface.

Conventional SMCs cannot tackle the mismatched uncertainty. A disturbance observer based OSOSMC is proposed here for nonlinear systems affected by mismatched uncertainties. The optimal sliding mode controller is designed for both linear and nonlinear uncertain systems affected by matched as well as mismatched types of uncertainties. Extensive simulation studies are conducted to evaluate performance of the optimal second order sliding mode controller (OSOSMC) and compared with some existing control schemes. The proposed OSOSMC design is extended to multi input multi output (MIMO) systems also. Stability of the proposed controller is established by using Lyapunov's criterion.

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Nomenclature

ASMC	Adaptive sliding mode controller
CLF	Control Lyapunav function
CSOSMC	Compensator based second order sliding mode controller
DOB-SMC	Disturbance observer based sliding mode controller
DOB-OSOSMC	Disturbance observer based optimal second order sliding mode controller
HJB	Hamilton-Jacobi-Bellman
HOSMC	Higher order sliding mode control
ISMC	Integral sliding mode control
LMI	Linear matrix inequality
LQR	Linear quadratic regulator
MIMO	Multi input multi Output
NTSMC	Nonsingular terminal sliding mode control
OASMC	Optimal adaptive sliding mode controller
OSMC	Optimal sliding mode control
OSOSMC	Optimal second order sliding mode controller
SDC	State dependent coefficient
SDRE	State dependent Riccati equation
SISO	Single input single output
SMC	Sliding mode controller
SOSMC	Second order sliding mode controller
TSMC	Terminal sliding mode control
TV	Total Variation

Mathematical Notations

A	System matrix of continuous time LTI system
B	Input matrix of continuous time LTI system
J	Performance index
$f(x), g(x)$	Nominal nonlinearity
Q, R	Weighting matrices
W, P	Symmetric positive definite matrix
V, V_1, V_2	Lyapunov function
$s(t)$	Integral sliding variables
$\sigma(t)$	Terminal sliding variables
$\theta(t)$	Conventional sliding variables
$x(t)$	State vector
$y(t)$	Output of the system
$x_d(t)$	Desired trajectory
$e(t)$	Error signal
$u(t)$	Control input
$sign(.)$	Signum function
$\eta, \epsilon, \alpha, \beta, \Xi$	Positive costant

List of Publications

Journal Publications

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3. Madhulika Das and Chitralekha Mahanta, "Optimal adaptive sliding mode controller for linear uncertain systems ", *International conference on signal processing, computing and control (ISPCC 2013)*, 26-28th September 2013, Shimla, India.

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1
Introduction

Contents

1.1 Introduction

Optimal control is an important area in modern control theory. Optimal control methodology provides a systematic approach to design a controller which satisfies predefined performance criteria like time minimization, cost minimization and error minimization. Optimal control is an well established methodology and has been proposed for both linear and nonlinear systems. For designing an optimal controller for linear systems, linear quadratic regulator (LQR) technique [12] is widely used because of its simplicity and systematic structure of design. The LQR has been designed to solve both finite and infinite horizon optimal control problems for linear systems by solving the algebraic Riccati equation [13]. Although designing an optimal controller for nonlinear systems is not as straightforward as in the case of linear systems, a number of methods are available in literature for designing optimal controller for nonlinear systems. One of the main approaches for designing an optimal controller for nonlinear systems is to solve the Hamilton-Jacobi-Bellman (HJB) partial differential equation [14]. Unfortunately, the HJB partial differential equation for nonlinear systems is difficult to be solved analytically and it is computationally intensive due to its high dimension. In recent practice, the state dependent Riccati equation (SDRE) [15–18] has been used to find solution for the optimal control problem of nonlinear systems. Due to the systematic design approach, SDRE based optimal control has gained wide attention within the control community. The SDRE based control methodology provides an extremely effective algorithm to design a nonlinear feedback control by accommodating the nonlinearity in the system. In this method the nonlinear matrix is factorized into the product of a matrix-valued function and the state vector. Hence, the nonlinear system is converted into a linear like structure having state dependent coefficient (SDC) matrices [19, 20]. The SDRE technique is used to design an optimal controller for nonlinear systems similar to the LQR technique being used to design an optimal controller for linear systems. In the case of SDRE technique, the weighing matrices are dependent on the states of the system. Another effectively used method to design optimal controller for nonlinear systems is the control Lyapunav function (CLF) based optimal control [21–23]. Unlike the Lyapunov function, the CLF is defined for the closed loop system. Sontag showed in [24] that if the CLF could be found for the nonlinear system, there would exist a feedback controller to make the system asymptotically stable. Moreover, CLF which solves the HJB equation also optimizes certain performance index. So, if CLF exists for a nonlinear system, then there also exists a feedback controller which stabilizes the nonlinear system and also optimizes certain performance criteria without actually solving the HJB equation. However, performance of an optimal controller is not guaranteed when the system is not in its nominal state. If the system is affected by uncertainty, performance of the optimal controller deteriorates and the system may even be led towards instability. In reality, almost all physical systems are affected by uncertainty caused by modeling error, parameter variation and external disturbances.

Designing an optimal controller for uncertain systems is a highly challenging task. Continuing research in that area has led to design of robust optimal controllers. One such robust control method is the H_∞ suboptimal control [13] which is a well known technique to handle the uncertain system while optimizing a defined performance criterion. Another popular way to make the optimal controller robust is to integrate the optimal controller with some other controllers which exhibit robust behavior towards uncertainty. In recent literature a number of such robust controllers like adaptive controller [25, 26], backstepping controller [27, 28] and model predictive controller [29, 30] have been proposed. Nonconventional methods like neural network and fuzzy logic based controllers [31, 32] have also evolved to tackle systems in uncertain environment. Still, demand for trivial details and complex design procedure are major drawbacks of these methods. In the field of robust control, the sliding mode control (SMC) [33–38] has drawn wide attention due to its inherent insensitivity to parametric variations and external disturbances. The SMC has been successfully applied in diverse areas like power converters [39], robotics [40], aircrafts [41], underwater vehicles [42] and spacecrafts [43]. The SMC is a special case of variable structure control which originated in the 70's [44, 45]. Typically in the SMC, the control input is realized in two steps. In the first step, the control input is designed to bring the nominal system onto the sliding surface. This phase is called the reaching phase [46] of the SMC and the control input designed for this phase is known as equivalent control [47]. In the reaching phase, the system is sensitive to uncertainties affecting it. In the second step, a discontinuous switching control is used to keep the system onto the sliding surface. After occurrence of sliding mode, the system becomes immune to the matched uncertainty [46], which is in the range space of the input matrix. In conventional sliding mode control, when sliding mode occurs, the system behaves as a reduced order system with respect to the original model.

I. Optimal Sliding Mode Controller

The primary concern of designing an SMC is to stabilize the system in an environment where uncertainties and external disturbances are dominant. In such cases, the main interest is the fast convergence of the system states which demands a high control gain to tackle uncertainties. However, during nominal operation of the uncertain system, stability is not the only concern of controller design. Performance criteria like minimization of the control input and fast convergence of the system states also need attention during nominal phase of operation. Hence, when the system is far away from the equilibrium where its nominal part is dominant, optimal control is a natural choice to drive the system optimally towards the equilibrium. On the other hand, in the neighborhood of the equilibrium where uncertainties are more dominant, the SMC fits in as an ideal controller to ensure robustness. Though optimal control and SMC are two different types of control strategy, it is possible to combine them. Combination of optimal control with SMC has given rise to optimal sliding mode control (OSMC). Active research is continuing to design optimal sliding mode control as evident in literature. J.Zhou et al. [48] proposed time optimal sliding mode control for hard disk drive. They used the time varying sliding surface to take the system towards the origin in minimum time. Effort was made to minimize the control effort for a linear system with matched uncertainty [49–53]. To design optimal sliding mode control, the first step is to design the optimal control part for stabilizing the nominal linear system while minimizing the given performance index. A popular method to design optimal control for nominal linear system is LQR technique. To combine the LQR methodology with the SMC, integral sliding mode control (ISMC) [54–56] has been used. Though the system's order is not reduced in ISMC, its main advantage is that it achieves robustness from the very beginning. As ISMC has no reaching phase, the system is immune to matched uncertainties from the starting. Another remarkable advantage of ISMC is that it can combine the SMC with different types of control strategy. Like in linear systems, different methods are proposed to design optimal sliding mode control for the nonlinear system too. In [57] state dependent Riccati equation (SDRE) based optimal controller was proposed for uncertain nonlinear system. The control Lyapunov function based optimal control was developed to design OSMC in [58]. In [59–61], an optimal sliding mode control design was proposed by solving the two point boundary value problem. Hamiltonian method was used to design the optimal control which was combined with the sliding mode control in [43, 62]. To minimize reaching time, optimal sliding mode control was proposed by Jafarian and Nazarzadeh [39]. In this method, Jafarian and Nazarzadeh designed the control law for infinite switching for the sliding mode control and then depending upon this control constraint, they proposed the time optimal control based on Pontryagin's minimum principle.

II. Chattering Mitigation

Though the optimal sliding mode controller (OSMC) counters the problem of high control gain and guarantees robustness for uncertain systems, implementation of the OSMC becomes difficult due to a major drawback known as chattering which is the high frequency chattering inherently present in SMCs. The main cause behind chattering is the fast dynamics which is usually neglected at the time of designing an ideal sliding mode. In ideal sliding mode, it is assumed that the switching frequency of the controller is infinite. However, due to the inertia of the actuator and the sensor and the presence of nonlinearity in the actual plant, switching in SMCs occurs at very high but finite frequency. As an effect, the sliding mode occurs in the small neighborhood of the sliding surface which is inversely proportional to the switching frequency of the controller. This high frequency switching in control is known as chattering phenomenon. The chattering phenomenon has serious harmful affects on the actuator as it leads to premature wear and tear or even breakdown of the system. A good number of methods have been proposed in the recent past to prevent chattering in the SMC [1,3,63–66]. One way to reduce chattering is to use the boundary layer technique where the sign function is approximated by a saturation function and the sliding function converges and remains within the sliding layer [10,63,64]. However, robustness of the controller has to be compromised in this method. However, the most recent and efficient solution to eliminate chattering in the control input is the higher order sliding mode control (HOSMC) methodology [1, 3, 65, 66]. The HOSMC retains the prime features of conventional SMCs and at the same time reduces chattering. For a $r-th$ order HOSMC, the derivatives of the $(r-1)-th$ control input are continuous everywhere except on the sliding surface. In HOSMC, not only the sliding manifold but its higher order derivatives also need to be zero [67–71]. However, the main difficulty in a higher order sliding mode is its high information demand. Among HOSMC methods, the second order sliding mode controller (SOSMC) [69, 72–76] is most widely accepted by the control community as it requires lesser information to design the controller. One recent approach of designing the second order sliding manifold is to combine two different sliding surfaces [77]. Commonly, a linear hyperplane is used to design the switching surface in the SMC. But linear sliding surface is generally unable to converge the system states to the equilibrium state in finite time. To achieve finite time convergence of the system states, the terminal sliding mode [78–80] was proposed. Because of assured finite time convergence, the terminal sliding mode controller has become increasingly popular among designers in recent years. However, conventional terminal sliding mode control is usually associated with the singularity problem. To overcome this problem, nonsingular terminal sliding mode control (NTSMC) was proposed [8,77,81]. The terminal sliding manifold has been combined with other sliding manifolds to develop a higher order sliding mode control which has features like robustness, finite time convergence and also reduces chattering in the control input.

1.2 Motivation

Literature reports that the OSMC is typically designed by combining the optimal controller with the conventional first order SMC. As a result, following drawbacks are observed:

(i) The OSMC is affected by chattering phenomenon.

(ii) The OSMC cannot tackle the mismatched uncertainty.

(iii) Bound of uncertainty should be known apriori to design the OSMC.

Above challenges associated with SMC design triggered continuous efforts among researchers for finding their solutions. To mitigate chattering in the conventional first order OSMC, effort is on to use continuous switching. Boundary layer method and the second order sliding mode controller (SOSMC) [74–76] are widely used for minimizing chattering in SMCs. In the boundary layer method, a saturation function is substituted for the sigmoid function used for switching in SMCs. This degrades the robustness which is the key feature of the SMC. The SOSMC has proved to be a better choice for designing OSMCs to mitigate chattering.

Uncertainty which is not in the range space of the input matrix of the system is known as the mismatched uncertainty. Conventional SMCs are not robust against mismatched uncertainties. However, research is continuing to design SMCs for controlling systems with mismatched uncertainties. Kwan [4] developed a SMC strategy for linear systems with mismatched uncertainty. Choi [82–84] proposed a linear matrix inequality (LMI) based SMC technique to handle mismatched uncertainty. Integral sliding mode control has also been used to handle mismatched uncertainty [56, 85, 86]. SMC based on output feedback was developed in [87, 88] for mitigating mismatched uncertainty. Recently, the disturbance observer [11, 89] was proposed to estimate the mismatched uncertainty affecting a nonlinear system and the estimation was used to design the SMC to stabilize the nonlinear system.

For designing SMCs, prior knowledge about the upper bound of the uncertainty is a necessary requirement which is not always available in practice. In absence of the advance knowledge about the upper bound of uncertainty, the SMC gain is generally chosen quite high resulting in a large control effort. On the other hand, if the SMC gain is chosen too small, the stability conditions may not be satisfied. To estimate the unknown upper bound of uncertainties, adaptive techniques were proposed using which a number of conventional and non-conventional control structures have been developed [64, 90–92]. Huang et al. [64] developed an adaptive SMC (ASMC) for the nonlinear system with uncertain parameters [93, 94]. Wai and Chang [90] proposed an ASMC using adaptive tuning approach to deal with unknown but bounded uncertainty. In [91] Wai proposed an ASMC where an adaptive technique was utilized to relax the requirement of the bound on the lumped uncertainty existing in the traditional sliding mode control. Plestan et al. [95] proposed an adaptive SMC where the adaptive gain value was not over estimated. Adaptive technique is also proposed for higher order sliding mode control [96–99]. In [92], Hu and Woo proposed a fuzzy supervisory sliding mode control combining fuzzy logic and neural network for controlling robotic manipulators.

Attempts continue to persist in finding solution to the above-mentioned difficulties because of their highly challenging nature. This thesis is also an attempt in that direction. The aim of this thesis is to design a chattering free optimal sliding mode controller (OSMC) for both linear and nonlinear uncertain systems. The controller is also improved to tackle matched and mismatched uncertainty.

1.3 Contributions of this Thesis

This thesis attempted to design a robust optimal control strategy with focus on control input minimization and immunity against system uncertainty caused by parametric change or external disturbance. As an outcome of this attempt, this thesis proposed the following:

I. First order optimal sliding mode controller for linear uncertain systems

The sliding mode is established if the switching gain is greater than upper bound of the matched uncertainty. Hence, Prior knowledge about the upper bound of the matched uncertainty is an essential prerequisite to design an OSMC. For designing an OSMC for linear systems having matched uncertainty with unknown upper bound, an optimal adaptive sliding mode controller (OASMC) is proposed here. To design the switching control, an adaptive tuning law is applied. The OASMC is designed for both stabilization and tracking applications of the uncertain system. It is a challenge to design a first order optimal sliding mode controller (OSMC) for linear systems affected by mismatched uncertainty. This thesis proposes disturbance observer based OSMC for linear systems affected by mismatched uncertainty. The proposed optimal control law is based on simple linear quadratic regulator (LQR) technique. The optimal controller is integrated with the SMC by designing an integral sliding surface which is designed by using the disturbance estimation. A disturbance observer is used to estimate the mismatched uncertainty. The proposed OSMC is applied for stabilizing linear systems affected by mismatched uncertainty.

II. Optimal second order sliding mode controller for linear uncertain systems

An optimal second order sliding mode control (OSOSMC) method is proposed for linear uncertain systems with an objective to minimize the control effort. The overall controller is obtained by integrating a linear quadratic regulator (LQR) based optimal control technique with the second order sliding mode controller. The optimal controller which minimizes the control input is imparted robustness against uncertainties by combining it with the SMC. The second order sliding mode strategy is realized by designing a non-singular terminal sliding mode control (NTSMC) based on the integral sliding variable. The NTSMC also guarantees finite time convergence of the proposed integral sliding variable and its first derivative. In this second order sliding mode control method, the discontinuous control input is derived by using the first derivative of the control input. Actual control input is obtained by integrating the discontinuous control input and hence becomes smooth and chattering free. The OSOSMC is first designed for single input single output (SISO) linear uncertain system and is further extended to decoupled multi input multi output (MIMO) linear uncertain system.

III. State dependent Riccati equation (SDRE) based optimal second order sliding mode controller for nonlinear uncertain systems

Designing an optimal controller for nonlinear uncertain systems is still a challenge. To find an optimal controller for nonlinear systems, the linear quadratic regulator (LQR) technique cannot be applied. Here, extended linearization is used to convert the nonlinear system into a linear like structure having state dependent coefficient (SDC) matrices. The control law for the nonlinear uncertain system is obtained in two steps. For the nominal part of the system, a state dependent Riccati equation (SDRE) based optimal controller is designed and for the uncertain part, a second order sliding mode control (SOSMC) strategy is used. The state dependent Riccati equation (SDRE) based optimal controller is designed similarly as linear quadratic regulator (LQR) based optimal control technique. However in this case system matrix and input distribution matrix are state dependent. Hence, The weighing matrices also chosen as state dependent matrix. To design OSOSMC, at first integral sliding variable is proposed to combine the optimal controller with SMC and then a NTSMC based on integral sliding variable is designed to develop the second order sliding mode methodology. The advantage of using nonlinear sliding surface like terminal sliding surface is that it converges the proposed integral sliding variable in finite time. The OSOSMC is developed for stabilization and trajectory tracking of nonlinear uncertain systems.

The chaotic system is a special case of nonlinear system exhibiting long term aperiodic behavior. Chaotic systems are highly sensitive to initial conditions of the system. The proposed state dependent Riccati equation (SDRE) based second order sliding mode controller is applied to stabilize certain chaotic systems.

IV. Control Lyapunov function (CLF) based optimal second order sliding mode controller for nonlinear uncertain systems

In practice, it is not possible to convert all nonlinear systems into linear like structure and design SDRE based optimal controller. To handle such cases, an optimal control strategy is developed by using the control Lyapunov function (CLF). For ensuring robustness of the designed optimal controller in presence of parametric uncertainty and external disturbances, a second order sliding mode control scheme is realized by designing a terminal sliding mode control based on an integral sliding variable. The resulting second order sliding mode can effectively reduce chattering in the control input.

It is difficult to design a SMC when a nonlinear system is affected by mismatched uncertainty. A disturbance observer based OSOSMC is proposed here for nonlinear systems affected by mismatched uncertainties at the minimum expense of control effort. At first, CLF based optimal controller is designed for the nominal part of the nonlinear system. A disturbance observer is designed to estimate the uncertainty. After designing the observer, an integral sliding variable is proposed based on the estimated values of uncertainty. To reduce the chattering phenomenon, the SOSMC is designed by developing a NTSMC based on integral sliding variable.

1.4 Thesis Organization

The thesis is organized as follows.

Chapter 2: This chapter presents a first order optimal sliding mode control method for linear systems affected by matched and mismatched uncertainties. The chapter contains two major sections. In the first section a first order optimal adaptive sliding mode controller is designed for linear systems affected by matched uncertainty where upper bound is unknown. The switching gain is adaptively tuned to design the sliding mode controller. Simulation studies demonstrate effectiveness of the proposed controller for both stabilization and trajectory tracking problems. In the second part a first order optimal sliding mode controller is proposed for linear systems affected by the mismatched uncertainty. The optimal controller is designed for the nominal part of the system and is based on linear quadratic regulator (LQR) technique. Then an integral sliding surface is designed based on the disturbance estimation. A disturbance observer is applied to estimate the mismatched uncertainty. From simulation result it can be concluded that the system states are bounded in spite of the presence of mismatched uncertainty.

Chapter 3: In this chapter an optimal second order sliding mode controller (OSOSMC) is proposed for linear uncertain systems. The optimal controller is designed for the nominal linear system using LQR technique. An integral sliding variable is designed to integrate the optimal controller with SMC to tackle the matched uncertainty affecting the linear system. As first order SMC contains undesired high frequency chattering in the control input, a second order sliding mode control methodology is proposed here by designing a non-singular terminal sliding mode control (NTSMC) based on the integral sliding variable. The chapter is divided into three sections. In the first section an OSOSMC is designed for the linear uncertain SISO system and applied for stabilization. In the second section, output tracking of a linear SISO system affected by the matched uncertainties is discussed. In the third section, the OSOSMC designed for the linear uncertain SISO system is extended to linear uncertain decoupled MIMO systems and applied for stabilization. Simulation results confirm that the proposed controller produces smother control input and requires lesser energy than some other existing methods.

Chapter 4: This chapter explains the state dependent Riccati equation (SDRE) based optimal second order sliding mode control (OSOSMC) strategy proposed for nonlinear uncertain systems and studies its utility when applied for controlling a chaotic system. To design the SDRE based optimal second order sliding mode controller for a nonlinear uncertain system, it should have linear like structure. Using extended linearization certain nonlinear systems can be represented as linear like structures having state dependent coefficient (SDC) matrices. The SDRE based optimal second order sliding mode controller is designed for those systems. The optimal controller is proposed for the nominal nonlinear system and then combined with a sliding mode controller by designing an integral sliding variable. A second order sliding mode control methodology is realized by designing a NTSMC based on integral sliding variable. The proposed OSOSMC is applied to stabilize the chaotic system which is a classic example of highly unstable nonlinear system. Some chaotic systems can be represented as linear like structures for stabilizing which the proposed SDRE based optimal second order sliding mode controller is successfully applied. Simulation results confirm effectiveness of the proposed controller.

Chapter 5: In this chapter a control Lyapunov function (CLF) based optimal second order sliding mode control is proposed for nonlinear systems affected by matched and mismatched uncertainties. The chapter is divided into two main sections. In the first section the CLF based OSOSMC is designed for nonlinear systems affected by the matched uncertainty using the integral sliding variable based NTSMC in a similar manner as followed in previous sections. In the next section the CLF based OSOSMC is proposed for the nonlinear system affected by the mismatched uncertainty which is estimated by using a disturbance observer. Extensive simulation study is conducted to validate performance of the proposed optimal second order sliding mode control strategy.

Chapter 6: In this chapter conclusions are drawn based on the research work done and suggestions for future directions of work in this area are outlined.

2

Optimal first order sliding mode controller for linear uncertain systems

Contents

2.1 Introduction

In the field of robust control, sliding mode control (SMC) [33,34,47] is highly appreciated due to its inherent robust behavior towards the matched uncertainty [46], which is in the range space of the input distribution matrix. Though the conventional SMC is strongly robust, it requires a high control input as the gain of the controller needs to be higher than the upper bound of the uncertainty. But a high control gain is undesirable as it may saturate the actuator and the cost of the controller also becomes high. So, an optimal sliding mode controller (OSMC) [49–53] has been developed to tackle uncertain systems with minimum expense of control energy. To develop an OSMC, the optimal controller is incorporated with the integral SMC. However, the OSMC suffers from the following drawbacks:

- The bound of the matched uncertainty should be known apriori.
- It cannot tackle the mismatched uncertainty.

To design the switching control of sliding mode, upper bound of the matched uncertainly should be known a priori. If the upper bound of the matched uncertainty is not known in advance, a popular choice is to choose high gain switching which, however, increases the control energy. To know the upper bound of the matched uncertainty, different adaptive tuning methods [64, 90–92] have been proposed in the literature. But many a times the bound of the matched uncertainty is over estimated, which is not desirable. In [95] Plestan et al. proposed an adaptive SMC to solve the problem of over estimation. Another drawback of the conventional SMC is that it cannot tackle the mismatched uncertainty. In recent literature, a few methods have been proposed to design sliding mode controllers for systems affected by mismatched uncertainties. Some such methods are the output feedback sliding mode controller [87, 88] and backstepping sliding mode controller [27, 28] which have been designed to handle mismatched uncertain systems. For linear systems affected by the norm bounded mismatched uncertainty, the sliding surface was designed by solving the linear inequality matrix in [82–84]. Observer based SMC has recently been proposed [11, 89] to tackle the mismatched uncertainty affecting the linear system. In [11, 89] a disturbance observer has been used to estimate the mismatched uncertainty and design the sliding surface based on the estimation.

In this chapter two design approaches are proposed for uncertain linear systems using first order optimal sliding mode controller. In the first design approach, a first order optimal adaptive sliding mode controller (OASMC) is proposed for the uncertain linear system affected by the matched uncertainty whose upper bound is unknown. The optimal controller is designed for the nominal linear system by using the LQR technique. Then an integral sliding mode controller is combined with the optimal controller. As the upper bound of the matched uncertainty is not known, an adaptive law is used to estimate the upper bound of the uncertainty which is needed for designing the switching control. In the second method a first order optimal sliding mode controller (OSMC) is proposed for linear systems affected by the mismatched uncertainty. The optimal controller is designed by using the well established linear quadratic regulator (LQR) technique for the nominal linear system. Then a disturbance observer is used to estimate the mismatched uncertainty and based on the estimation, an integral sliding surface based sliding mode controller is integrated with the optimal controller.

The outline of this chapter is as follows. In Section 2.2 a first order optimal adaptive sliding mode controller (OASMC) is developed for the linear system affected by the matched uncertainty whose upper bound is unknown. The proposed OASMC is applied for stabilization and tracking problems. Simulation results confirm satisfactory performance of the proposed controller. In Section 2.3 a first order optimal sliding mode controller is proposed for the linear system affected by the mismatched uncertainty. The designing procedure of the optimal controller using the LQR technique and the sliding surface design based on the disturbance observer method are elaborated here. The proposed controller is applied for stabilization problem. Simulation results validate the effectiveness of the proposed controller.

2.2 First order optimal adaptive sliding mode controller for the linear uncertain system

In this section a first order optimal adaptive sliding mode controller (OASMC) is proposed for the linear system affected by the matched uncertainty whose upper bound is unknown. LQR technique is used to design the optimal controller for the nominal linear system and an integral sliding mode controller is combined with the optimal controller to impart robustness. An adaptive tuning law is used here to design the sliding mode controller. The proposed OASMC is designed for both stabilization and tracking problems.

I. Stabilization problem

In stabilization, the system states are forced to converge to the equilibrium state. The proposed OASMC is designed to bring the system states to the equilibrium state using minimum control effort.

The design process is discussed in the following sections.

A. Problem statement

A linear uncertain system is defined as

$$\begin{aligned}
\dot{x}_i(t) &= x_{i+1}(t), \quad i = 1, 2, \cdots, n-1 \\
\dot{x}_n(t) &= a_1 x_1(t) + a_2 x_2(t) + \cdots + a_n x_n(t) + b_1 u(t) + \Delta a x(t) + \Delta b_1 u(t) + \omega_1(t) \\
y(t) &= x_1(t)
\end{aligned} \tag{2.1}$$

where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ is the state vector, $a_1, a_2, \cdots, a_n,\ b_1$ are known integers and the perturbation in system is defined as $\Delta a = [\Delta a_1 \ \ \Delta a_2 \ \ \cdots \ \ \Delta a_n]$ and Δb_1. The disturbance of the system is defined as $\omega_1(t)$. Uncertain part of the system is bounded and satisfies the matched condition but the bound of the uncertainty is unknown. The output of the system is denoted by $y(t)$.

The linear uncertain system (2.1) can be written as

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + \Delta A x(t) + \Delta B u(t) + \omega(t) \\
y(t) &= x_1(t)
\end{aligned} \tag{2.2}$$

$$\text{where } A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ddots & 1 \\ a_1 & a_2 & \cdots & a_n \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_1 \end{bmatrix}$$

.

Uncertainties affecting the system are defined as

$$\Delta A = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ddots & 0 \\ \Delta a_1 & \Delta a_2 & \cdots & \Delta a_n \end{bmatrix}, \ \Delta B = \begin{bmatrix} 0 \\ 0 \\ \Delta b_1 \end{bmatrix}, \ \omega(t) = \begin{bmatrix} 0 \\ 0 \\ \omega_1(t) \end{bmatrix}$$

.

The objective is to design an optimal sliding mode controller for the above uncertain system to achieve stabilization and tracking at the expense of minimum control input. An adaptive tuning law is used to design a sliding mode controller with unknown upper bound of the matched uncertainty.

B. Optimal controller design

Optimal sliding mode control design is divided into two parts. In the first part optimal control for the nominal system is designed and in the second part the sliding surface as well as the switching control are designed. Hence the control input $u(t)$ in equation (2.2) is obtained as $u(t) = u_1(t) + u_2(t)$ where $u_1(t)$ is the optimal control applied to the nominal system and $u_2(t)$ is the sliding mode control to tackle uncertainties.

The optimal control law for the nominal system is designed by using conventional linear quadratic regulator (LQR) technique as discussed below.

Neglecting the uncertainties, the state equation of system (2.2) becomes

$$\dot{x}(t) = Ax(t) + Bu_1(t) \tag{2.3}$$

The performance index to be minimized for the optimal control is defined as

$$J = \int_0^\infty \left(x(t)^T Qx(t) + u_1(t)^T Ru_1(t)\right) \, dt \tag{2.4}$$

where $Q \in \mathcal{R}^{n\times n}$ and $R \in \mathcal{R}$ are positive definite weighing matrices. The optimal control law $u_1(t)$ is given by

$$u_1(t) = -R^{-1}B^T Px(t) = -Kx(t) \tag{2.5}$$

where $K = R^{-1}B^T P$ and P is a symmetric, positive definite matrix which is the solution of the algebraic Riccati equation

$$A^T P + PA + Q - PBR^{-1}B^T P = 0 \tag{2.6}$$

C. Adaptive sliding mode controller design

After designing the optimal controller, the uncertain system (2.2) can be defined as

$$\dot{x}(t) = Ax(t) - BKx(t) + Bu_2(t) + \Delta Ax(t) + \Delta Bu(t) + \omega(t) \tag{2.7}$$

To combine the optimal controller with a sliding mode controller (SMC), an integral sliding surface

$s(t)$ is designed as

$$s(t) = Cx(t) - Cx(0) - \int_0^t C[Ax(\tau) - BKx(\tau)]d\tau = 0 \tag{2.8}$$

where $C \in \mathcal{R}^{1\times n}$, $x(0)$ is the initial state and CB is considered as nonsingular. Here the upper bound of the system uncertainty is unknown. So, the controller gain is designed using the adaptive law proposed by Plestan et al. [95]. The adaptation law by Plestan et al. [95] guarantees a real sliding mode. The gain $\Psi(t)$ is found by using the following tuning law [95],

$$\dot{\Psi}(t) = \begin{cases} \Xi|s(t)|sign(|s(t)| - \epsilon), & \text{if } \Psi(t) > \nu; \\ \nu, & \text{if } \Psi(t) \leq \nu. \end{cases} \tag{2.9}$$

where $\Xi > 0$, $\epsilon > 0$ and $\nu > 0$ are very small with $\Psi(0) > 0$. Parameter ν is introduced to get only positive values for Ψ.

Hence, the control law $u_2(t)$ is designed as

$$u_2(t) = -(CB)^{-1}\Psi(t)sign(s(t)) \tag{2.10}$$

The gain $\Psi(t)$ has an upper bound meaning that there exists a positive constant $\widehat{\Psi}$ so that $\Psi(t) \leq \widehat{\Psi}, \ \forall t > 0$.

Stability analysis of the sliding surface

Let us consider the Lyapunov function $V_1(t)$ as

$$\begin{aligned} V_1(t) &= \frac{1}{2}s(t)^2 + \frac{1}{2\Upsilon}(\Psi(t) - \widehat{\Psi})^2 \quad \text{where } \Upsilon > 0 \\ \dot{V}_1(t) &= s(t)\dot{s}(t) + \frac{1}{\Upsilon}(\Psi(t) - \widehat{\Psi})\dot{\Psi}(t) \\ &= s(t)[CBu_2(t) + C(\Delta Ax(t) + \Delta Bu(t) + \omega(t))] + \frac{1}{\Upsilon}(\Psi(t) - \widehat{\Psi})\Xi|s(t)|sign(|s(t)| - \epsilon) \\ &= s(t)(-\Psi(t)sign(s(t)) + \zeta(x(t), u(t), \omega(t))) + \frac{1}{\Upsilon}(\Psi(t) - \widehat{\Psi})\Xi|s(t)|sign(|s(t)| - \epsilon) \end{aligned} \tag{2.11}$$

where $\zeta(x(t), u(t), \omega(t)) = C(\Delta A x(t) + \Delta B u(t) + \omega(t))$

$$
\begin{aligned}
\dot{V}_1(t) &\leq -\Psi(t)|s(t)| + \zeta(x(t), u(t), \omega(t))|s(t)| + \frac{1}{\Upsilon}(\Psi(t) - \widehat{\Psi})\Xi|s(t)|sign(|s(t)| - \epsilon) \\
&\leq -\Psi(t)|s(t)| + \zeta(x(t), u(t), \omega(t))|s(t)| + \widehat{\Psi}|s(t)| - \widehat{\Psi}|s(t)| \\
&\quad + \frac{1}{\Upsilon}(\Psi(t) - \widehat{\Psi})\Xi|s(t)|sign(|s(t)| - \epsilon) \\
&\leq (\zeta(x(t), u(t), \omega(t)) - \widehat{\Psi})|s(t)| + (\Psi(t) - \widehat{\Psi})(-|s(t)| + \frac{1}{\Upsilon}\Xi|s(t)|sign(|s(t)| - \epsilon)) \\
&\leq (\zeta(x(t), u(t), \omega(t)) - \widehat{\Psi})|s(t)| + (\Psi(t) - \widehat{\Psi})(-|s(t)| + \frac{1}{\Upsilon}\Xi|s(t)|sign(|st)| - \epsilon)) \\
&\quad + \beta_\Psi|\Psi(t) - \widehat{\Psi}| - \beta_\Psi|\Psi(t) - \widehat{\Psi}|
\end{aligned}
$$

where $\beta_\Psi > 0$

$$
\begin{aligned}
\dot{V}_1(t) &\leq -(-\zeta(x(t), u(t), \omega(t)) + \widehat{\Psi})|s(t)| - \beta_\Psi|\Psi(t) - \widehat{\Psi}| - |\Psi - \widehat{\Psi}|(-|s(t)| \\
&\quad + \frac{1}{\Upsilon}\Xi|s(t)|sign(|s(t)| - \epsilon) - \beta_\Psi) \\
&\leq -\beta_s|s(t)| - \beta_\Psi|\Psi(t) - \widehat{\Psi}| - \Gamma
\end{aligned} \tag{2.12}
$$

where

$$\beta_s = (-\zeta(x(t), u(t), \omega(t)) + \widehat{\Psi}) > 0 \tag{2.13}$$

and

$$\Gamma = |\Psi(t) - \widehat{\Psi}|(-|s(t)| + \frac{1}{\Upsilon}\Xi|s(t)|sign(|s(t)| - \epsilon) - \beta_\Psi) \tag{2.14}$$

Suppose $s(t) \neq 0$. From the dynamics of $\Psi(t)$ and for bounded uncertainty $\zeta(x(t), u(t), \omega(t))$, it follows that $\Psi(t)$ is increasing and there exists a time t^* such that

$$\widehat{\Psi} = \Psi(t^*) > \zeta(x(t^*), u(t^*), \omega(t^*)) \tag{2.15}$$

The proof is discussed in [95].

Hence,

$$\begin{aligned}
\dot{V}_1(t) &\leq -\beta_s\sqrt{2}\frac{|s(t)|}{\sqrt{2}} - \beta_\Psi\sqrt{2\Upsilon}\frac{|\Psi(t)-\widehat{\Psi}|}{\sqrt{2\Upsilon}} - \Gamma \\
&\leq -\min\{\beta_s\sqrt{2}, \beta_\Psi\sqrt{2\Upsilon}\}(\frac{|s(t)|}{\sqrt{2}} + \frac{|\Psi(t)-\widehat{\Psi}|}{\sqrt{2\Upsilon}}) - \Gamma \\
&\leq -\widehat{\beta}V_1^{\frac{1}{2}} - \Gamma
\end{aligned}$$

where $\widehat{\beta} = \sqrt{2}\min\{\beta_s, \beta_\Psi\sqrt{\Upsilon}\}$

Case 1. Suppose $|s(t)| > \epsilon$. Then from (2.14), it is found that Γ is positive if

$$-|s(t)| + \frac{1}{\Upsilon}\Xi|s(t)| - \beta_\Psi > 0$$

or,

$$\Upsilon < \frac{\Xi|s(t)|}{|s(t)| + \beta_\Psi} \tag{2.16}$$

It is always possible to choose Υ such that the inequality condition (2.16) is satisfied. Hence, it can be written that

$$\begin{aligned}
\dot{V}_1 &\leq -\widehat{\beta}V_1^{\frac{1}{2}} - \Gamma \\
&\leq -\widehat{\beta}V_1^{\frac{1}{2}}
\end{aligned}$$

Therefore, finite time convergence of $s(t)$ to a domain $|s(t)| \leq \epsilon$ is guaranteed.

Case 2. Suppose $|s(t)| < \epsilon$. So, Γ can be negative. It signifies that $\dot{V}_1(t)$ would be sign indefinite and it is impossible to conclude about the stability. Therefore, $|s(t)|$ can increase over ϵ. As soon as $|s(t)|$ becomes greater than ϵ, $\dot{V}_1(t) \leq -\widehat{\beta}V_1^{\frac{1}{2}}$ and $V_1(t)$ starts decreasing.

II. Tracking problem

In a tracking problem, the system response is made to follow certain desired trajectory. The approach here is to convert the system dynamic model into an error dynamic model in error coordinates. Then the controller is designed in the error domain with an aim to converge the error to zero. Thus the tracking control problem is easily converted to a regulatory control problem. Here an optimal adaptive sliding mode controller (OASMC) is designed to track the desired system trajectory at the expense of minimum control effort. The output $y(t) = x_1(t)$ is made to follow the desired trajectory

$x_{d1}(t)$. The tracking error $e(t)$ can be defined as

$$e(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_n(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} - \begin{bmatrix} x_{d1}(t) \\ x_{d1}^{(1)}(t) \\ \vdots \\ x_{d1}^{(n-1)}(t) \end{bmatrix} \tag{2.17}$$

where $x_{d1}^{(1)}(t), x_{d1}^{(2)}(t), \cdots x_{d1}^{(n-1)}(t)$ are the first and successive time derivatives of $x_{d1}(t)$. Hence, system (2.2) can be expressed in the error domain as follows,

$$\dot{e}(t) = \begin{bmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \\ \vdots \\ \dot{e}_n(t) \end{bmatrix} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} - \begin{bmatrix} x_{d1}^{(1)}(t) \\ x_{d1}^{(2)}(t) \\ \vdots \\ x_{d1}^{(n)}(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ x_3(t) \\ \vdots \\ a_1x_1(t) + a_2x_2(t) \cdots + a_nx_n(t) + b_1u(t) \end{bmatrix} - \begin{bmatrix} x_{d1}^{(1)}(t) \\ x_{d1}^{(2)}(t) \\ \vdots \\ x_{d1}^{(n)}(t) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ddots & 0 \\ \Delta a_1 & \Delta a_2 & \cdots & \Delta a_n \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \Delta b_1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ \omega_1(t) \end{bmatrix}$$

$$= \begin{bmatrix} x_2(t) \\ x_3(t) \\ \vdots \\ a_1x_1(t) + a_2x_2(t) \cdots + a_nx_n(t) \end{bmatrix} - \begin{bmatrix} x_{d1}^{(1)}(t) \\ x_{d1}^{(2)}(t) \\ \vdots \\ a_1x_{d1}(t) + a_2x_{d1}^{(1)}(t) + \cdots + a_nx_{d1}^{(n-1)}(t) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ a_1x_{d1}(t) + a_2x_{d1}^{(1)}(t) + \cdots + a_nx_{d1}^{(n-1)}(t) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ x_{d1}^{(n)}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_1 \end{bmatrix} u(t)$$

$$+ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ddots & 0 \\ \Delta a_1 & \Delta a_2 & \cdots & \Delta a_n \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \Delta b_1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ \omega_1(t) \end{bmatrix}$$

$$= \begin{bmatrix} e_2(t) \\ e_3(t) \\ \vdots \\ a_1e_1(t) + a_2e_2(t) \cdots + a_ne_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ a_1x_{d1}(t) + a_2x_{d1}^{(1)}(t) + \cdots + a_nx_{d1}^{(n-1)}(t) - x_{d1}^{(n)}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_1 \end{bmatrix} u(t)$$

$$+ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ddots & 0 \\ \Delta a_1 & \Delta a_2 & \cdots & \Delta a_n \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \Delta b_1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ \omega_1(t) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ a_1 & \cdots & a_{n-1} & a_n \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ a_1x_{d1}(t) + a_2x_{d1}^{(1)}(t) \cdots + a_nx_{d1}^{(n-1)}(t) - x_{d1}^{(n)}(t) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ddots & 0 \\ \Delta a_1 & \Delta a_2 & \cdots & \Delta a_n \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \Delta b_1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ \omega_1(t) \end{bmatrix}$$

$$= Ae(t) + Bu(t) + \widetilde{A}(t) + \Delta Ax(t) + \Delta Bu(t) + \omega(t) \tag{2.18}$$

Here $\widetilde{A}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ a_1x_{d1}(t) + a_2x_{d1}^{(1)}(t) + \cdots + a_nx_{d1}^{(n-1)}(t) - x_{d1}^{(n)}(t) \end{bmatrix}$ is a known matrix and it satisfies the matching condition. So, it will be taken care while designing the sliding mode controller by combining with the uncertainty.

The control input $u(t)$ in (2.18) is obtained as $u(t) = u_1(t) + u_2(t)$ where $u_1(t)$ is the optimal control applied to the nominal system and $u_2(t)$ is the sliding mode control to tackle uncertainties.

The optimal control law $u_1(t)$ is designed by using the LQR technique for the nominal system. Hence, neglecting the uncertainties, (2.18) takes the form

$$\dot{e}(t) = Ae(t) + Bu_1(t) \tag{2.19}$$

The performance index to be minimized for the optimal control is defined as

$$J = \int_0^{\infty} (e(t)^T Q e(t) + u_1(t)^T R u_1(t)) dt \tag{2.20}$$

where $Q \in \mathcal{R}^{n\times n}$ and $R \in \mathcal{R}$ are positive definite weighing matrices.

The optimal control law $u_1(t)$ is given by

$$u_1(t) = -R^{-1}B^T P e(t) = -Ke(t) \tag{2.21}$$

where $K = R^{-1}B^T P$ and P is a symmetric positive definite matrix which is the solution of the algebraic Riccati equation

$$A^T P + PA + Q - PBR^{-1}B^T P = 0 \tag{2.22}$$

Now for the uncertain system (2.18), an integral sliding surface is designed as

$$s(t) = Ce(t) - Ce(0) - \int_0^t C[Ae(\tau) + Bu_1(\tau)]d\tau = 0 \tag{2.23}$$

where $C \in \mathcal{R}^{1\times n}$ and CB is nonsingular. The sliding mode control $u_2(t)$ is designed as

$$u_2(t) = -(CB)^{-1}\Psi(t) sign(s(t)) \tag{2.24}$$

where $\Psi(t)$ is adaptively tuned controller gain [95]. The tuning law used for stabilization and discussed in the previous section is also applied for the tracking problem.

III. Simulation Results

Let us consider the following mathematical model of a mass-spring-damper system,

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -5 & -0.5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0.5\sin t \end{bmatrix} \\ y(t) = x_1(t) \end{aligned} \tag{2.25}$$

Here, state vector $x(t) = [x_1(t) \quad x_2(t)]^T$ where $x_1(t)$ is the position of the mass and $x_2(t)$ is the velocity of the mass. The control input $u(t)$ is the force applied to the system. System matrix $A = \begin{bmatrix} 0 & 1 \\ -5 & -0.5 \end{bmatrix}$, input matrix $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and the uncertain part of the system is defined as $\zeta(x(t), u(t), \omega(t)) = \begin{bmatrix} 0 \\ 0.5\sin t \end{bmatrix}$. The initial condition is $x(0) = [1 \quad 2]^T$

The proposed first order optimal adaptive sliding mode controller (OASMC) is now used for stabilization of the above linear uncertain system (2.25).

For the stabilization problem, the weighing matrices Q and R in (2.4) are chosen as follows,

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, R = 1.$$

The state feedback gain matrix K in (2.5) for the given system is found as

$$K = \begin{bmatrix} 0.0990 & 0.7033 \end{bmatrix}.$$

In the first order OASMC (2.8), the value of C is chosen as $[2 \;\; 1]$. Parameters in the gain adaptation law (2.9) are chosen as $\Psi(0) = 0.9, \;\; \Xi = 0.5, \;\; \epsilon = 0.8, \;\; \nu = .08$.

The performance of the proposed OASMC is compared with that of a conventional SMC. The conventional sliding surface $\theta(t) = 0$ is designed as

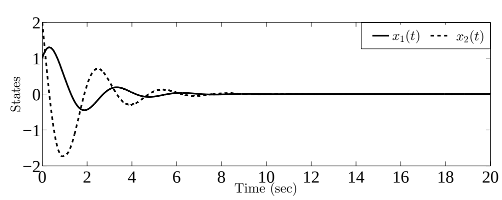
$$\theta(t) = Cx(t) = [2 \quad 1]x(t) = 0 \tag{2.26}$$

and the sliding mode control law $u(t)$ is chosen as

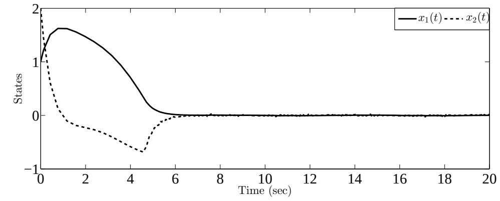
$$u(t) = \begin{cases} -(CB)^{-1}CAx(t), & \theta(t) = 0 \; ; \\ -(CB)^{-1}sign(\theta(t)), & \theta(t) \neq 0. \end{cases} \tag{2.27}$$

The performance of the proposed OASMC is also compared with that of the integral SMC proposed by Laghrouche et al. [1] which is described in Appendix A.1. The states x_1 and x_2 obtained by using the proposed OASMC, the conventional SMC and the integral SMC proposed by Laghrouche et al. [1] are shown in Figure 2.1. The control inputs obtained by using these three controllers are shown in Figure 2.2. It is observed in these plots that although the convergence time taken by the states to reach the origin is almost the same for all the controllers, the proposed OASMC spends lesser control input compared to the other two controllers.

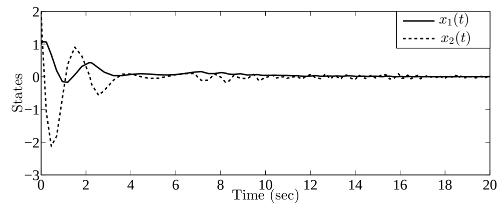
Now the output tracking problem for the same system (2.25) is considered. The desired output for tracking is chosen as $x_{d1}(t) = 0.5\sin(\sqrt{5}t)$. The system can be defined in the error domain as



(a) Proposed OASMC

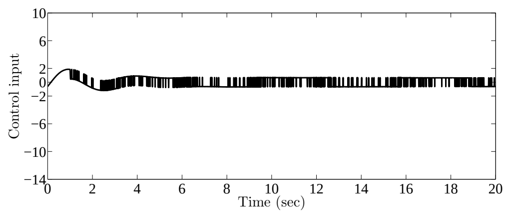


(b) Conventional SMC

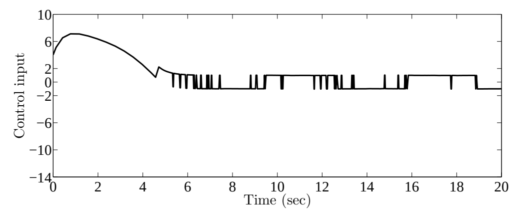


(c) Integral SMC proposed by Laghrouche et al. [1]

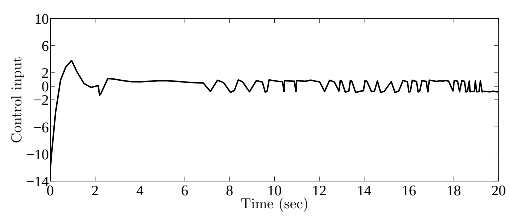
Figure 2.1: States obtained by applying the proposed OASMC, conventional SMC and integral SMC proposed by Laghrouche et al. [1] to stabilize the linear uncertain system



(a) Proposed OASMC



(b) Conventional SMC



(c) Integral SMC proposed by Laghrouche et al. [1]

Figure 2.2: Control input obtained by applying the proposed OASMC, conventional SMC and integral SMC proposed by Laghrouche et al. [1] to stabilize the linear uncertain system

$$\dot{e}(t) = \begin{bmatrix} 0 & 1 \\ -5 & -0.5 \end{bmatrix} e(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0.5\sin t \end{bmatrix} + \begin{bmatrix} 0 \\ -5x_{d1}(t) - 0.5x_{d1}^{(1)}(t) - x_{d1}^{(2)}(t) \end{bmatrix} \tag{2.28}$$

where $\widetilde{A}(t) = \begin{bmatrix} 0 \\ -5x_{d1}(t) - 0.5x_{d1}^{(1)}(t) - x_{d1}^{(2)}(t) \end{bmatrix}$

For the proposed OASMC, weighing matrices Q and R in (2.20) are chosen as follows:

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; R = 1.$$

The state feedback gain matrix K in (2.21) for the above controller is found as

$$K = \begin{bmatrix} 0.0990 & 0.7033 \end{bmatrix}.$$

In the first order OASMC (2.23), the value of C is chosen as $[2 \quad 1]$ and parameters in the gain adaptation law (2.9) are chosen as $\Psi(0) = 1.25,\ \Xi = 0.5,\ \epsilon = 0.8,\ \nu = .08$.

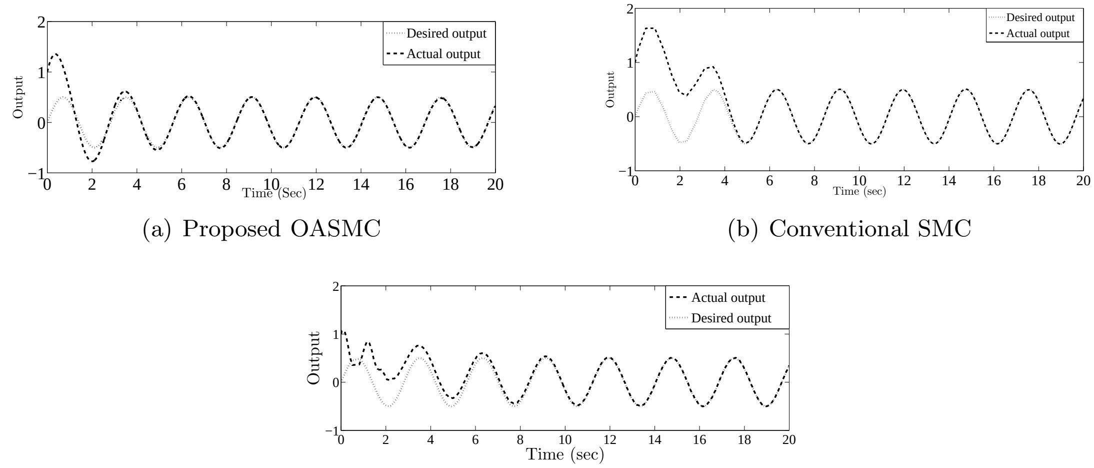
The proposed OASMC is compared with the conventional SMC whose sliding surface $\theta(t) = 0$ is designed as

$$\theta(t) = Ce(t) = [2 \quad 1]e(t) = 0 \tag{2.29}$$

and the sliding mode control law is chosen as

$$u(t) = \begin{cases} -(CB)^{-1}(CAe(t) + C\widetilde{A}(t)), & \theta(t) = 0\ ; \\ -(CB)^{-1}sign(\theta(t)), & \theta \neq 0. \end{cases} \tag{2.30}$$

The performance of the proposed OASMC is next compared with that of the integral SMC proposed by Laghrouche et al. [1] discussed in Appendix A.1. Figure 2.3 shows the state x_1 obtained by using the proposed OASMC, the conventional SMC and the integral SMC proposed by Laghrouche et al. [1] for the tracking problem. The control inputs obtained by using these controllers are shown in Figure 2.4. From these figures it is evident that the proposed OASMC uses lesser control input compared to the other two controllers while offering similar convergence speed of the state. Tables 2.1 and 2.2 compare the control energies by computing the second norm of the control input till 20 sec for both the stabilization and tracking problem applications. It is clear from Tables 2.1 - 2.2 that the proposed OASMC is able to reduce the control effort for comparable performance standard.

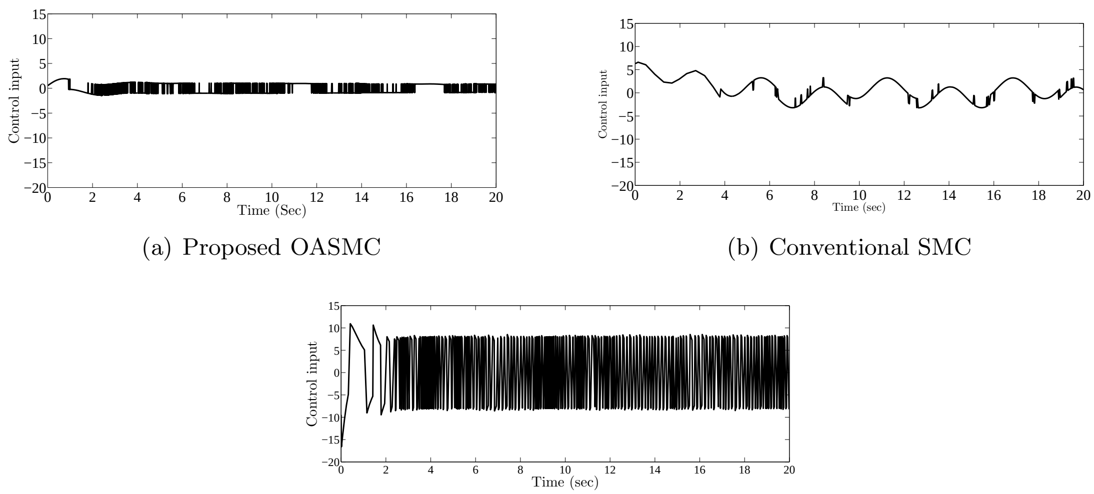


(a) Proposed OASMC

(b) Conventional SMC

(c) Integral SMC proposed by Laghrouche et al. [1]

Figure 2.3: State $x_1(t)$ obtained by applying the proposed OASMC, conventional SMC and integral SMC proposed by Laghrouche et al. [1] for tracking the linear uncertain system



(a) Proposed OASMC

(b) Conventional SMC

(c) Integral SMC proposed by Laghrouche et al. [1]

Figure 2.4: Control input obtained by applying the proposed OASMC, conventional SMC and integral SMC proposed by Laghrouche et al. [1] for tracking the linear uncertain system

Table 2.1: Comparison of control energy of conventional SMC, integral SMC proposed by Laghrouche et al. [1] and the proposed OASMC for the stabilization problem

Method	Control Energy
Conventional SMC	41.81
Integral SMC proposed by Laghrouche et al. [1]	22.59
Proposed OASMC	10.41

Table 2.2: Comparison of control energy of conventional SMC, integral SMC proposed by Laghrouche et al. [1] and the proposed OASMC for the tracking problem

Method	Control Energy
Conventional SMC	33.60
Integral SMC proposed by Laghrouche et al. [1]	25.56
Proposed OASMC	14.02

2.3 Disturbance observer based first order optimal sliding mode controller

A first order optimal sliding mode controller is proposed for the linear system affected by mismatched uncertainty. The optimal controller is designed for the nominal nonlinear system using the LQR technique as discussed in the previous section. As the sliding mode controller (SMC) cannot tackle the mismatched uncertainty, it is estimated by using a disturbance observer. A first order sliding mode methodology is proposed based on an integral sliding surface.

I. Problem statement

A nonlinear system with mismatched uncertainty is considered as given below:

$$\dot{x}(t) = Ax(t) + Bu(t) + gd(t) \tag{2.31}$$

where $x(t) \in \mathcal{R}^n$ is the state vector and $u(t) \in \mathcal{R}$ is the control input. Further A, B are the system matrix and input distribution matrix respectively. Moreover, $d(t)$ represents the mismatched uncertainty affecting the system as matrix g is not in the range space of B.

Assumption: The disturbance $d(t)$ is unknown but bounded and $\dot{d}(t) = 0$.

The objective is to design a sliding mode controller for the system (2.31) affected by the mismatched uncertainty with minimum expense of control input. The control input $u(t)$ is divided into two parts. At first, the optimal controller $u_1(t)$ is designed for the nominal part of the system using LQR technique. In the second part, the disturbance observer based sliding mode controller $u_2(t)$ is designed to tackle the mismatched uncertainty.

II. Optimal controller design

Neglecting the uncertainty, (2.31) takes the form

$$\dot{x}(t) = Ax(t) + Bu_1(t) \tag{2.32}$$

The performance index to be minimized for the optimal control is defined as

$$J = \int_0^\infty \left(x(t)^T Q x(t) + u_1(t)^T R u_1(t) \right) dt \tag{2.33}$$

where $Q \in \mathcal{R}^{n\times n}$ and $R \in \mathcal{R}$ are positive definite weighing matrices.

The optimal control law $u_1(t)$ is constructed as

$$u_1(t) = -R^{-1}B^T P x(t) = -Kx(t) \tag{2.34}$$

where $K = R^{-1}B^T P$ and P is a symmetric, positive definite matrix which is the solution of the algebraic Riccati equation [100]

$$A^T P + PA + Q - PBR^{-1}B^T P = 0 \tag{2.35}$$

III. Sliding mode controller based on disturbance observer

An integral sliding mode controller (ISMC) is combined with the optimal controller designed above to impart robustness. However, a conventional ISMC cannot tackle the mismatched uncertainty. So, an integral sliding surface is designed based on disturbance estimation by using a nonlinear disturbance observer (DOB) [11] defined as follows:

$$\begin{aligned} \dot{p}(t) &= -\rho g p(t) - \rho[g\rho x(t) + Ax(t) + Bu(t)] \\ \widehat{d}(t) &= p(t) + \rho x(t) \end{aligned} \tag{2.36}$$

where $\widehat{d}(t)$ denotes the estimation of the disturbance $d(t)$. Further, $p(t)$, ρ represent the internal state of the nonlinear disturbance observer and the observer gain respectively. Here ρ is chosen such that ρg becomes positive definite. It implies that

$$\dot{e}_d(t) + \rho g e_d(t) = 0 \tag{2.37}$$

is asymptotically stable, where $e_d(t) = \widehat{d}(t) - d(t)$.

or,

$$\lim_{t\to\infty} e_d(t) = 0 \tag{2.38}$$

The integral sliding surface $s(t) = 0$ is chosen as follows:

$$s(t) = C[x(t) - x(0) - \int_0^t \dot{\Phi}(\tau)d\tau + g\widehat{d}(t)] = 0 \tag{2.39}$$

where C is a design parameter which is so chosen such that CB is invertible and $x(0)$ is the initial state vector. Moreover, $\dot{\Phi}(t)$ is defined as

$$\dot{\Phi}(t) = Ax(t) + Bu_1(t) \tag{2.40}$$

The sliding mode control is obtained as

$$u_2(t) = -(CB)^{-1}(\eta sign(s(t)) + Cg\widehat{d}(t)) \tag{2.41}$$

where $\eta \geq 0$.

<u>Stability analysis of the sliding surface</u>

The Lyapunov function is defined as

$$\begin{aligned}
V_1(t) &= \frac{1}{2}s^2(t) \\
\dot{V}_1(t) &= s(t)\dot{s}(t) \\
&= s(t)(CAx(t) + CBu_1(t) + CBu_2(t) + Cgd(t) \\
&\quad -CAx(t) - CBu_1(t) + Cg\dot{\widehat{d}}(t)) \\
&= s(t)(CBu_2(t) + Cgd(t) + Cg\dot{\widehat{d}}(t)) \\
&= s(t)(-\eta sign(s(t)) - Cg\widehat{d}(t) + Cgd(t) + Cg\dot{\widehat{d}}(t)) \\
&= s(t)(-\eta sign(s(t)) + Cg\dot{\widehat{d}}(t) - Cg(\widehat{d}(t) - d(t))) \\
&\leq -\eta|s(t)| + Cg\dot{\widehat{d}}(t)|s(t)| - Cg(\widehat{d}(t) - d(t))|s(t)| \\
&\leq -\eta|s(t)| + Cg\dot{\widehat{d}}(t)|s(t)| - Cge_d(t)|s(t)|
\end{aligned} \tag{2.42}$$

Now form (2.36), it can be found that $\dot{\hat{d}}(t) = -\rho g e_d(t)$.

Hence, the above equation can be written as

$$\begin{aligned}
\dot{V}_1(t) &\leq -\eta|s(t)| - Cg\rho g e_d(t)|s(t)| - Cge_d(t)|s(t)| \\
&\leq -\eta|s(t)| - |Cge_d(t)||s(t)|(\rho g + 1)
\end{aligned} \tag{2.43}$$

The designed parameter ρ is chosen such a way that $\rho g > 0$ then $\eta|s(t)| + |Cge_d(t)||s(t)|(\rho g + 1) = \chi > 0$. Hence,

$$\begin{aligned}
\dot{V}_1(t) &\leq -\chi|s(t)| \\
\dot{V}_1(t) &\leq -\chi|\sqrt{2V_1(t)}| \quad \text{as} \quad V_1(t) = \frac{1}{2}s^2(t)
\end{aligned} \tag{2.44}$$

Therefore, finite time stability [101] of the sliding surface is guaranteed.

IV. Simulation Results

Stabilization problem of a linear system affected by the mismatched uncertainty is considered as given below:

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) + d(t) \\
\dot{x}_2(t) &= x_3(t) \\
\dot{x}_3(t) &= u(t)
\end{aligned} \tag{2.45}$$

where mismatched disturbance $d(t) = 0.6$ is applied after 6 sec and initial state $x(0) = [1 \ 0 \ -1]^T$. To minimize the control input, performance index is chosen as

$$J = \int_0^\infty \left(x^T(t) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t) + u_1^2(t) \right) dt \tag{2.46}$$

Using the LQR technique, the feedback control $u_1(t)$ is found as

$$u_1(t) = \begin{bmatrix} 3.1623 & 6.3333 & 4.7609 \end{bmatrix} x(t) \tag{2.47}$$

Design parameters of the proposed disturbance observer based first order optimal sliding mode controller are chosen as follows:

$C = [2 \quad 3 \quad 1]$, $\rho = [15 \quad 0 \quad 0]$ and $\eta = 0.6$.

The performance of the proposed disturbance observer based OSMC is compared with that of a disturbance observer based conventional SMC. The conventional sliding surface $\theta(t) = 0$ is designed as

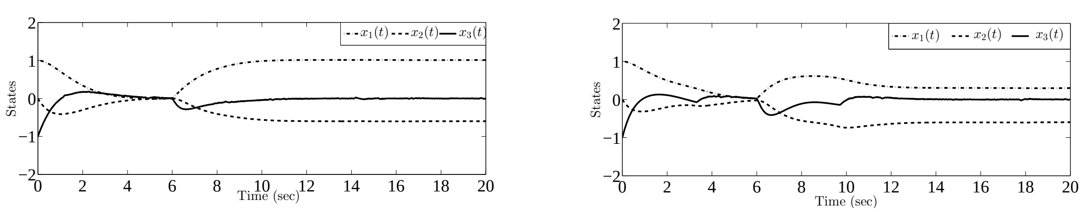
$$\theta(t) = C(x(t) + gd(t)) = [2 \quad 3 \quad 1](x(t) + [1 \quad 0 \quad 0]^T d(t)) = 0 \tag{2.48}$$

and the sliding mode control law $u(t)$ is chosen as

$$u(t) = -(CB)^{-1}[CAx(t) + 0.6sign(\theta(t)) + g\dot{\hat{d}}(t)] \tag{2.49}$$

Disturbance observer for the conventional SMC is designed similarly as designed for the proposed disturbance observer based OSMC.

The states obtained by using the proposed disturbance observer based OSMC and the disturbance observer based SMC are shown in Figure 2.5. The control inputs obtained by using these controllers are shown in Figure 2.6. It is observed in these plots that although the convergence time taken by the states to reach the origin is almost the same for both the controllers, the proposed disturbance observer based OSMC spends lesser control input compared to the disturbance observer based SMC. However, it is observed that in the case of the proposed OSMC, the steady-state errors for $x_1(t)$ and $x_2(t)$ are little higher that those of the conventional SMC.

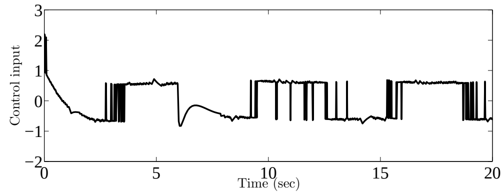


(a) Proposed disturbance observer based OSMC

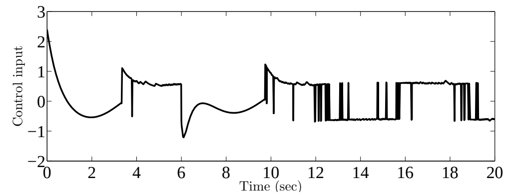
(b) Disturbance observer based SMC

Figure 2.5: States obtained by applying the proposed disturbance observer based OSMC and disturbance observer based SMC

Table 2.3 compares the control indices of both the proposed OSMC and SMC in terms of total variation (TV) and control energy.



(a) Proposed disturbance observer based OSMC



(b) Disturbance observer based SMC

Figure 2.6: Control input obtained by applying the proposed disturbance observer based OSMC and disturbance observer based SMC

Table 2.3: Comparison of control indices of the proposed disturbance observer based OSMC and disturbance observer based SMC

Method	Total Variation (TV)	Control Energy
Disturbance observer based SMC	30.51	8.69
Proposed disturbance observer based OSMC	29.73	8.21

2.4 Summary

In this chapter, initially a first order optimal adaptive sliding mode controller (OASMC) is designed for the linear system affected by matched uncertainty with unknown upper bound. Next, a disturbance observer based first order optimal sliding mode controller is proposed for the linear system affected by the mismatched uncertainty. In both these cases, the optimal controller is designed by using the LQR technique for the nominal linear system and then an integral sliding mode controller is combined. An adaptive gain tuning method is used to tackle the unknown upper bound of the matched uncertainty for designing the OASMC. The proposed OASMC is applied for stabilization and trajectory tracking problems. Compared to conventional SMCs and the integral SMC proposed by Laghrouche et al. [1], the proposed OASMC spends lower control energy while maintaining similar performance standard. In the second part of the work, a disturbance observer is used to estimate the mismatched uncertainty and based on the estimation, the integral sliding mode controller is designed. The proposed controller is applied for the stabilization problem and its performance is compared with a disturbance observer based conventional sliding mode controller. From simulation results it is observed that the proposed optimal SMC requires lesser control effort while offering almost similar performance as that of the conventional SMC.

3

Optimal second order sliding mode controller (OSOSMC) for linear uncertain systems

Contents

3.1 Introduction

In order to minimize the control effort required to control linear uncertain systems, optimal sliding mode control (OSMC) technique [49–53] has evolved. In OSMC, the optimal control method is integrated with a first order sliding mode controller (SMC) by designing an integral sliding surface. However, the main drawback of the OSMC is the high frequency chattering which is inherent in the sliding mode. Chattering occurs in the SMC due to the sign function in the switching control. To reduce it, boundary layer method [10,63,64] has been used where the discontinuous function is replaced by a continuous approximation of the sign function. Drawback of this technique is that the robustness of the sliding mode controller has to be partially compromised. A more effective method to reduce chattering in the control input has been to use higher order sliding mode controllers (HOSMC) [1,3,65,66]. HOSMC comprises all the advantages of the conventional SMC while at the same time it mitigates chattering. Among HOSMC methods, second order sliding mode controller (SOSMC) [74–76] is widely used because of its low information demand.

In this chapter, an optimal second order sliding mode controller (OSOSMC) is proposed for controlling linear systems affected by matched uncertainties. An optimal controller [102] is designed for the nominal part of the system by using the well established linear quadratic regulator (LQR) [12] technique. To tackle the uncertainty, a second order sliding mode controller (SOSMC) is integrated with the optimal controller by designing an integral sliding surface. The SOSMC strategy is developed by designing a terminal sliding surface [77, 79] based on an integral sliding variable. The proposed SOSMC mitigates chattering in the control input and also converges the sliding variables in finite time. Apart from stabilization, the proposed OSOSMC is also applied for tracking problem where the system is converted into the error domain. The OSOSMC is designed to converge the error to zero using minimum control effort. The optimal sliding mode controller designed for the single input single output (SISO) system is also extended for the decoupled multi input multi output (MIMO) system. To demonstrate the effectiveness of the proposed controller, it is compared with other existing controllers designed for similar purpose.

The outline of this chapter is as follows. In Section 3.2, the design procedure of the optimal second order sliding mode controller (OSOSMC) for stabilizing a single input single output (SISO) linear uncertain system is explained. Output tracking of linear uncertain systems using the proposed OSOSMC is discussed in Section 3.3. In Section 3.4, an OSOSMC is designed to stabilize a linear uncertain MIMO system. A brief summary of the chapter is given in Section 3.5.

3.2 Stabilization of linear uncertain single input single output (SISO) system

Stability is the main concern while designing a controller for an uncertain system. However, very high control input increases the cost of the control system. Moreover, actuator saturation due to the high control gain leads the system towards instability. Hence, minimization of the control input is also a major demand for designing the control law. For achieving control input minimization while ensuring stability of a linear uncertain system, an optimal controller is integrated with a second order sliding mode controller (SOSMC). The design procedure of the proposed OSOSMC is discussed in the following section.

I. Problem statement

Let us consider a linear uncertain system described as follows:

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A(t))x(t) + (B + \Delta B(t))u(t) + \zeta(t) \\ y(t) &= Cx(t) \end{aligned} \tag{3.1}$$

where $x(t) \in \mathcal{R}^n$ is the measurable state vector, $u(t) \in \mathcal{R}$ is the control input, $\Delta A(t)$, $\Delta B(t)$ are parametric uncertainties and $\zeta(t)$ is the exogenous disturbance affecting the system. Further, $y(t)$ is the detectable state vector and A, B, C are known real constant matrices with appropriate dimensions.

Assumptions: The following are the assumptions made:

• A, B pair is controllable.

• All the states are observable.

• System uncertainties $\Delta A(t)$, $\Delta B(t)$ and disturbance $\zeta(t)$ are unknown but bounded and their time derivatives exist. These uncertainties and disturbance satisfy the matching condition. So, it can be written that

$$\Delta A(t)x(t) + \Delta B(t)u(t) + \zeta(t) = Bd(t) \tag{3.2}$$

where $d(t)$ denotes the uncertain part of the system (3.1).

So, system (3.1) can be written as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B(u(t) + d(t)) \\ y(t) &= Cx(t) \end{aligned} \tag{3.3}$$

The objective of the proposed control method is to design a robust optimal controller for the linear uncertain system (3.3). The specific aim is to realize stabilization for the linear uncertain system (3.3) at the expense of minimum control input. To achieve this goal, an optimal controller is combined with the sliding mode control (SMC). The control input $u(t)$ is obtained as

$$u(t) = u_1(t) + u_2(t) \tag{3.4}$$

where $u_1(t)$ is the equivalent control required to bring system (3.3) onto the sliding surface. The equivalent control is designed for the nominal part of the system using optimal control strategy. Further, $u_2(t)$ is the switching control which keeps the linear uncertain system (3.3) onto the sliding surface and leads the system states towards the equilibrium.

II. Optimal controller design

At first the optimal control $u_1(t)$ is designed for (3.3) considering nominal condition. Hence, neglecting the uncertain part, (3.3) becomes

$$\dot{x}(t) = Ax(t) + Bu_1(t) \tag{3.5}$$

The performance index J is defined as

$$J = \int_0^\infty \left[x^T(t)Qx(t) + u_1^T(t)Ru_1(t)\right] dt \tag{3.6}$$

where $Q \in \mathcal{R}^{n\times n}$ and $R \in \mathcal{R}$ are a positive definite weighing matrix. The optimal control law $u_1(t)$ is obtained as

$$u_1(t) = -R^{-1}B^TPx(t) = -Kx(t) \tag{3.7}$$

where $K = R^{-1}B^TP$ and P is a symmetric, positive definite matrix which is the solution of the algebraic Riccati equation [100]

$$A^TP + PA + Q - PBR^{-1}B^TP = 0 \tag{3.8}$$

The nominal system (3.5) is stabilized by the optimal control $u_1(t)$ which is obtained by minimizing the performance index (3.6).

Remark 1: In (3.6), weighing matrices Q and R need to be chosen suitably to find the optimal control law ensuring the stability of the system.

Remark 2: To design an optimal control law, full knowledge of the system is considered as necessary requirement. If the system is affected by uncertainties and disturbances, the optimal controller loses its effectiveness. An effective method to tackle uncertainties is to integrate the optimal controller with the sliding mode control (SMC) to ensure robustness.

III. Sliding mode controller design

In order to stabilize the linear uncertain system (3.3) with minimum control effort, the optimal controller is integrated with the SMC. The integration of the optimal controller with the SMC is realized by designing an integral sliding surface. The optimal control strategy described in the earlier section can be easily incorporated into an integral sliding surface based SMC [103] as explained below.

Conventionally, integral sliding variable $s(t)$ [57] is defined as

$$s(t) = G\left[x(t) - x_0 - \int_0^t \dot{\varphi}(\tau)d\tau\right] \tag{3.9}$$

where G is a design parameter chosen in such a way that GB is invertible, x_0 is the initial condition and

$$\dot{\varphi}(t) = Ax(t) + Bu_1(t) \tag{3.10}$$

Hence,

$$\dot{s}(t) = G\left[\dot{x}(t) - \dot{\varphi}(t)\right] \tag{3.11}$$

Then by using (3.3) and (3.10), (3.11) can be written as

$$\begin{aligned}\dot{s}(t) &= G\left[Ax(t) + B(u_1(t) + u_2(t)) + Bd(t) - Ax(t) - Bu_1(t)\right]\\ &= G\left[Bu_2(t) + Bd(t)\right]\end{aligned} \tag{3.12}$$

As the reaching phase is eliminated in the conventional integral sliding mode control (ISMC) [103], the system is onto the sliding manifold from the very beginning. In the ISMC, the switching control is designed based on $\eta-$reachability condition defined as

$$\dot{s}(t) < -\rho sgn(s(t)) \tag{3.13}$$

where $\rho > 0$ and

$$sgn(s(t)) = \begin{cases} 1, & s(t) > 0 \\ -1, & s(t) < 0 \\ 0, & s(t) = 0 \end{cases} \tag{3.14}$$

So, from (3.12) and (3.13) it is found that

$$u_2(t) < -(GB)^{-1}\left[\rho sgn(s(t)) + GBd(t)\right] \tag{3.15}$$

From (3.15) it is clear that the switching control $u_2(t)$ of conventional ISMC is influenced by the sign function of the sliding variable. Due to the sign function, chattering is prevalent in the control input $u_2(t)$. The chattering in control input degrades the performance of the sliding mode controller. To reduce this chattering phenomenon, second order methodology is proposed here. The proposed second order sliding mode controller is designed in two steps. In the first part, an integral sliding variable $s(t)$ is proposed as follows,

$$s(t) = G\left[x(t) - \int_0^t \dot{\varphi}(\tau)d\tau\right] \tag{3.16}$$

with

$$\dot{s}(t) = G\left[Bu_2(t) + Bd(t)\right] \tag{3.17}$$

In the proposed integral sliding variable (3.16), initial condition x_0 is eliminated as the design procedure does not require information about initial states. Unlike the conventional ISMC, in the proposed ISMC, the system is not onto the sliding surface from the very beginning. In order to bring the system onto the sliding surface in finite time, a nonsingular terminal sliding mode controller is designed based on the proposed integral sliding variable $s(t)$. The integral sliding variable based terminal sliding mode controller imparts second order sliding methodology which reduces the chattering.

Moreover, the integral sliding mode based terminal sliding variable converges to zero in finite time. The nonsingular terminal sliding surface [77, 79] is defined as

$$\sigma(t) = s(t) + \delta \dot{s}(t)^{\frac{\alpha}{\beta}} = 0 \tag{3.18}$$

where δ is the switching gain chosen such that

$$\delta > 0 \tag{3.19}$$

and α, β are selected in such a way that these satisfy the following conditions:

$$\alpha, \beta \in \{2n + 1 : \ n \, is \, an \, integer\} \tag{3.20}$$

and

$$1 < \frac{\alpha}{\beta} < 1.5 \tag{3.21}$$

The integral sliding variable $s(t)$ based nonsingular terminal sliding surface $\sigma(t) = 0$ gives rise to a second order SMC. Using the constant plus proportional reaching law [104] yields

$$\dot{\sigma}(t) = -\eta_1 sgn(\sigma(t)) - \varepsilon_1 \sigma(t) \tag{3.22}$$

where $\eta_1 > 0$ and $\varepsilon_1 > 0$. Taking the first time derivative of the terminal sliding variable $\sigma(t)$ gives

$$\begin{aligned} \dot{\sigma}(t) &= \dot{s}(t) + \delta \frac{\alpha}{\beta} \dot{s}(t)^{\frac{\alpha}{\beta}-1} \ddot{s}(t) \\ &= \delta \frac{\alpha}{\beta} \dot{s}(t)^{\frac{\alpha}{\beta}-1} (\frac{\beta}{\delta \alpha} \dot{s}(t)^{2-\frac{\alpha}{\beta}} + \ddot{s}(t)) \end{aligned} \tag{3.23}$$

For the parameters α, β satisfying (3.20) and (3.21), it can be shown [79] that

$$\begin{aligned} \dot{s}(t)^{\frac{\alpha}{\beta}-1} > 0 \quad for \quad \dot{s}(t) \neq 0 \\ \dot{s}(t)^{\frac{\alpha}{\beta}-1} = 0 \quad only \, for \quad \dot{s}(t) = 0 \end{aligned} \tag{3.24}$$

Further, from (3.19), (3.20) and (3.24), it can be observed that the term $\delta\frac{\alpha}{\beta}\dot{s}(t)^{\frac{\alpha}{\beta}-1}$ in (3.23) is always positive and hence can be substituted by a scalar $\eta_2 > 0$ for $\dot{s}(t) \neq 0$. Hence (3.23) can be written as

$$\dot{\sigma}(t) = \eta_2(\frac{\beta}{\delta\alpha}\dot{s}(t)^{2-\frac{\alpha}{\beta}} + \ddot{s}(t)) \tag{3.25}$$

Substituting the value of $\dot{\sigma}(t)$ from (3.25), (3.22) can be expressed as

$$\begin{aligned} \eta_2(\frac{\beta}{\delta\alpha}\dot{s}(t)^{2-\frac{\alpha}{\beta}} + \ddot{s}(t)) &= -\eta_1 sgn(\sigma(t)) - \varepsilon_1\sigma(t) \\ or, \frac{\beta}{\delta\alpha}\dot{s}(t)^{2-\frac{\alpha}{\beta}} + \ddot{s}(t) &= -\eta sgn(\sigma(t)) - \varepsilon\sigma(t) \end{aligned} \tag{3.26}$$

where $\eta = \frac{\eta_1}{\eta_2} > 0$ and $\varepsilon = \frac{\varepsilon_1}{\eta_2} > 0$. Then (3.26) can be rewritten as

$$\ddot{s}(t) = -\eta sgn(\sigma(t)) - \varepsilon\sigma(t) - \frac{\beta}{\delta\alpha}\dot{s}(t)^{2-\frac{\alpha}{\beta}} \tag{3.27}$$

Differentiating (3.17) gives rise to

$$\ddot{s}(t) = G\left[B\dot{u}_2(t) + B\dot{d}(t)\right] \tag{3.28}$$

Hence from (3.27) and (3.28), the switching control law is designed as follows:

$$u_2(t) = -\int_0^t (GB)^{-1}\left[\frac{\beta}{\delta\alpha}\dot{s}(\tau)^{2-\frac{\alpha}{\beta}} + \eta sgn(\sigma(\tau)) + \varepsilon\sigma(\tau)\right]d\tau \tag{3.29}$$

where the design parameters η and ε are chosen in such a way that $|GB\dot{d}(t)| < \eta$ [66] and $\varepsilon > 0$.

Theorem 1: The sliding variables converge to zero in finite time if the second order sliding variables are chosen as (3.16), (3.18) and the control law is designed as

$$u(t) = u_1(t) + u_2(t) \tag{3.30}$$

where $u_1(t)$ is the optimal control defined in (3.7) to stabilize the nominal system and $u_2(t)$ is the switching control defined in (3.29).

Proof:

In the following proof finite time convergence of sliding variable $\sigma(t)$ is established first and then it is shown that the convergence time of the integral sliding variable $s(t)$ is also finite. The Lyapunov function is considered as

$$
\begin{aligned}
V_1(t) &= \frac{1}{2}\sigma(t)^2 \\
\dot{V}_1(t) &= \sigma(t)\dot{\sigma}(t)
\end{aligned} \tag{3.31}
$$

Using (3.23) and (3.28) in (3.31) yields

$$
\begin{aligned}
\dot{V}_1(t) &= \sigma(t)[\dot{s}(t) + \frac{\delta\alpha}{\beta}\dot{s}(t)^{\frac{\alpha}{\beta}-1}\ddot{s}(t)] \\
&= \sigma(t)[\dot{s}(t) + \frac{\alpha\delta}{\beta}\dot{s}(t)^{\frac{\alpha}{\beta}-1}(GB\dot{u}_2(t) + GB\dot{d}(t))]
\end{aligned} \tag{3.32}
$$

Taking derivative of $u_2(t)$ in (3.29) and substituting in (3.32) yields

$$
\begin{aligned}
\dot{V}_1(t) &= \sigma(t)[\dot{s}(t) + \frac{\alpha\delta}{\beta}\dot{s}(t)^{\frac{\alpha}{\beta}-1}(-\frac{\beta}{\delta\alpha}\dot{s}(t)^{2-\frac{\alpha}{\beta}} - \eta sgn(\sigma(t)) - \varepsilon\sigma(t) + GB\dot{d}(t))] \\
&= \sigma(t)[\frac{\alpha\delta}{\beta}\dot{s}(t)^{\frac{\alpha}{\beta}-1}(-\eta sgn(\sigma(t)) - \varepsilon\sigma(t) + GB\dot{d}(t))] \\
&= \frac{\alpha\delta}{\beta}\dot{s}(t)^{\frac{\alpha}{\beta}-1}[-|\sigma(t)|\eta - \varepsilon\sigma(t)^2 + \sigma(t)GB\dot{d}(t)] \\
&\leq \frac{\alpha\delta}{\beta}\dot{s}(t)^{\frac{\alpha}{\beta}-1}[-|\sigma(t)|\eta - \sigma(t)^2\varepsilon + |\sigma(t)||GB\dot{d}(t)|] \\
&\leq \frac{\alpha\delta}{\beta}\dot{s}(t)^{\frac{\alpha}{\beta}-1}[-\eta - |\sigma(t)|\varepsilon + |GB\dot{d}(t)|]|\sigma(t)| \\
&\leq -\frac{\alpha\delta}{\beta}\dot{s}(t)^{\frac{\alpha}{\beta}-1}[\eta + |\sigma(t)|\varepsilon - |GB\dot{d}(t)|]|\sigma(t)| \\
&\leq -\kappa|\sigma(t)|
\end{aligned} \tag{3.33}
$$

where $\kappa = \frac{\alpha\delta}{\beta}\dot{s}(t)^{\frac{\alpha}{\beta}-1}[\eta + |\sigma(t)|\varepsilon - |GB\dot{d}(t)|]$, which is a positive integer as $\eta > |G\widehat{B}\dot{d}(t)|$ and $\varepsilon > 0$. Moreover, in [66] Wang et al. showed that $\dot{s}^{\frac{\alpha}{\beta}-1}(t) > 0$ for $|s(t)| \neq 0$. So, $\kappa > 0$ when $|s(t)| \neq 0$.

Hence,

$$
\dot{V}_1 \leq -\kappa|\sqrt{2V_1}| \quad \text{as} \quad V_1 = \frac{1}{2}\sigma(t)^2 \tag{3.34}
$$

$$
\leq -\overline{\kappa}|\sqrt{V_1}| \quad \text{where} \quad \overline{\kappa} = \kappa|\sqrt{2}| > 0 \tag{3.35}
$$

Hence, the finite time convergence [101, 105] of sliding variable $\sigma(t)$ to zero is guaranteed.

Moreover, it can also be proved that the integral sliding variable $s(t)$ converges to zero in finite time. Suppose in time t_r, $\sigma(t)$ reaches zero from $\sigma(0) \neq 0$ and $\sigma(t) = 0 \; \forall \; t > t_r$. So, once $\sigma(t)$ reaches zero it remains at zero and based on (3.33), $s(t)$ will converge to zero in time t_s. The total time required from $\sigma(0) \neq 0$ to $s(t_s)$ can be calculated as follows:

From (3.18), it follows that

$$\begin{aligned} s(t) + \delta \dot{s}(t)^{\frac{\alpha}{\beta}} &= 0 \\ or,\ (\frac{s}{\delta})^{\frac{\beta}{\alpha}} &= -(\dot{s}(t)^{\frac{\alpha}{\beta}})^{\frac{\beta}{\alpha}} \\ or,\ \frac{1}{\delta^{\frac{\beta}{\alpha}}} s(t)^{\frac{\beta}{\alpha}} &= -\dot{s}(t) \end{aligned} \tag{3.36}$$

As α, β are chosen according to (3.20) and (3.21), (3.36) can be written as

$$\begin{aligned} \frac{1}{\delta^{\frac{\beta}{\alpha}}} s(t)^{\frac{\beta}{\alpha}} &= -\frac{ds(t)}{dt} \\ or,\ \frac{1}{\delta^{\frac{\beta}{\alpha}}} dt &= -\frac{ds(t)}{s(t)^{\frac{\beta}{\alpha}}} \\ or,\ \int_{t_r}^{t_s} dt &= -\delta^{\frac{\beta}{\alpha}} \int_{s(t_r)}^{s(t_s)} \frac{ds(t)}{s(t)^{\frac{\beta}{\alpha}}} \\ or,\ t_s - t_r &= -\frac{\alpha}{\alpha - \beta} \delta^{\frac{\beta}{\alpha}} \left[s(t_s)^{\frac{\alpha-\beta}{\alpha}} - s(t_r)^{\frac{\alpha-\beta}{\alpha}} \right] \end{aligned} \tag{3.37}$$

At time t_s sliding variable $s(t_s) = 0$. So, (3.37) gives rise to

$$t_s = t_r + \frac{\alpha}{\alpha - \beta} \delta^{\frac{\beta}{\alpha}} s(t_r)^{\frac{\alpha-\beta}{\alpha}} \tag{3.38}$$

Hence, $s(t)$ and $\dot{s}(t)$ converge to zero in finite time.

Remark 3: It is observed from (3.29) and (3.17) that ε is a parameter which determines the control input which in turn decides the convergence rate of the sliding variable. It is evident that a high value of ε will force the system states to converge to the origin at a faster rate. Consequently, it will demand a very high control input which is not feasible in most practical situations. Thus the parameter ε cannot be selected to be very large. In practice, ε is to be chosen appropriately by striking a balance between the response speed and the magnitude of the control input.

IV. Simulation results

To demonstrate the effectiveness of the proposed optimal second order sliding mode controller, simulation studies are conducted for stabilization of the inverted pendulum system and triple integrator system.

Example 1. Stabilization of the inverted pendulum system

The inverted pendulum system is shown in Figure 3.1 where

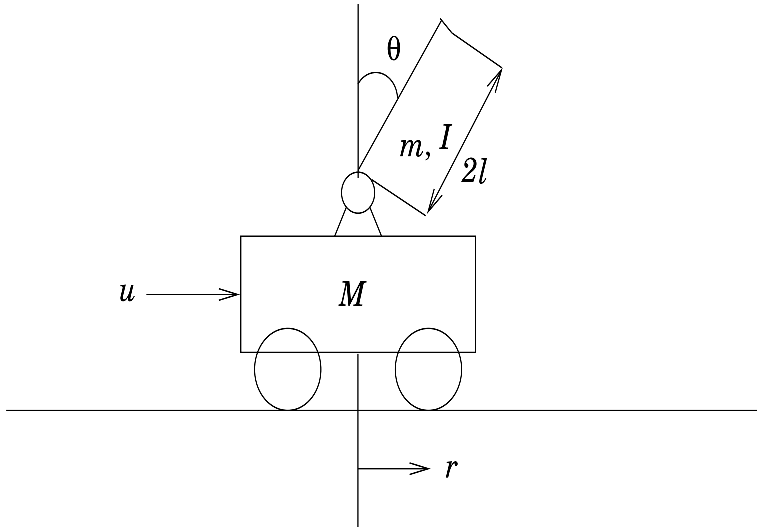


Figure 3.1: Inverted pendulum system

the cart having mass M is displaced by an amount of r due to the external force u. As a consequence, the pendulum with mass m mounted on the cart has an angular displacement θ. The length of the pendulum is $2l$ and I represents the inertia of the pendulum.

The state space model of the inverted pendulum system described above is given by [2]

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1.933 & -1.987 & 0.009 \\ 0 & 36.977 & 6.258 & -0.173 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.320 \\ -1.009 \end{bmatrix} (u(t) + d(t)) \tag{3.39}$$

where states $x_1(t)$, $x_2(t)$, $x_3(t)$ and $x_4(t)$ represent the cart displacement r, angular displacement of the pendulum θ, cart velocity $\dot{r}$ and angular velocity of the pendulum $\dot{\theta}$ respectively. Further, $d(t)$ is the external disturbance chosen as $1.5sin(\pi t/3) + 2cos(t)$. The problem is to stabilize the above system (3.39) with the minimum control input.

To minimize the control input, the performance index J is chosen as

$$J = \int_0^\infty \left(x^T(t) \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t) + u_1^T(t) 0.1 u_1(t) \right) dt \tag{3.40}$$

Using the LQR technique, the feedback control $u_1(t)$ is found as

$$u_1(t) = \begin{bmatrix} -10.00 & -117.16 & -10.78 & -19.93 \end{bmatrix} x(t) \tag{3.41}$$

Design parameters of the proposed optimal second order sliding mode controller (OSOSMC) are chosen as follows:

$G = [0 \quad 0 \quad 0 \quad 1]$, $\alpha = 7$, $\beta = 5$, $\delta = 0.15$, $\eta = 1.5$ and $\varepsilon = 0.01$.

The simulation results obtained by applying the proposed optimal second order sliding mode controller (OSOSMC) are compared with those obtained by using the compensator-based second order sliding mode controller (CSOSMC) designed by Chang [2]. The compensator-based second order sliding mode controller (CSOSMC) is discussed in Appendix A.2. The states $x_1(t)$, $x_2(t)$ and the control inputs obtained by applying the proposed OSOSMC and the compensator-based SOSMC proposed by Chang [2] are shown in Figures 3.2- 3.3 and Figures 3.4-3.5 respectively.

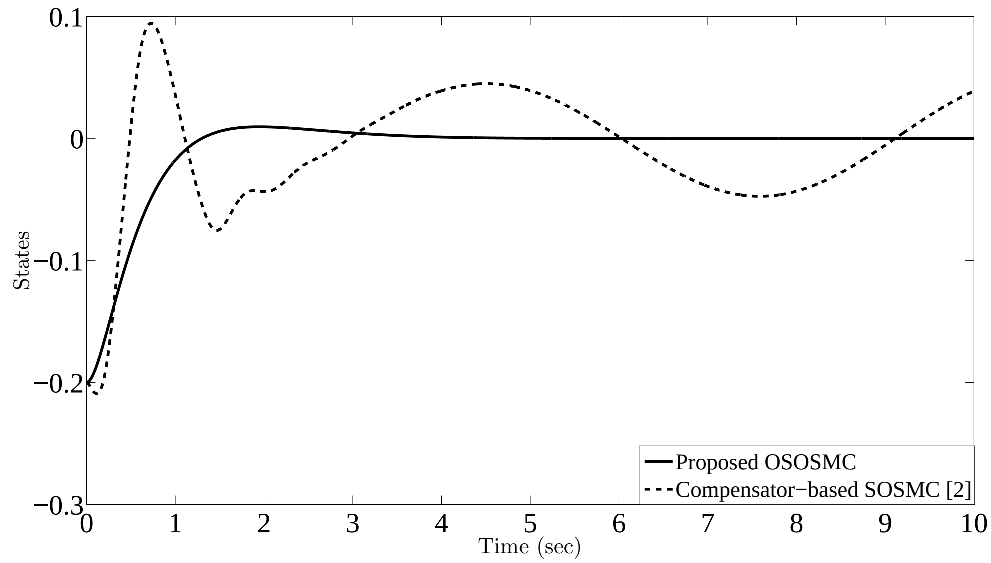


Figure 3.2: State $x_1(t)$ obtained by applying the proposed OSOSMC and compensator-based SOSMC proposed by Chang [2]

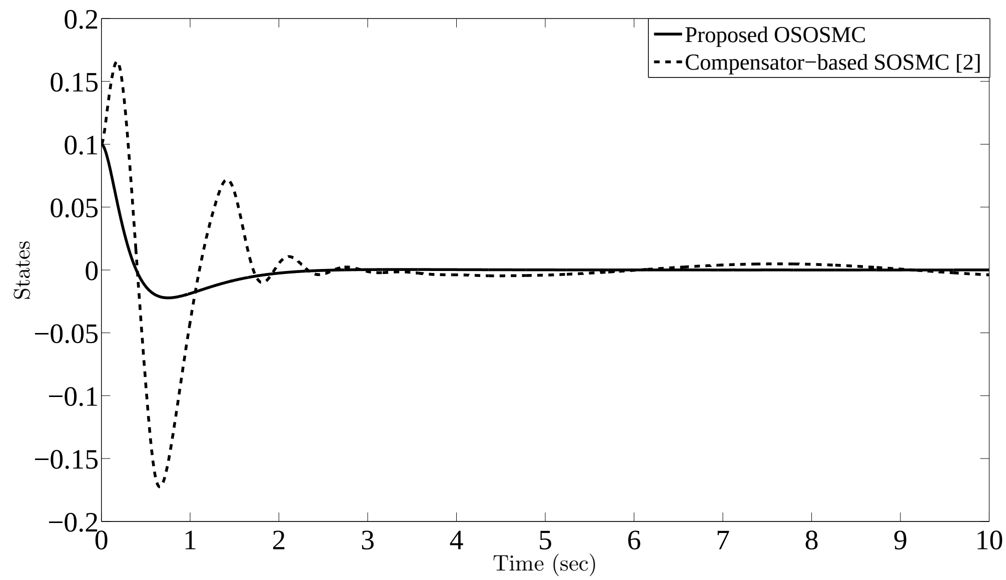


Figure 3.3: State $x_2(t)$ obtained by applying the proposed OSOSMC and compensator-based SOSMC proposed by Chang [2]

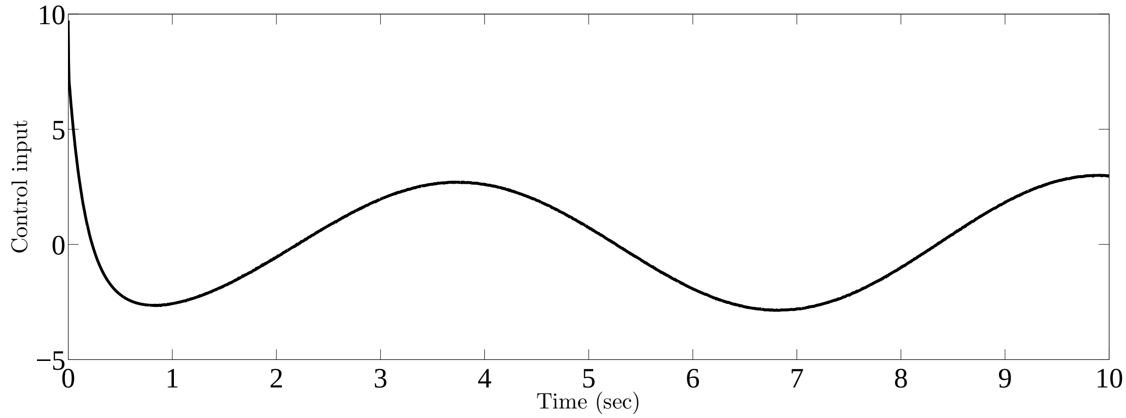


Figure 3.4: Control input obtained by applying the proposed OSOSMC

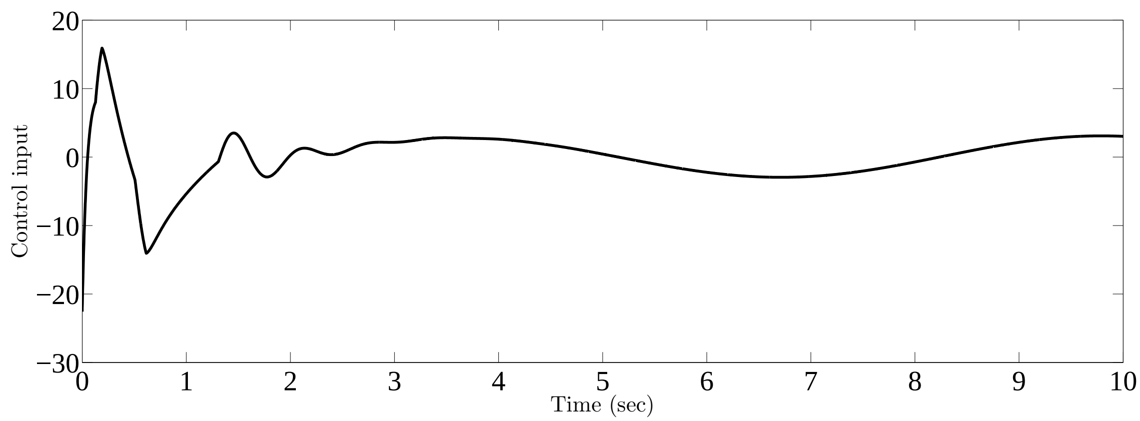


Figure 3.5: Control input obtained by applying the compensator-based SOSMC proposed by Chang [2]

In order to evaluate the controller performance, the total variation (TV) [106] of the control input $u(t)$ is computed as follows:

$$TV = \sum_{i=1}^{n} |u_{i+1}(t) - u_i(t)| \tag{3.42}$$

where n is the number of samples. Also, the energy of the control input $u(t)$ is calculated by using the 2-norm method. Table 3.1 shows the TV and the 2-norm of the control input calculated for the period from 0 to 10 sec with a sampling time of 0.1 sec. It is clear from Table 3.1 that the proposed

OSOSMC is able to produce a smoother control input with substantial reduction in the control effort than the CSOSMC proposed by Chang [2].

Table 3.1: Comparison of Control Indices for Inverted Pendulum

Method	Total Variation (TV)	Control Energy
Proposed OSOSMC	29.36	65.20
CSOSMC [2]	111.95	115.78

Although stabilization of the uncertain system with minimum control input is the main aim of the controller design, satisfactory transient and steady state performances of the controlled system are also very much desired. Transient performance specifications like rise time, settling time and steady state performance index like steady state error are compared for the proposed OSOSMC and CSOSMC [2] which are tabulated in Table 3.2. From 3.2 it is evident that the proposed OSOSMC exhibits superior performance than Chang's CSOSMC [2].

Table 3.2: Comparison of Performance Indices for Inverted Pendulum

Performance specification	OSOSMC		CSOSMC [2]	
	x_1	x_2	x_1	x_2
Rise time (sec)	0.83	0.27	0.22	0.08
Settling time (sec)	0.70	0.19	1.70	1.54
Steady state error	0	0	0.045	0.005

Example 2. Stabilization of triple integrator system

The triple integrator system affected by uncertainty [4] is described as

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= x_3(t)\\ \dot{x}_3(t) &= u(t) + d(t)\\ y(t) &= x_1(t)\end{aligned} \tag{3.43}$$

where $x_1(t),\ x_2(t),\ x_3(t)$ are three states of the system and $u(t)$ is the control input. The matched uncertainty of the system is denoted as $d(t) = \sin(10x_1)$ and the output of the system is $y(t)$. To minimize the control input, the performance index is chosen as

$$J = \int_0^\infty \left(x^T(t) \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t) + u_1^2(t) \right) dt \tag{3.44}$$

Using the LQR technique, the feedback control $u_1(t)$ is found as

$$u_1(t) = \begin{bmatrix} 3.1623 & 6.3333 & 4.7609 \end{bmatrix} x(t) \tag{3.45}$$

Design parameters of the proposed optimal second order sliding mode controller (OSOSMC) are chosen as follows:

$G = [0 \;\; 0 \;\; 1]$, $\alpha = 7$, $\beta = 5$, $\delta = 0.15$, $\eta = 1.2$ and $\varepsilon = 0.01$.

The simulation results obtained by applying the proposed optimal second order sliding mode controller are compared with those obtained by using the second-order sliding mode controller (SOSMC) designed by Defoort et al. proposed in [3] and Mondal and Mahanta proposed in [4]. These two control methods are discussed in Appendix A.3 and A.4. System states obtained by using the proposed OSOSMC are compared with the states obtained by using the second-order sliding mode controllers (SOSMCs) proposed by Defoort et al. [3] and Mondal and Mahanta [4] in Figures 3.6 - 3.8. The control inputs are compared in Figures 3.9-3.11.

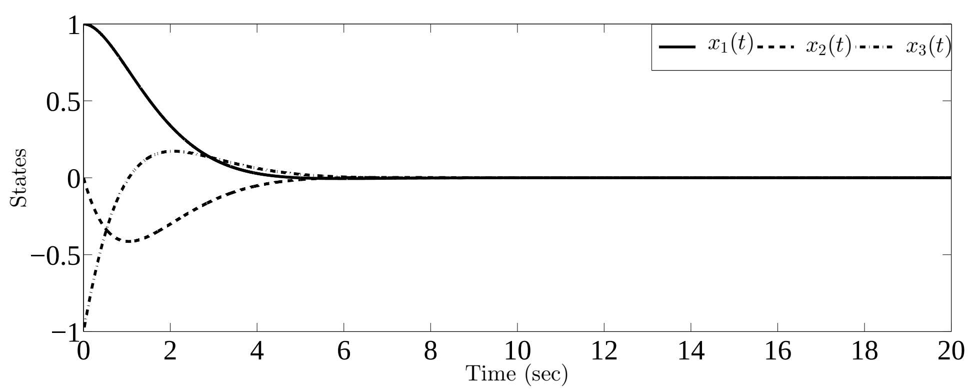


Figure 3.6: States obtained by applying the proposed OSOSMC

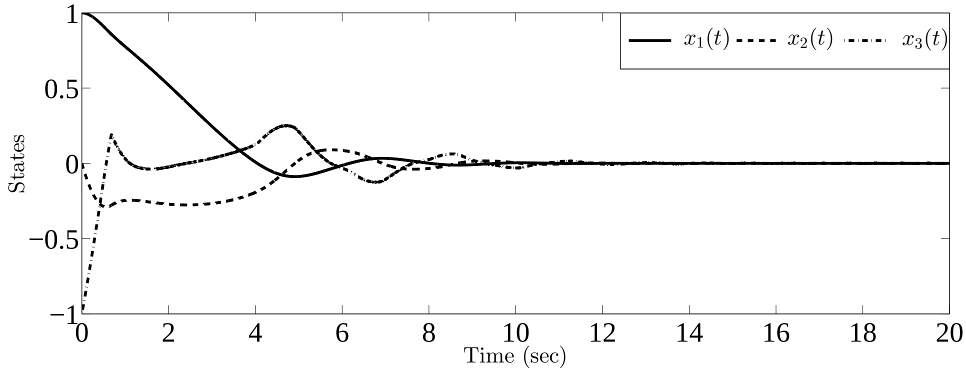


Figure 3.7: States obtained by applying the SOSMC proposed by Defoort et al. [3]

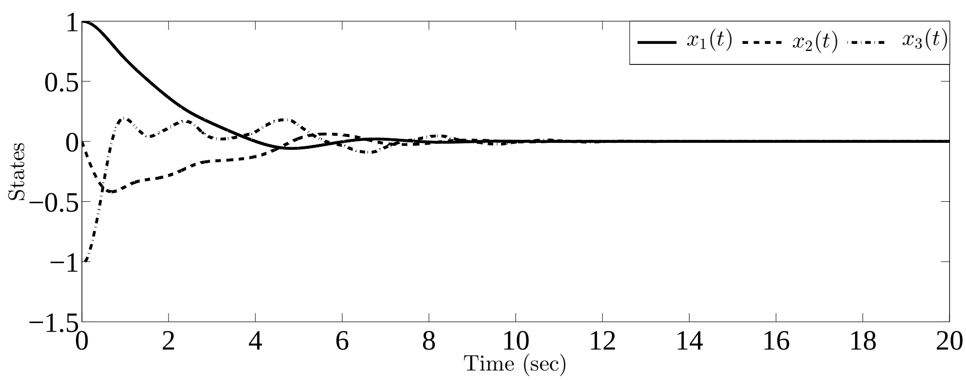


Figure 3.8: States obtained by applying the SOSMC proposed by Mondal and Mahanta [4]

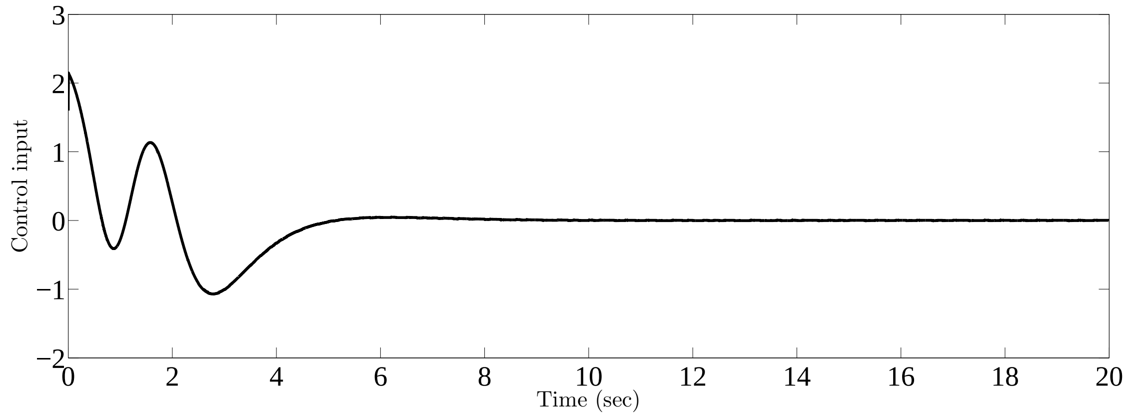


Figure 3.9: Control input obtained by applying the proposed OSOSMC

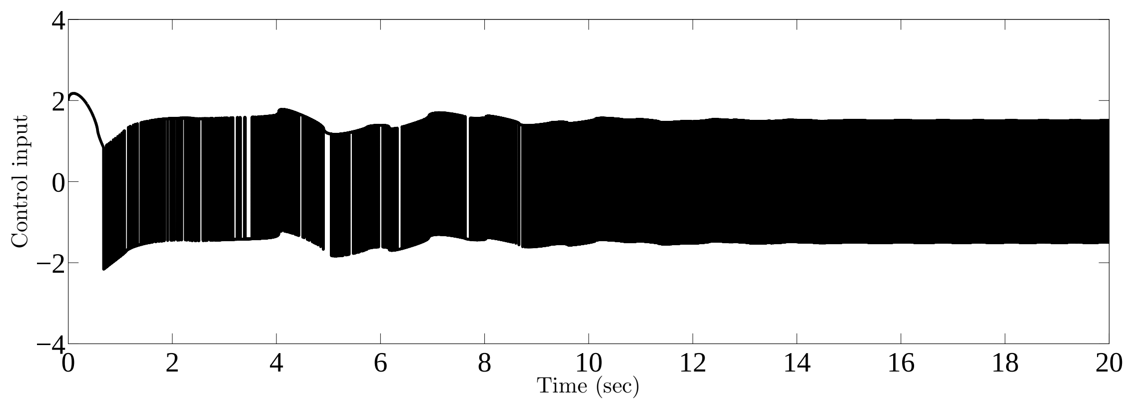


Figure 3.10: Control input obtained by applying the SOSMC proposed by Defoort et al. [3]

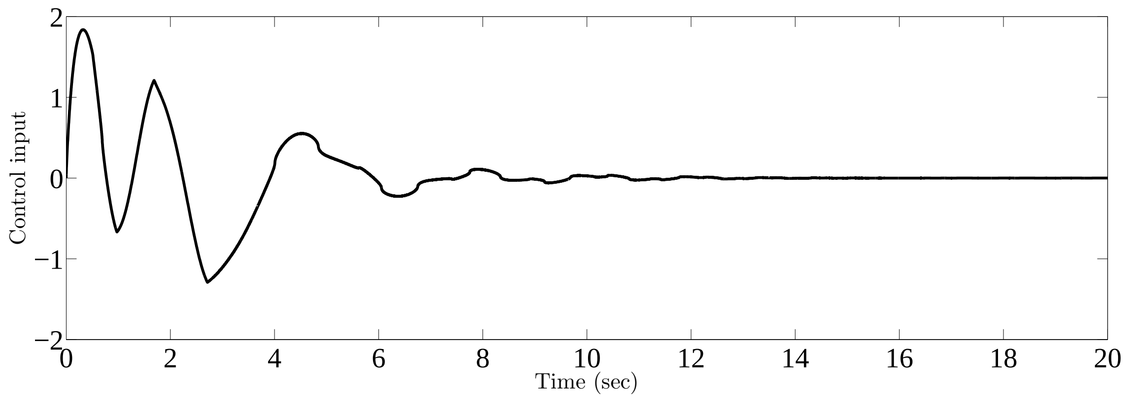


Figure 3.11: Control input obtained by applying the SOSMC proposed by Mondal and Mahanta [4]

Table 3.3 shows the TV and the 2-norm of the control input calculated for the period from 0 to 20 sec with a sampling time of 0.1 sec. It is clear from Table 3.3 that the proposed OSOSMC is able to produce a smoother control input with substantial reduction in the control effort than the SOSMCs proposed by Defoort et al. [3] and Mondal and Mahanta [4].

Table 3.3: Comparison of Control Indices for Triple Integrator System

Method	Total Variation (TV)	Control Energy
Proposed OSOSMC	9.80	17.53
SOSMC proposed by Defoort et al. [3]	2393.00	67.74
SOSMC proposed by Mondal and Mahanta [4]	12.67	18.88

Transient performance specifications like rise time and settling time are compared for the proposed OSOSMC and SOSMC proposed by Defoort et al. [3] and Mondal et al. [4] which are tabulated in Table 3.4. The comparison shows that the proposed OSOSMC exhibits superior performance than SOSMCs proposed by Defoort et al. [3] and Mondal and Mahanta [4].

Table 3.4: Comparison of Performance Indices for Triple Integrator System

Performance specification	Proposed OSOSMC	SOSMC proposed by Defoort et al. [3]	SOSMC proposed by Mondal and Mahanta [4]
Rise time (sec)	2.6	3.0	2.8
Settling time (sec)	4	7.4	6.8
Steady state error	0	0	0

3.3 Tracking of linear uncertain single input single output (SISO) system

Trajectory tracking by the uncertain system is another challenging problem. To design the tracking controller, the system is represented in the error domain and the controller is designed to converge the error to zero. Thereby the tracking problem is converted into a regulatory problem and the OSOSMC is designed in the error domain.

I. Problem statement

Let us consider an uncertain linear system given by

$$\begin{aligned} \dot{x}_i(t) &= x_{i+1}(t) \quad i = 1, 2, \cdots, n-1 \\ \dot{x}_n(t) &= ax(t) + bu(t) + d(t) \\ y(t) &= x_1(t) \end{aligned} \tag{3.46}$$

where $\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_i(t) \\ \vdots \\ x_n(t) \end{bmatrix} = x(t) \in \mathcal{R}^n$ is the state vector, $u(t) \in \mathcal{R}$ is the input and a is a row vector defined as $a = [a_1 \;\; a_2 \;\; \cdots \;\; a_n]$. Further, elements $a_1, \; a_2, \; \cdots \; a_n$ are known and b is a known integer. The system output is $y(t) \in \mathcal{R}$ and uncertainty affecting the system is denoted by $d(t)$.

Assumption:

• It is assumed that the bounded uncertainty $d(t)$ satisfies the matching condition and its first time derivative exists.

The objective is to design a controller for trajectory tracking of the linear uncertain system (3.46) with minimum control expense. To achieve this target, a robust controller is designed by combining the optimal controller built for the nominal linear system with a second order sliding mode controller. As such, the control input $u(t)$ is obtained by combining two different control laws and can be defined as

$$u(t) = u_1(t) + u_2(t) \tag{3.47}$$

where $u_1(t)$ is the optimal control designed to track the nominal system and the control input $u_2(t)$ is a second order sliding mode controller which is designed to tackle uncertainties in the system.

II. Optimal controller design

The optimal controller is designed for the nominal system by neglecting the uncertain part. So, system (3.46) can be written as

$$\begin{aligned} \dot{x}_i(t) &= x_{i+1}(t) \quad i = 1, 2, \cdots, n-1 \\ \dot{x}_n(t) &= ax(t) + bu_1(t) \end{aligned} \tag{3.48}$$

The proposed objective is to track the output of the system ($y(t) = x_1(t)$) along the desired trajectory $x_d(t)$. So, the tracking error is defined by

$$e(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_n(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} - \begin{bmatrix} x_d(t) \\ x_d^{(1)}(t) \\ \vdots \\ x_d^{(n-1)}(t) \end{bmatrix} \tag{3.49}$$

where $x_d^{(1)}(t), x_d^{(2)}(t), \cdots, x_d^{(n-1)}(t)$ are the successive derivatives of the desired trajectory $x_d(t)$. The error dynamics can be defined as

$$\dot{e}(t) = \begin{bmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \\ \vdots \\ \dot{e}_n(t) \end{bmatrix} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} - \begin{bmatrix} x_d^{(1)}(t) \\ x_d^{(2)}(t) \\ \vdots \\ x_d^{(n)}(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ x_3(t) \\ \vdots \\ a_1x_1(t) + a_2x_2(t) \cdots + a_nx_n(t) + bu_1(t) \end{bmatrix} - \begin{bmatrix} x_d^{(1)}(t) \\ x_d^{(2)}(t) \\ \vdots \\ x_d^{(n)}(t) \end{bmatrix}$$

$$= \begin{bmatrix} x_2(t) \\ x_3(t) \\ \vdots \\ a_1x_1(t) + a_2x_2(t) \cdots + a_nx_n(t) \end{bmatrix} - \begin{bmatrix} x_d^{(1)}(t) \\ x_d^{(2)}(t) \\ \vdots \\ a_1x_d(t) + a_2x_d^{(1)}(t) + \cdots + a_nx_d^{(n-1)}(t) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ a_1x_d(t) + a_2x_d^{(1)}(t) + \cdots + a_nx_d^{(n-1)}(t) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ x_d^{(n)}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b \end{bmatrix} u_1(t) =$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ a_1 & \cdots & a_{n-1} & a_n \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b \end{bmatrix} u_1(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ a_1x_d(t) + a_2x_d^{(1)}(t) \cdots + a_nx_d^{(n-1)}(t) - x_d^{(n)}(t) \end{bmatrix}$$

$$= Ae(t) + Bu_1(t) + \widetilde{A}(t) \tag{3.50}$$

where $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ a_1 & \cdots & a_{n-1} & a_n \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b \end{bmatrix}$

and $\widetilde{A}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ a_1x_d(t) + a_2x_d^{(1)}(t) + \cdots + a_nx_d^{(n-1)}(t) - x_d^{(n)}(t) \end{bmatrix}$.

Further, $\widetilde{A}(t)$ is in the range space of the input distribution matrix B and it is taken care of by the sliding mode controller to be designed in the next section. Hence, to design an optimal controller, the nominal part of the error dynamics (3.50) is written as

$$\dot{e}(t) = Ae(t) + Bu_1(t) \tag{3.51}$$

To minimize the control effort, the performance index is chosen as

$$J = \int_0^\infty \{e^T(t)Qe(t) + u_1^T(t)Ru_1(t)\}dt \tag{3.52}$$

where $Q \in \mathcal{R}^{n\times n}$ and $R \in \mathcal{R}$ are a positive definite weighing matrix. Using the LQR method, the optimal control law $u_1(t)$ is obtained as

$$u_1(t) = -R^{-1}B^T Pe(t) \tag{3.53}$$

where P is a symmetric, positive definite matrix which is the solution of the algebraic Riccati equation (3.8).

III. Sliding mode controller design

In presence of uncertainties, the error dynamics can be defined as

$$\dot{e}(t) = Ae(t) + B(u_1(t) + u_2(t)) + \widetilde{A}(t) + d(t) \tag{3.54}$$

In order to track the linear uncertain system (3.54) with minimum control expense, the optimal controller is combined with the SOSMC. The design procedure of the SOSMC is similar as described in Section 3.2.

The integral sliding surface is designed as

$$s(t) = G\left[e(t) - \int_0^t \dot{\varphi}(\tau)d\tau\right] = 0 \tag{3.55}$$

where G is chosen such that GB is invertible and

$$\dot{\varphi}(t) = Ae(t) + Bu_1(t) \tag{3.56}$$

The non-singular terminal sliding surface is designed as

$$\sigma(t) = s(t) + \delta\dot{s}(t)^{\frac{\alpha}{\beta}} = 0 \tag{3.57}$$

where δ, α and β satisfy the conditions discussed in 3.19 to 3.21. To tackle the uncertainties, the switching control $u_2(t)$ is designed as

$$u_2(t) = -\int_0^t (GB)^{-1}\left[\frac{\beta}{\delta\alpha}\dot{s}(\tau)^{2-\frac{\alpha}{\beta}} + \eta sgn(\sigma(\tau)) + \varepsilon\sigma(\tau)\right] d\tau \tag{3.58}$$

where $\eta > |\widetilde{A}(t) + d(t)|$ and $\varepsilon > 0$.

IV. Simulation results

To demonstrate the effectiveness of the proposed OSOSMC, tracking control of a magnetic levitation (maglev) system is considered. The proposed OSOSMC is applied for suspension control of the magnetic levitation (maglev) vehicle model [5].

The system dynamics of the maglev [5] is described by

$$\begin{aligned} \dot{x}(t) &= [A + \Delta A(L_r)]x(t) + [B + \Delta B(L_r)]u(t) \\ y(t) &= Cx(t) \end{aligned} \tag{3.59}$$

where $x(t) = \begin{bmatrix} x_1(t) \\ \dot{x}_1(t) \\ \ddot{x}_1(t) \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 57000 & 1938 & -16 \end{bmatrix}$, $\Delta A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -57000L_r & 1624L_r & 16L_r \end{bmatrix}$,
$B = \begin{bmatrix} 0 \\ 0 \\ -14.25 \end{bmatrix}$, $\Delta B = \begin{bmatrix} 0 \\ 0 \\ 14.25L_r \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.
Further, L_r is the parametric uncertainty considered as $-\frac{1}{2} \le L_r \le \frac{1}{2}$. The problem is to track the output $y(t) = x_1(t)$ which is the vertical displacement of point mass. The output is expected to follow the desired trajectory $x_d(t)$.

The objective is to track the desired trajectory with minimum control effort. At first, the desired trajectory is considered as $x_d(t) = 1$ having all its time derivatives as zero. So, the tracking error is defined as

$$e(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{3.60}$$

The performance index J is chosen as

$$J = \int_0^t \left[e^T(\tau) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} e(\tau) + u_1^T(\tau) u_1(\tau) \right] d\tau \tag{3.61}$$

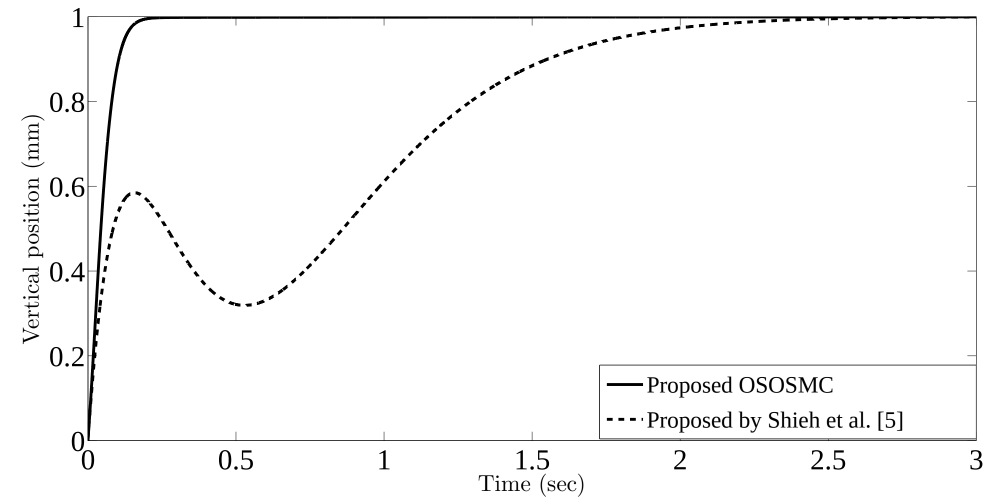
Using the LQR technique, the feedback control $u_1(t)$ is found as

$$u_1(t) = \begin{bmatrix} 8000 & 438.3 & 119.12 & 6.9 \end{bmatrix} e(t) \tag{3.62}$$

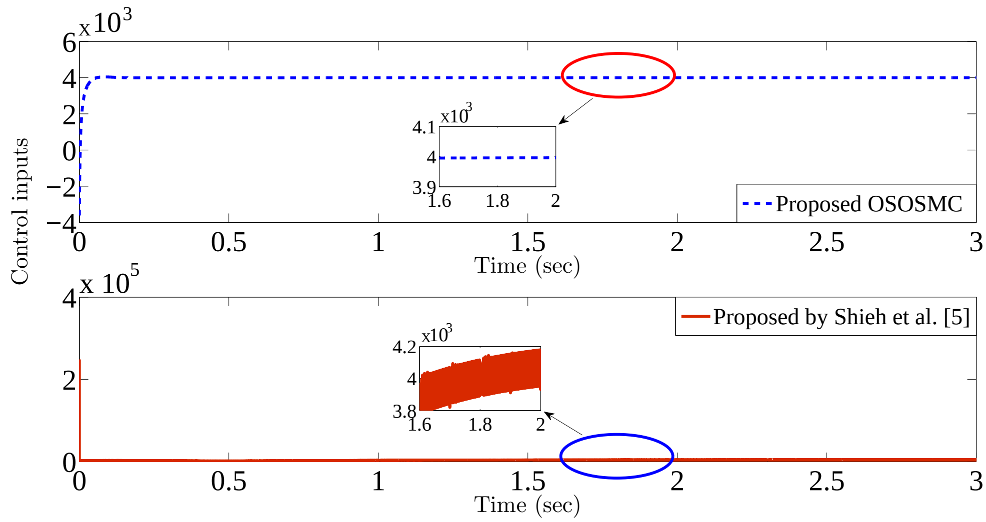
Design parameters of the proposed optimal second order sliding mode controller (OSOSMC) are chosen as follows:

$G = [0 \;\; 0 \;\; 1]$, $\alpha = 7$, $\beta = 5$, $\delta = 0.15$, $\eta = 6$ and $\varepsilon = .3$.

The results obtained by applying the proposed optimal second order sliding mode controller (OSOSMC) are compared with those obtained by using the robust output tracking control designed by Shieh et al. [5] which is discussed in Appendix A.5. The output $y(t) = x_1(t)$ and the control inputs obtained by applying the OSOSMC and the controller proposed by Shieh et al. [5] are shown in Figure 3.12. It is observed that the proposed OSOSMC can successfully track the desired trajectory whereas the controller proposed by Shieh et al. [5] is unable to track the desired trajectory faithfully. Moreover the control input in the case of Shieh et al.'s controller [5] contains considerable chattering.



(a) State $x_1(t)$



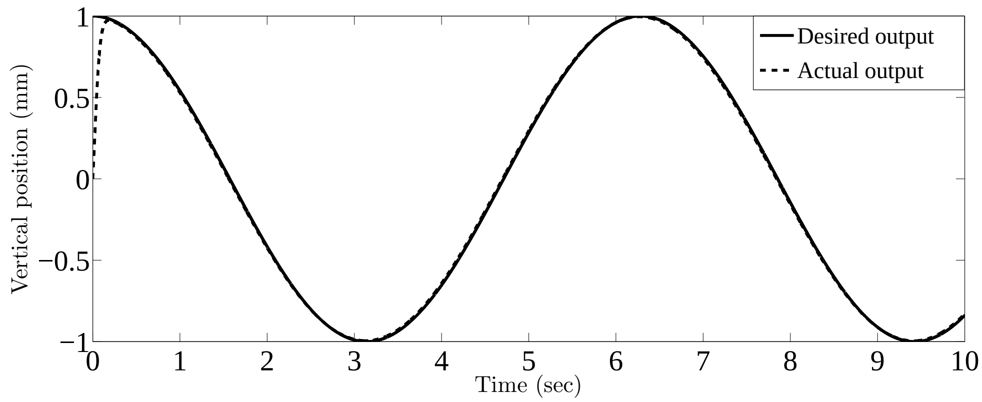
(b) control inputs

Figure 3.12: Tracking of the maglev system for desired trajectory $x_d(t) = 1$ using proposed OSOSMC and the method proposed by Shieh et al. [5]

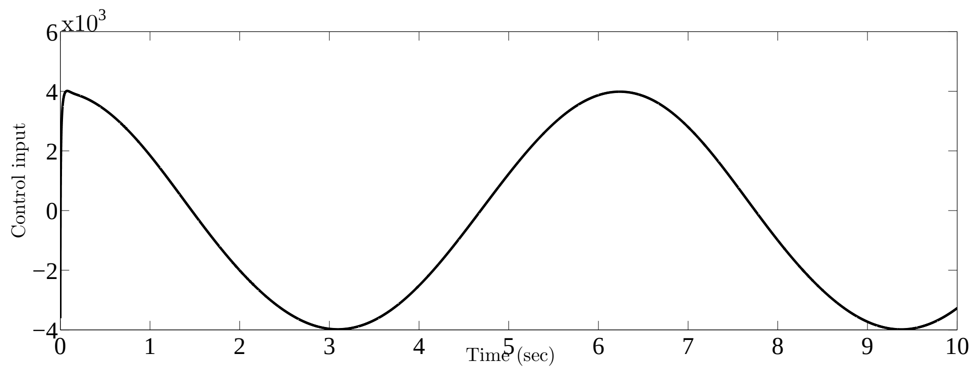
Next, the desired trajectory is chosen as $x_d(t) = \cos t$. The successive derivatives of the desired trajectory $x_d(t)$ are $\dot{x}_d(t) = -\sin t$, $\ddot{x}_d(t) = -\cos t$. The tracking error is defined as

$$e(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} - \begin{bmatrix} \cos t \\ -\sin t \\ -\cos t \end{bmatrix} \tag{3.63}$$

The actual and desired states and the control input obtained by applying the proposed OSOSMC are shown in Figure 3.13. It is found that the proposed OSOSMC can successfully track the desired trajectory $x_d(t) = \cos t$ whereas the robust output tracking control designed by Shieh et al. [5] fails to track it and becomes unstable.



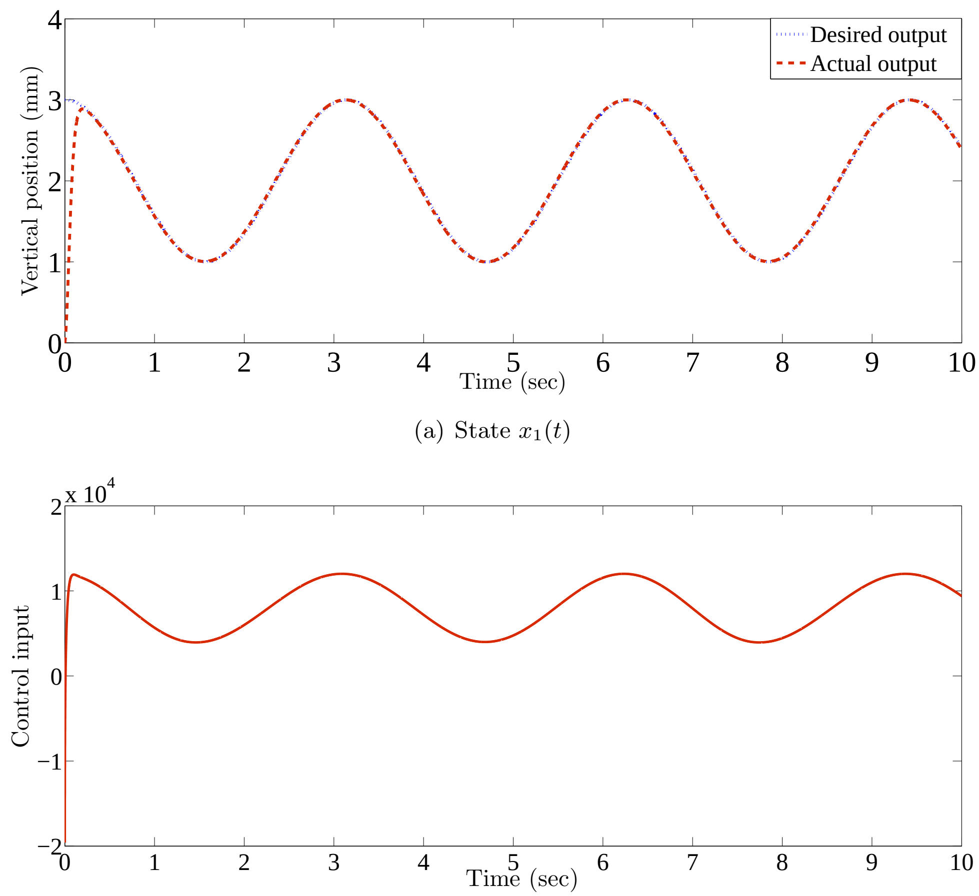
(a) State $x_1(t)$



(b) Control input

Figure 3.13: Tracking of the maglev system for desired trajectory $x_d(t) = cost$ using proposed OSOSMC

Another type of desired trajectory $x_d(t) = 2 + \cos 2t$ is now considered for tracking. The actual and desired states and the control input obtained by applying the proposed OSOSMC are shown in Figure 3.14. It is found from Figure 3.14 that the proposed OSOSMC can successfully track this trajectory also.



(a) State $x_1(t)$

(b) Control input

Figure 3.14: Tracking of the maglev system for desired trajectory $x_d(t) = 2 + \cos 2t$ using proposed OSOSMC

Table 3.5 compares the total variation (TV) and the 2-norm of the control input for the proposed OSOSMC and the robust output tracking control proposed by Shieh el al. [5]. The control indices are computed for the period from 0 to 3 sec with a sampling time of 0.1 sec for the desired trajectory $x_d(t) = 1$. For the desired trajectories $x_d(t) = cost$ and $x_d(t) = 2 + \cos 2t$, the control indices are computed for the period from 0 to 10 sec with a sampling time of 0.1 sec. It is clear from Table 3.5 that for tracking $x_d(t) = 1$, the proposed OSOSMC requires substantially lesser control effort than that of the robust controller proposed in [5]. Also, the control input of the proposed OSOSMC is significantly smoother than that of Shieh et al. [5] in this case. For tracking the time dependent trajectories $x_d(t) = cost$ and $x_d(t) = 2 + \cos 2t$, the robust output tracking control proposed by Shieh el al. [5] failed whereas the proposed OSOSMC is able to track these successfully.

Table 3.5: Comparison of Control Indices for Maglev System

Method	Total Variation (TV)	Control Energy
Robust output tracking control [5] ($x_d(t) = 1$)	2,50,010	2,49,050
Proposed OSOSMC ($x_d(t) = 1$)	7,698	22,191
Robust output tracking control [5] ($x_d(t) = cost$)	Fails to track	Fails to track
Proposed OSOSMC ($x_d(t) = cost$)	15,563	15,564
Robust output tracking control [5] ($x_d(t) = 2 + \cos 2t$)	Fails to track	Fails to track
Proposed OSOSMC ($x_d(t) = 2 + \cos 2t$)	88,647	82,248

The transient and steady state performances of the proposed OSOSMC and the robust output tracking controller proposed by Shieh et al. [5] are tabulated in Table 3.6. From this table it is evident that the proposed OSOSMC shows superior transient and steady state performances than those obtained by using Shieh et al.'s controller [5] for tracking $x_d(t) = 1$. Moreover, Shieh et al.'s method [5] fails to track when the desired trajectory is a time varying function as is observed in the cases of $x_d(t) = cost$ and $x_d(t) = 2 + \cos 2t$.

Table 3.6: Comparison of Performance Indices for Maglev System

Method	Rise time (sec)	Settling time (sec)	Steady state error (mm)
Robust output tracking control [5] ($x_d(t) = 1$)	2.320	2.320	0
Proposed OSOSMC ($x_d(t) = 1$)	0.150	0.150	0
Robust output tracking control [5] ($x_d(t) = \cos\ t$)	Fails	to track	the trajectory
Proposed OSOSMC ($x_d(t) = \cos\ t$)	0.094	0.230	0.005
Robust output tracking control [5] ($x_d(t) = 2 + \cos 2t$)	Fails	to track	the trajectory
Proposed OSOSMC ($x_d(t) = 2 + \cos 2t$)	1.45	1.45	0.006

3.4 Stabilization of linear uncertain multi input multi output (MIMO) system

The optimal second order sliding mode control (OSOSMC) method designed to stabilize the linear uncertain single input single output (SISO) system can be extended to the linear decoupled uncertain multi input multi output (MIMO) system. Similar to the design procedure discussed earlier in Section 3.2, the optimal controller part is designed based on the LQR technique and the SOSMC is combined with the optimal controller to impart robustness. The design method for the OSOSMC for decoupled MIMO systems is discussed now.

I. Problem statement

A linear uncertain MIMO system is described as

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + B(u(t) + d(t)) \\
y(t) &= Cx(t)
\end{aligned}
\tag{3.64}
$$

where $x(t) \in \mathcal{R}^n$ is the state vector and $u(t) \in \mathcal{R}^m$ is the control input. Further, A, B, C are the known matrices with appropriate dimensions and $d(t)$ represents uncertainty affecting the system. Output of the system is denoted by $y(t) \in \mathcal{R}^p$.

Assumptions: Following are the assumptions made:

- A, B pair is controllable.
- All the states are observable.
- Uncertainty $d(t)$ satisfies the matching condition.

The objective is to design an optimal sliding mode controller for the linear uncertain MIMO system (3.64). The control process is divided into two steps, viz. (i) Designing an optimal controller for the nominal system and (ii) Designing the sliding mode controller to tackle uncertainties affecting the system. So, the control input $u(t)$ is defined as

$$
u(t) = u_1(t) + u_2(t) \tag{3.65}
$$

where $u_1(t)$ is called the equivalent control and $u_2(t)$ is known as switching control.

II. Optimal controller design

The optimal controller is designed for the nominal part of the system. Neglecting the uncertainty, the nominal part of the system (3.64) can be written as

$$
\dot{x}(t) = Ax(t) + Bu_1(t) \tag{3.66}
$$

Performance index J is chosen to minimize the control input and is defined as follows:

$$
J = \int_0^\infty [x(t)^T Q x(t) + u_1(t)^T R u_1(t)] dt \tag{3.67}
$$

where $Q \in \mathcal{R}^{n \times n}$ and $R \in \mathcal{R}^{m \times m}$ are a positive definite weighing matrix. The optimal control law $u_1(t)$ is obtained as

$$u_1(t) = -R^{-1}B^T Px(t) = -Kx(t) \tag{3.68}$$

where $K = R^{-1}B^T P$ and P is a symmetric positive definite matrix which is the solution of the algebraic Riccati equation [100]

$$A^T P + PA + Q - PBR^{-1}B^T P = 0 \tag{3.69}$$

The controller designed above minimizes the control input $u_1(t)$. However, the main disadvantage of the optimal controller is that it requires the exact model of the system apriori. Hence, in presence of system uncertainties, performance of the optimal controller degrades and may even lead to instability. So, a sliding mode controller is integrated with the optimal controller to make it immune to system uncertainties.

III. Sliding mode controller design

The main feature of the sliding mode control (SMC) is that once the system is brought onto the sliding surface, it becomes insensitive to any matched uncertainty affecting the system. This is the motivation for combining a sliding mode control (SMC) scheme with the optimal controller. In the proposed SMC, an integral sliding surface is used which is given as follows:

$$s(t) = G[x(t) - \int_0^t \Phi(\tau)d\tau] = 0 \tag{3.70}$$

where $G \in \mathcal{R}^{m\times n}$ is designed in such a way that GB is invertible and $\Phi(t)$ is defined as

$$\Phi(t) = Ax(t) + Bu_1(t) \tag{3.71}$$

The sliding hyperplane $\mathcal{S} = 0$ for the MIMO system is defined as

$$\mathcal{S} = \bigcap_{i=1}^{m} \mathcal{S}_i \tag{3.72}$$

where $\mathcal{S}_i$ is the i-th row of $\mathcal{S}$ and is given by

$$\mathcal{S}_i = \{x(t) \in \mathcal{R}^n \quad : \quad s_i(t) = 0\} \tag{3.73}$$

Here $s_i(t)$ is the i-th row of matrix $s(t) \in \mathcal{R}^m$.

To eliminate chattering in the sliding mode controller, a second order sliding mode is realized by using a terminal sliding surface based on the integral sliding variable discussed above. The terminal sliding mode ensures finite time convergence of the sliding variables. The terminal sliding variable $\sigma(t)$ is designed as

$$\sigma(t) = \begin{bmatrix} \sigma_1(t) & \sigma_2(t) & \cdots & \sigma_i(t) & \cdots & \sigma_m(t) \end{bmatrix}^T \tag{3.74}$$

$$\text{where} \quad \sigma_i(t) = s_i(t) + \delta_i \dot{s}_i(t)^{\frac{\alpha_i}{\beta_i}}; \quad i = 1, 2, \cdots, m \tag{3.75}$$

Here $\delta_i,\ \alpha_i,\ \beta_i$ are design parameters of the ith terminal sliding variable and $i = 1,\ 2,\ ...,\ m$. These parameters should satisfy the following conditions

$$\delta_i > 0 \tag{3.76}$$

$$\alpha_i, \beta_i \in \{2n+1 : \ n \, is \, an \, integer\} \tag{3.77}$$

and

$$1 < \frac{\alpha_i}{\beta_i} < 1.5; \quad i = 1, 2, \cdots, m \tag{3.78}$$

Switching control $u_2(t)$ is defined as $u_2(t) = [u_{21}(t) \ \ u_{22}(t) \ \ ... \ \ u_{2i}(t) \ \ ... \ \ u_{2m}(t)]$.

Using the constant plus proportional reaching law [104], the switching control u_{2i} is designed using the approach discussed in Section 3.2 and is given by

$$u_{2i}(t) = -\int_0^t (G_i B_i)^{-1} \left[\frac{\beta_i}{\delta_i \alpha_i} \dot{s}_i(\tau)^{2-\frac{\alpha_i}{\beta_i}} + \eta_i \frac{|\sigma_i(\tau)|}{\sigma_i(\tau)} + \varepsilon_i \sigma_i(\tau) \right] d\tau \tag{3.79}$$

where G_i is the i-th row and B_i is the i-th column of the design parameter G and input distribution matrix B respectively. The design parameters η_i and ε_i are chosen in such a way that $||G_i B_i \dot{d}_i(t)|| < \eta_i$ [66] and $\varepsilon_i > 0$.

Stability analysis of sliding surfaces:

Let us consider the Lyapunov function $V_i(t)$ as

$$\begin{aligned}
V_i(t) &= \frac{1}{2}\sigma_i^2(t) \\
\dot{V}_i(t) &= \sigma_i(t)\dot{\sigma}_i(t) \\
&= \sigma_i(t)[\dot{s}_i(t) + \frac{\alpha_i\delta_i}{\beta_i}\dot{s}_i(t)^{\frac{\alpha_i}{\beta_i}-1}\ddot{s}_i(t)] \\
&= \sigma_i(t)[\dot{s}_i(t) + \frac{\alpha_i\delta_i}{\beta_i}\dot{s}_i(t)^{\frac{\alpha_i}{\beta_i}-1}(G_iB_i\dot{u_{2i}}(t) + G_iB_i\dot{d}_i(t))] \\
&= \sigma_i(t)[\dot{s}_i(t) + \frac{\alpha_i\delta_i}{\beta_i}\dot{s}_i(t)^{\frac{\alpha_i}{\beta_i}-1}(-\frac{\beta_i}{\delta_i\alpha_i}\dot{s}_i(t)^{2-\frac{\alpha_i}{\beta_i}} - \eta_i\frac{|\sigma_i(t)|}{\sigma_i(t)} - \varepsilon_i\sigma_i(t) + G_iB_i\dot{d}_i(t))] \\
&= \sigma_i(t)[\frac{\alpha_i\delta_i}{\beta_i}\dot{s}_i(t)^{\frac{\alpha_i}{\beta_i}-1}[-\eta_i\frac{|\sigma_i(t)|}{\sigma_i(t)} - \varepsilon_i\sigma_i(t) + G_iB_i\dot{d}_i(t)] \\
&= \frac{\alpha_i\delta_i}{\beta_i}\dot{s}_i(t)^{\frac{\alpha_i}{\beta_i}-1}[-\eta_i|\sigma_i(t)| - \varepsilon_i\sigma_i^2(t) + G_iB_i\dot{d}_i(t)\sigma_i(t)] \\
&\leq \frac{\alpha_i\delta_i}{\beta_i}\dot{s}_i(t)^{\frac{\alpha_i}{\beta_i}-1}[-\eta_i|\sigma_i(t)| - \varepsilon_i\sigma_i^2(t) + G_iB_i\dot{d}_i|\sigma_i(t)|] \\
&\leq -\frac{\alpha_i\delta_i}{\beta_i}\dot{s}_i(t)^{\frac{\alpha_i}{\beta_i}-1}[\eta_i + \varepsilon_i|\sigma_i(t)| - G_iB_i\dot{d}_i]|\sigma_i(t)| \\
&\leq -\kappa|\sigma_i(t)| \qquad (3.80) \\
& \qquad (3.81)
\end{aligned}$$

where $\kappa = \frac{\alpha_i\delta_i}{\beta_i}\dot{s}_i(t)^{\frac{\alpha_i}{\beta_i}-1}[\eta_i + \varepsilon_i|\sigma_i(t)| - G_iB_i\dot{d}_i]$ is positive for $s_i(t) \neq 0$. In [66] Wang et al. showed that $\dot{s}_i^{\frac{\alpha_i}{\beta_i}-1}(t) > 0$ for $|s(t)| \neq 0$.

Hence,

$$\begin{aligned}
\dot{V}_i(t) &\leq -\kappa|\sqrt{2V_i}| \quad \text{as} \quad V_i(t) = \frac{1}{2}\sigma_i^2(t) \qquad (3.82) \\
& \qquad (3.83)
\end{aligned}$$

Hence $\sigma_i(t)$ converges to zero in finite time [101, 105]. Moreover, it can also be shown that integral sliding variable $s_i(t)$ converges to 0 in finite time. Suppose within the finite time t_r, $\sigma_i(t)$ reaches zero from $\sigma_i(0) \neq 0$ and $\sigma_i(t) = 0 \ \forall \ t > t_r$. So, once $\sigma_i(t)$ reaches zero, it remains at zero and based on (3.82), $s_i(t)$ will converge to zero in time t_s. The total time required from $\sigma_i(0) \neq 0$ to $s_i(t_s)$ can be computed as described below.

From (3.75), it follows that

$$\begin{aligned} s_i(t) + \delta_i \dot{s}_i(t)^{\frac{\alpha_i}{\beta_i}} &= 0 \\ or,\ (\frac{s_i}{\delta_i})^{\frac{\beta_i}{\alpha_i}} &= -(\dot{s}_i(t)^{\frac{\alpha_i}{\beta_i}})^{\frac{\beta_i}{\alpha_i}} \\ or,\ \frac{1}{\delta_i^{\frac{\beta_i}{\alpha_i}}} s_i(t)^{\frac{\beta_i}{\alpha_i}} &= -\dot{s}_i(t) \end{aligned} \tag{3.84}$$

As α_i, β_i are chosen according to (3.77) and (3.78), (3.84) can be written as

$$\begin{aligned} \frac{1}{\delta_i^{\frac{\beta_i}{\alpha_i}}} s_i(t)^{\frac{\beta_i}{\alpha_i}} &= -\frac{ds_i(t)}{dt} \\ or,\ \frac{1}{\delta_i^{\frac{\beta_i}{\alpha_i}}} dt &= -\frac{ds_i(t)}{s_i(t)^{\frac{\beta_i}{\alpha_i}}} \\ or,\ \int_{t_r}^{t_s} dt &= -\delta_i^{\frac{\beta_i}{\alpha_i}} \int_{s_i(t_r)}^{s_i(t_s)} \frac{ds_i(t)}{s_i(t)^{\frac{\beta_i}{\alpha_i}}} \\ or,\ t_s - t_r &= -\frac{\alpha_i}{\alpha_i - \beta_i} \delta_i^{\frac{\beta_i}{\alpha_i}} \left[s_i(t_s)^{\frac{\alpha_i - \beta_i}{\alpha_i}} - s_i(t_r)^{\frac{\alpha_i - \beta_i}{\alpha_i}} \right] \end{aligned} \tag{3.85}$$

At time t_s, sliding variable $s_i(t_s) = 0$. So, (3.85) gives rise to

$$t_s = t_r + \frac{\alpha_i}{\alpha_i - \beta_i} \delta_i^{\frac{\beta_i}{\alpha_i}} s_i(t_r)^{\frac{\alpha_i - \beta_i}{\alpha_i}} \tag{3.86}$$

Hence, $s_i(t)$ and $\dot{s}_i(t)$ converge to zero in finite time. As every $s_i(t)$ converges to zero in finite time, $s(t)$ also reaches zero in finite time.

IV. Simulation results

The proposed controller is applied to stabilize a linear uncertain MIMO system and the performances of the controller are compared with some other existing controllers.

Example

An unstable batch reactor with matched uncertainty is considered whose state space model [6] is described as

$$\dot{x}(t) = \begin{bmatrix} 1.3800 & -0.2077 & 6.7150 & -5.6760 \\ -0.5814 & -4.2900 & 0 & 0.6750 \\ 1.0670 & 4.2370 & -6.6540 & 5.8930 \\ 0.0480 & 4.2730 & 1.3430 & -2.1040 \end{bmatrix} x(t) \quad + \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix} \left(\begin{bmatrix} u_a(t) \\ u_b(t) \end{bmatrix} + \begin{bmatrix} 1.5\sin t \\ \cos(\pi t) \end{bmatrix} \right) \tag{3.87}$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t)$$

where $x(t)$ is the state of the system. Two control inputs of the MIMO system are $u_a(t)$ and $u_b(t)$. Uncertainty of the system is defined as $\begin{bmatrix} 1.5\sin t \\ \cos(\pi t) \end{bmatrix}$

To design optimal controller weighing matrices of the performance index J in (3.67) are chosen as $Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $R = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$.

Using the LQR technique, the feedback control $u_1(t)$ is found as

$$u_1(t) = \begin{bmatrix} 0.5621 & 0.3773 & 0.4741 & 0.0381 \\ -1.7915 & -0.1654 & -1.2527 & 0.7666 \end{bmatrix} x(t) \tag{3.88}$$

Design parameters of the proposed optimal second order sliding mode controller are chosen as follows:

$G = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$, $\alpha_1 = \alpha_2 = 7$, $\beta_1 = \beta_2 = 5$, $\delta_1 = \delta_2 = 0.15$, $\eta_1 = \eta_2 = 3$, $\varepsilon_1 = \varepsilon_2 = 0.1$.

The proposed OSOSMC is applied to stabilize the system (3.87). The results obtained by using the proposed OSOSMC are compared with the results obtained by applying the dynamic sliding mode controller proposed by Chang [6] which is discussed in appendix A.6. States $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$ obtained by applying the proposed OSOSMC and the SMC proposed by Chang [6] are shown in Figures.3.15 - 3.18. In Figure 3.19 and 3.20, the control inputs obtained by using the proposed

OSOSMC and the SMC proposed by Chang [6] are compared.

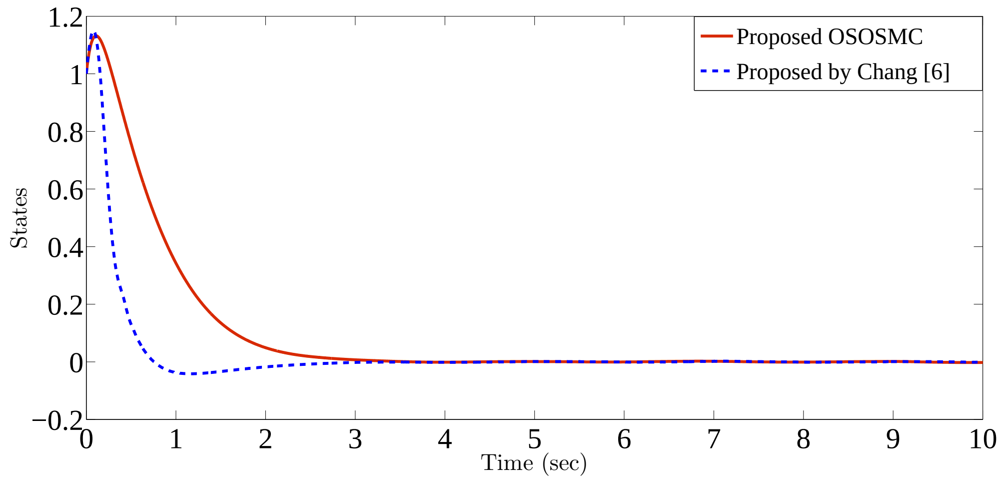


Figure 3.15: State $x_1(t)$ obtained by applying the proposed OSOSMC and the SMC proposed by Chang [6]

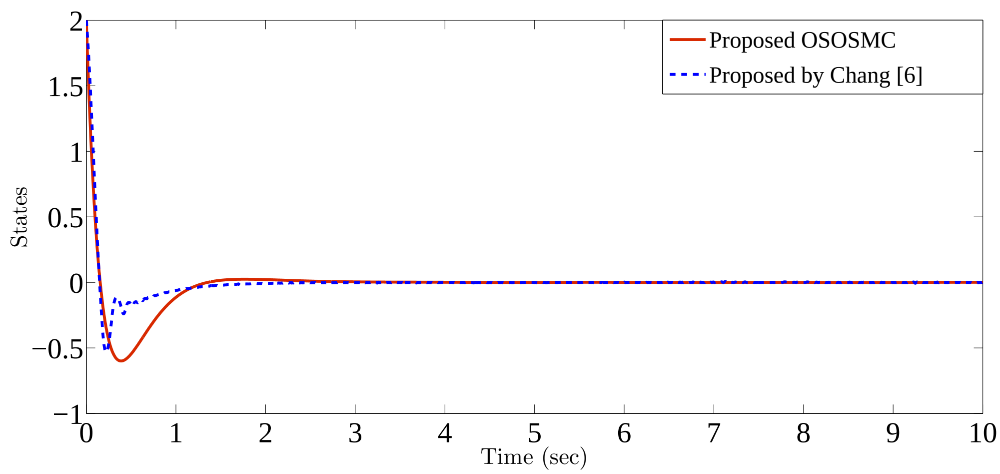


Figure 3.16: State $x_2(t)$ obtained by applying the proposed OSOSMC and the SMC proposed by Chang [6]

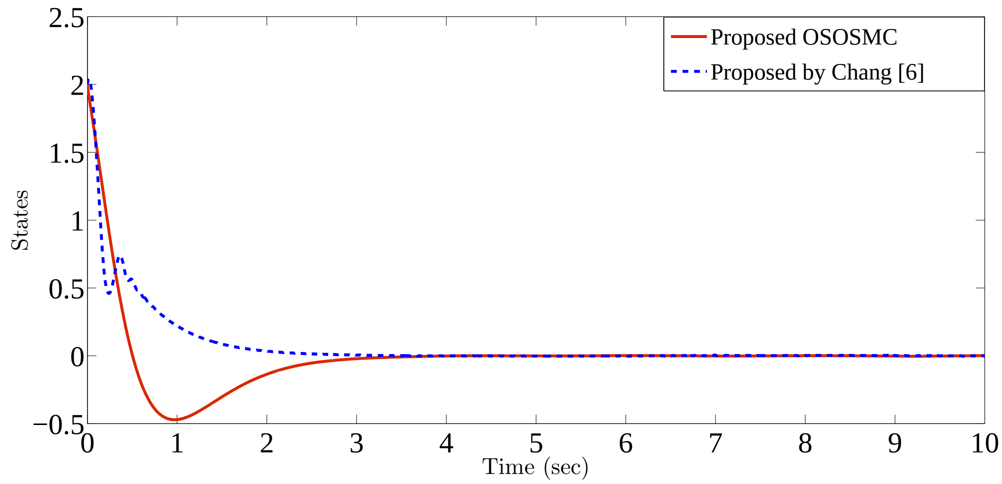


Figure 3.17: State $x_3(t)$ obtained by applying the proposed OSOSMC and the SMC proposed by Chang [6]

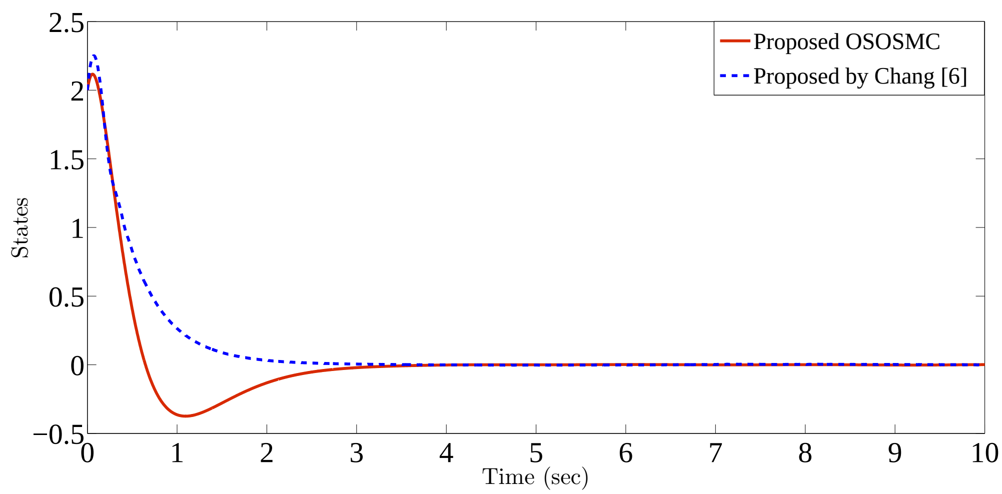


Figure 3.18: State $x_4(t)$ obtained by applying the proposed OSOSMC and the SMC proposed by Chang [6]

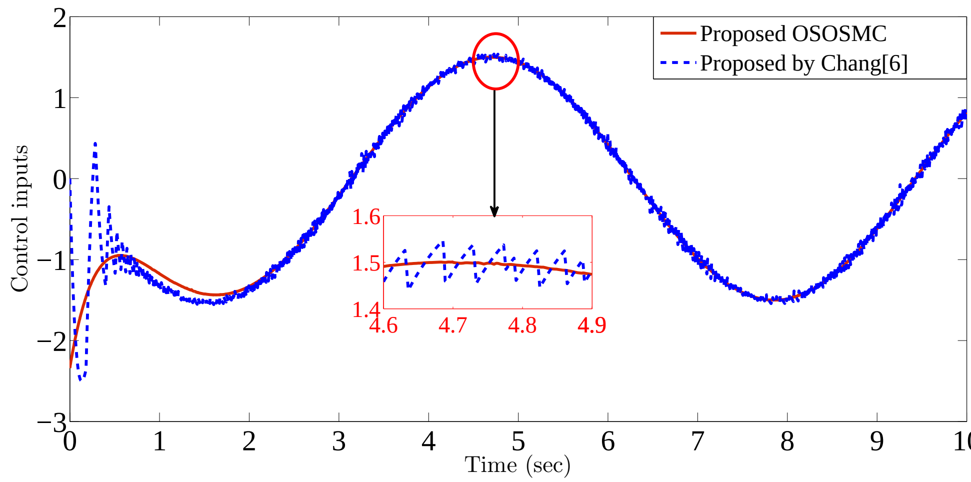


Figure 3.19: Control input $u_a(t)$ obtained by applying the proposed OSOSMC and the SMC proposed by Chang [6]

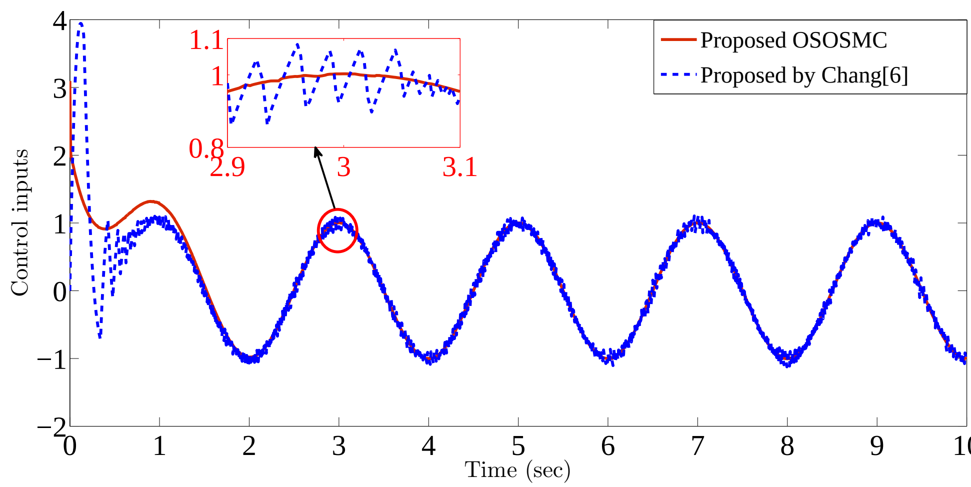


Figure 3.20: Control input $u_b(t)$ obtained by applying the proposed OSOSMC and the SMC proposed by Chang [6]

From above figures it is clear that the proposed OSOSMC achieves similar performance as that of the dynamic sliding mode controller proposed by Chang [6], but at the cost of much reduced control input. Moreover, the control input in the case of the SMC by Chang [6] is full of chattering whereas the proposed OSOSMC offers a smooth chattering free control input. In Table 3.7, the Total variation

(TV) and the 2-norm of the control input for the proposed OSOSMC and the SMC by Chang [6] are compared. Here the TV and control energy are calculated for the period from 0 to 10 sec with a sampling time of 0.01 sec. It is clear from Table 3.7 that the proposed OSOSMC is capable to produce a smoother control input with lesser control energy than the SMC proposed by Chang [6].

Table 3.7: Comparison of Control Indices for MIMO System in Example 1

Control input	Method	Total Variation (TV)	Control Energy
$u_a(t)$	Proposed OSOSMC	10.12	33.92
	SMC proposed by Chang [6]	56.80	34.47
$u_b(t)$	Proposed OSOSMC	21.01	24.88
	SMC proposed by Chang [6]	84.77	26.42

The performances of the system obtained by applying proposed OSOSMC and SMC proposed by Chang [6] are tabulated in Table 3.8.

Table 3.8: Comparison of Performance Indices for MIMO System in Example I

Performance specification	Proposed OSOSMC				SMC [6]			
	x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4
Rise time (sec)	1.28	0.11	0.38	0.38	0.36	1.29	0.98	0.93
Settling time (sec)	2.37	1.17	2.65	2.61	1.81	2.00	1.91	1.84
Steady state error	0	0	0	0	0	0	0	0

3.5 Summary

In this chapter an optimal second order sliding mode controller (OSOSMC) is proposed for linear uncertain systems. The optimal controller is designed for the nominal part of the linear uncertain system using the linear quadratic regulator (LQR) technique. In order to make the optimal controller robust against uncertainties affecting the system, a sliding mode control (SMC) method is integrated with the optimal controller by using an integral sliding surface. The sliding mode control (SMC) scheme is developed by designing a nonsingular terminal sliding surface based on the integral sliding variable and thereby giving rise to a second order sliding mode control strategy. The advantage of using the nonsingular terminal sliding surface is that it converges the integral sliding variable and its first derivative to the equilibrium in finite time. The OSOSMC designed for stabilization of the linear uncertain system can also be used for output tracking. To design the tracking controller, the tracking problem is converted into a regulatory problem by transforming the system into the error coordinates and then the OSOSMC is designed in the error domain. The OSOSMC designed for the linear single input single output (SISO) uncertain system is extended for the linear decoupled multi input multi output (MIMO) uncertain system also. Simulation studies are conducted to compare the proposed controller with other existing controllers developed to control linear uncertain systems. From simulation results it is observed that the control input in the case of the proposed OSOSMC is smoother and requires lesser energy than some existing control methods developed for similar purpose.

4

State dependent Riccati equation (SDRE) based optimal second order sliding mode controller for nonlinear uncertain systems

Contents

4.1 Introduction

Designing an optimal controller for linear systems is well defined in literature [12,13]. But it is still a challenge to design an optimal controller for nonlinear systems. One way to design optimal controller for nonlinear systems is to solve the Hamilton Jacobi Bellman (HJB) [14] partial differential equation. Unfortunately, analytical solution of the HJB partial differential equation has not been possible. Numerical solution of the HJB equation is computationally intensive due to its high dimension. A recent popular approach to design optimal control for nonlinear systems is by solving the state dependent Riccati equation (SDRE) [15–18]. For a systematic design approach, SDRE based optimal control has earned good attention within the control community. In this method the nonlinear matrix is factorized into the product of a matrix valued function and the state vector. Hence, the nonlinear system is converted into a linear like structure having state dependent coefficient (SDC) matrices [19, 20]. Moreover, choice of different weighing matrices is possible offering a wide range of design flexibility. Hence SDRE is a good option for designing optimal controller for nonlinear systems. For the nonlinear uncertain system, design procedure of the optimal sliding mode control is divided into two steps. In the first step, an optimal controller is designed for the nominal part of the nonlinear uncertain system using SDRE and in the second part the sliding mode controller (SMC) [33–35, 47] is combined with the optimal controller by designing an integral sliding surface [54–56]. As integral sliding mode has no reaching phase, the system becomes robust from the very beginning. However, the integral SMC is affected by high frequency oscillations known as chattering which degrades the system performance and may even lead the system towards instability. Chattering occurs in first order SMC due to the discontinuous switching control. Designing higher order SMC [1,3,65,66] is one effective way to reduce chattering in the SMC. Among higher order SMCs, second order sliding mode controller (SOSMC) is easy to implement because of its lesser information requirement.

Chaos is an interesting nonlinear phenomenon. It exhibits unpredictable and irregular dynamics depending on its initial conditions because a small change in the initial states can lead to extraordinary different state trajectories. In many engineering applications like lasers [107], Colpitt's oscillators [108], nonlinear circuits [109] and communication [110], chaotic behavior is visible. In the recent past, chaotic systems such as Lorenz system [111], Chua's circuit [112], Chen system [113] were proposed and their complex behaviors were studied. In 1990 chaos control was first considered by Ott et al. [114]. Many control strategies such as adaptive control [25,115], backstepping control [116,117] and observer based control [118, 119] were developed to control the chaotic system. Sliding mode controllers [120, 121] have been effectively applied to control the chaotic system. But the conventional SMC requires a high control energy. Some chaotic systems can be represented as linear like structure having SDC matrices. For those system SDRE based optimal control can be designed and integrated with the SMC.

In the first part of this chapter an optimal second order sliding mode controller for nonlinear uncertain systems is proposed. The optimal controller is designed for the nominal nonlinear system by solving the SDRE. For designing, the nonlinear system is converted into a linear like structure having state dependent coefficient (SDC) matrices. The SDRE based optimal controller design is similar to the linear quadratic regulator (LQR) technique used to design the optimal controller for linear systems. After designing the optimal controller, it is combined with a sliding mode controller (SMC) by using an integral sliding surface. Based on the integral sliding variable, a non-singular terminal sliding mode controller (TSMC) is developed which imparts second order characteristic to the SMC. Further, it guarantees finite time convergence of the integral sliding variable and its first derivative to the equilibrium. The proposed optimal second order sliding mode controller (OSOSMC) is applied for stabilization and tracking problems of nonlinear uncertain systems. Simulation results confirm that performance of the proposed controller is better than some existing control methodologies developed for similar systems. The proposed controller is applied to stabilize certain chaotic systems which is described in the second part of this chapter.

The outline of the chapter is as follows. In Section 4.2 optimal second order sliding mode controller (OSOSMC) for nonlinear uncertain systems is developed. In Section 4.3 the OSOSMC is proposed to stabilize chaotic systems. In Section 4.4 a brief summary of the chapter is presented.

4.2 Optimal second order sliding mode controller for nonlinear uncertain systems

Optimal second order sliding mode controller design is divided into two parts. In the first part the optimal controller is designed for the nominal nonlinear system by solving state dependent Riccati equation (SDRE) and then in the second part a second order sliding mode controller is designed to tackle uncertainties.

I. Problem statement

Let us consider an uncertain nonlinear system

$$\dot{x}(t) = f(x) + g(x)u(t) + d(t) \tag{4.1}$$

where $f(x) = \begin{bmatrix} x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \\ a(x)x(t) \end{bmatrix}$, $g(x) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b(x) \end{bmatrix}$. and $d(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ d_1(t) \end{bmatrix}$.

Functions $f : \mathcal{R}^n \to \mathcal{R}^n$ and $g : \mathcal{R}^n \to \mathcal{R}^n$ are continuous for $t \in [0, \infty)$ and $g(x) \neq 0 \ \forall x$. Here, $a(x)$ is a nonlinear function defined as $a(x) = [a_1(x) \ \ a_2(x) \ \ , \cdots , \ \ a_n(x)]$ and nonlinear function $b(x)$ is associated with the control input. The uncertainty affecting the system is denoted by $d_1(t)$.

Assumptions:

• It is assumed that the bounded uncertainty $d_1(t)$ satisfies the matching condition and its first time derivative also exists.

The objective is to design a controller for stabilization and trajectory tracking of the nonlinear uncertain system (4.1) with minimum control expense. To achieve this target, a robust controller is designed by combining the optimal controller designed for the nominal nonlinear system with a second order sliding mode controller. As such, the control input $u(t)$ is the combination of two different control laws and can be defined as

$$u(t) = u_1(t) + u_2(t) \tag{4.2}$$

where $u_1(t)$ is the optimal control designed to stabilize or track the nominal nonlinear system and the control input $u_2(t)$ is a second order sliding mode controller which is designed to tackle uncertainties in the system.

To design an optimal controller based on the state dependent Riccati equation (SDRE), the nonlinear system (4.1) is transformed into a linear like structure by using extended linearization [19]. For this linearized system, the equilibrium point is at $x = 0$ where $f(0) = 0$.

A. Extended linearization

The process of factorizing a nonlinear system into a linear like structure which contains state dependent coefficient (SDC) matrix [20], is called extended linearization [19]. This process is also known as apparent linearization or state dependent coefficient parametrization. Considering the assumption

$$f \in \mathbb{C}^1 \quad \text{and} \quad f(0) = 0, \tag{4.3}$$

a continuous nonlinear function $f(x)$ can be written as [19, 20]

$$f(x) = A(x)x(t) \tag{4.4}$$

where $A(x) \in \mathcal{R}^{n\times n}$ is a SDC matrix. Moreover, $A(x)$ cannot be determined uniquely if n is more than 1. Hence, extended linearization of nonlinear system (4.1), under the assumption on function $f(x)$ can be written as

$$\dot{x}(t) = A(x)x(t) + g(x)(u_1(t) + u_2(t)) + d(t) \tag{4.5}$$

where $A(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ a_1(x) & \cdots & a_{n-1}(x) & a_n(x) \end{bmatrix}$ and $\begin{bmatrix} a_1(x) & \cdots & a_{n_1}(x) & a_n(x) \end{bmatrix} = a(x)$ denotes the nonlinearity of the system matrix. The above equation (4.5) represents a linear structure with SDC matrices $A(x)$ and $g(x)$.

Remark 1: If pairs $A(x)$ and $g(x)$ are point wise controllable in the linear sense in a region Θ $\forall x \in \Theta$, then the SDC representation (4.5) is the controllable parameterization of the nonlinear system (4.1) in region Θ [19].

Remark 2: If the eigen values λ_i of $A(x)$ are such that $Re[\lambda_i(A(x))] < 0 \quad \forall x(t) \in \Theta$, then the SDC representation (4.5) is point wise Hurwitz in the region Θ [19].

B. System defined in error domain

The proposed objective is to track the output of the system ($y(t) = x_1(t)$) along the desired trajectory $x_d(t)$. So, the tracking error is defined by

$$e(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_n(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} - \begin{bmatrix} x_d(t) \\ x_d^{(1)}(t) \\ \vdots \\ x_d^{(n-1)}(t) \end{bmatrix} \tag{4.6}$$

where $x_d^{(1)}(t), x_d^{(2)}(t), \cdots, x_d^{(n-1)}(t)$ are the successive derivatives of the desired trajectory $x_d(t)$.

The error dynamics can be defined as

$$
\begin{aligned}
\dot{e}(t) = \begin{bmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \\ \vdots \\ \dot{e}_n(t) \end{bmatrix} &= \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} - \begin{bmatrix} x_d^{(1)}(t) \\ x_d^{(2)}(t) \\ \vdots \\ x_d^{(n)}(t) \end{bmatrix} \\
&= \begin{bmatrix} x_2(t) \\ x_3(t) \\ \vdots \\ a_1(x)x_1(t) + \cdots + a_{n-1}(x)x_{n-1}(t) + a_n(x)x_n(t) + b(x)(u_1(t) + u_2(t)) + d_1(t) \end{bmatrix} - \begin{bmatrix} x_d^{(1)}(t) \\ x_d^{(2)}(t) \\ \vdots \\ x_d^{(n)}(t) \end{bmatrix} \\
&= \begin{bmatrix} x_2(t) \\ x_3(t) \\ \vdots \\ a_1(x)x_1(t) + \cdots + a_n(x)x_n(t) \end{bmatrix} - \begin{bmatrix} x_d^{(1)}(t) \\ x_d^{(2)}(t) \\ \vdots \\ a_1(x)x_d(t) + \cdots + a_n(x)x_d^{(n-1)}(t) \end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ a_1(x)x_d(t) + \cdots + a_n(x)x_d^{(n-1)}(t) - x_d^{(n)}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b(x) \end{bmatrix} (u_1(t) + u_2(t)) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ d_1(t) \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ a_1(x) & \cdots & a_{n-1}(x) & a_n(x) \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b(x) \end{bmatrix} (u_1(t) + u_2(t)) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ d_1(t) \end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ a_1(x)x_d(t) + \cdots + a_n(x)x_d^{(n-1)}(t) - x_d^{(n)}(t) \end{bmatrix} = A(x)e(t) + g(x)(u_1(t) + u_2(t)) + d(t) + \widetilde{A}(t)
\end{aligned}
\tag{4.7}
$$

where $\widetilde{A}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ a_1(x)x_d(t) + \cdots + a_n(x)x_d^{(n-1)}(t) - x_d^{(n)}(t) \end{bmatrix}$ is a known matrix and it is in the range space of the input distribution matrix $g(x)$. It will be taken care of by designing a sliding mode controller to be explained later.

II. Optimal control

The optimal controller is designed for the nominal system. Hence, neglecting the uncertain part, (4.7) can be written as

$$\dot{e}(t) = A(x)e(t) + g(x)u_1(t) \tag{4.8}$$

To minimize the control effort, the performance index is chosen as

$$J = \int_0^\infty \{e^T(t)Q(x)e(t) + u_1^T(t)R(x)u_1(t)\}dt \tag{4.9}$$

where weighing matrices $Q : \mathcal{R}^n \to \mathcal{R}^{n\times n}$ and $R : \mathcal{R} \to \mathcal{R}$ are state dependent. At the time of designing, $Q(x)$ and $R(x)$ are chosen as a positive definite matrix. Optimal feedback control $u_1(t)$ is designed in such a way that it minimizes the cost function (4.9) subject to the nonlinear differential constraint (4.8) and also converges the system (4.8) to zero. This requires,

$$\lim_{t\to\infty} e(t) = 0$$

This is an optimal control problem based on state dependent Riccati equation (SDRE).

The basic approach of designing an optimal controller for nonlinear systems using SDRE methodology is by using extended linearization. Now, emulating the linear quadratic regulator method [18], the feedback control is designed as follows:

$$u_1(t) = -R^{-1}(x)g^T(x)P(x)e(t) \tag{4.10}$$

where $P(x)$ is the symmetric positive definite matrix which is the solution of the continuous time state dependent Riccati equation

$$A^T(x)P(x) + P(x)A(x) - P(x)g(x)R^{-1}(x)g^T(x)P(x) + Q(x) = 0 \tag{4.11}$$

Hence, the closed loop system (4.8) can be written as

$$\begin{aligned}\dot{e}(t) &= A(x)e(t) - g(x)R^{-1}(x)g^T(x)P(x)e(t)\\ &= [A(x) - g(x)R^{-1}(x)g^T(x)P(x)]e(t)\\ &= \widehat{A}(x)e(t)\end{aligned} \tag{4.12}$$

where $\widehat{A}(x) = A(x) - g(x)R^{-1}(x)g^T(x)P(x)$.

The control action is obtained by solving the linear quadratic optimal control problem as state dependent coefficient matrices which are considered as constant. To design the optimal controller based on SDRE method, HJB equation need not be solved. This is the key benefit of the SDRE based method.

A. Stability analysis

The conditions to be satisfied for guaranteeing stability of the nominal system are as follows:

Condition 1: $A(.)$, $B(.)$, $Q(.)$ and $R(.)$ are continuous matrix valued functions in the region $\mathcal{R}^n$ [122].

Condition 2: $A(x), B(x)$ is point wise stabilizable state dependent coefficient (SDC) matrix pair [122].

The closed loop system considered is

$$\dot{e}(t) = \widehat{A}(x)e(t) \tag{4.13}$$

where $\widehat{A}(x)$ is the closed loop SDC matrix as described in (4.12). With Condition 1, $P(x)$ is continuous. Hence $\widehat{A}(x)$ is also continuous in the region $\mathcal{R}^n$. By using mean value theorem, $\widehat{A}(x)$ can be written as

$$\widehat{A}(x) = \widehat{A}(0) + \frac{\partial \widehat{A}(z)}{\partial e}e(t) \tag{4.14}$$

where $\frac{\partial \widehat{A}(z)}{\partial e}$ generates a tensor and the vector z is that point on the line segment joining the origin 0 and $e(t)$ [122].

Using (4.14), (4.13) can be written as

$$\begin{aligned}\dot{e}(t) &= \widehat{A}(0)e(t) + e^T(t)\frac{\partial \widehat{A}(z)}{\partial e}e(t) \\ &= \widehat{A}(0)e(t) + \Gamma(e,z)||e(t)|| \end{aligned} \tag{4.15}$$

where $\Gamma(e,z) \triangleq \frac{1}{||e(t)||}e^T(t)\frac{\partial \widehat{A}(z)}{\partial e}e(t)$, such that

$$\lim_{|e(t)|\to 0} \Gamma(e,z) = 0$$

Here the constant stable coefficient matrix $\widehat{A}(0)$ dominates the higher order term. Now as $\widehat{A}(x)$ is linearizable and linearized $\widehat{A}(0)$ has a stable equilibrium point, $\widehat{A}(0)$ has negative eigenvalues meaning that $\widehat{A}(0)$ is a Hurwitz matrix. Hence $\widehat{A}(x)$ is also stable [122].

To prove global stability, the Lyapunov function is chosen as

$$\begin{aligned} V_1(t) &= \tfrac{1}{2}e^T(t)e(t) \\ \dot{V}_1(t) &= e^T(t)\dot{e}(t) \\ &= e^T(t)\widehat{A}(x)e(t) \end{aligned} \tag{4.16}$$

As $\widehat{A}(x)$ is stable, $\dot{V}_1(t) < 0$. Hence, the closed loop system is globally stable.

B. Minimization of the performance index

To check the optimality of the proposed controller, a commonly used method is to find the solution of the HJB equation. The Hamiltonian is given by

$$H = \lambda^T[A(x)e(t) + g(x)u_1(t)] + [e^T(t)Q(x)e(t) + u_1^T(t)R(x)u_1(t)] \tag{4.17}$$

where λ is a lagrangian multiplier and the HJB equation is given by

$$\lambda^T[A(x)e(t) + g(x)u_1(t)] + [e^T(t)Q(x)e(t) + u_1^T(t)R(x)u_1(t)] = 0 \tag{4.18}$$

The Hamiltonian dynamics in Lagrangian manifold are defined as

$$\dot{e}(t) = \frac{\partial H}{\partial \lambda}; \qquad \dot{\lambda} = -\frac{\partial H}{\partial e} \tag{4.19}$$

The optimal control $u_1^*(t)$ is found from $\frac{\partial H}{\partial u_1^*} = 0$. Hence the optimal controller is defined by

$$u_1^*(t) = -\frac{1}{2}R^{-1}(x)g^T(x)\lambda \tag{4.20}$$

Substituting the optimal value of $u_1(t)$ in (4.18), it is found that

$$\lambda^T A(x)e(t) - \frac{1}{4}\lambda^T g(x)R^{-1}(x)g^T(x)\lambda + e^T(t)Q(x)e(t) = 0 \tag{4.21}$$

The optimal cost is the solution of (4.21). From (4.19) it can be written [123] that

$$\lambda = 2P(x)e(t) \tag{4.22}$$

where $P(x)$ is the symmetric positive definite matrix.

Substituting the value of λ in (4.21), it can be written that

$$\begin{aligned}
&e^T(t)2P(x)A(x)e(t) - e^T(t)P(x)g(x)R^{-1}(x)g^T(x)P(x)e(t) + e^T(t)Q(x)e(t) = 0 \\
&or,\ e^T(t)[A^T(x)P(x) + P(x)A(x) - P(x)g(x)R^{-1}(x)g^T(x)P(x) + Q(x)]e(t) = 0 \\
&or,\ A^T(x)P(x) + P(x)A(x) - P(x)g(x)R^{-1}(x)g^T(x)P(x) + Q(x) = 0 \qquad (4.23)
\end{aligned}$$

Above equation (4.23) is a state dependent Riccati equation. So, the optimal control $u_1^*(t)$ can be obtained as

$$\begin{aligned}
u_1^*(t) &= -\frac{1}{2}R^{-1}(x)g^T(x)2P(x)e(t) \\
&= -R^{-1}(x)g^T(x)P(x)e(t) \qquad (4.24)
\end{aligned}$$

Hence, the controller defined in (4.10) minimizes the given performance index (4.9).

III. Sliding mode control

The uncertain system in the error domain (4.7) can be described as

$$\dot{e}(t) = A(x)e(t) + g(x)u_1(t) + g(x)u_2(t) + \zeta(t) \tag{4.25}$$

where $\zeta(t) = \widetilde{A}(t) + d(t)$.

Using the optimal control for the nominal part of the system using (4.24), the uncertain system (4.25) can be written as follows:

$$\dot{e}(t) = A(x)e(t) - g(x)R^{-1}(x)g^T(x)P(x)e(t) + g(x)u_2(t) + \zeta(t) \tag{4.26}$$

As both $\widetilde{A}(t)$ and uncertainty $d(t)$ are in the range space of the input matrix $g(x)$, $\zeta(t)$ satisfies the matching condition [46].

A sliding mode control (SMC) mechanism is now combined with the optimal controller to make it robust against uncertainties affecting the system (4.26).

The second order sliding mode control is designed by constructing a nonsingular TSMC which is based on ISMC. The ISMC proposed here does not required the knowledge of initial condition. This is an achievement of proposed controller. But the integral sliding variable is not equal to zero initially. Hence, the system is not onto the sliding surface from the beginning. To make the integral sliding variable zero that is to bring the system onto the sliding surface in finite time the nonsingular TSMC is proposed based on the integral sliding variables.

To design a second order sliding mode, the integral sliding variable is designed as follows:

$$s(t) = G\left[e(t) - \int_0^t (A(x)e(\tau) + g(x)u_1(\tau))d\tau\right] \tag{4.27}$$

and

$$\dot{s}(t) = G\left[\dot{e}(t) - A(x)e(t) - g(x)u_1(t)\right] \tag{4.28}$$

Using (4.25), (4.28) can be written as

$$\dot{s}(t) = G\left[g(x)u_2(t) + \zeta(t)\right] \tag{4.29}$$

Commonly used linear manifolds cannot assure finite time convergence of the sliding variable. However, non-singular terminal sliding mode [77,79] can achieve finite time convergence of the system dynamics. Therefore, a non-singular terminal manifold $\sigma(t)$ is proposed based on the integral sliding variable $s(t)$ and its first derivative $\dot{s}(t)$ giving rise to a second order sliding mode control (SOSMC). This SOSMC guarantees convergence of $s(t)$ and $\dot{s}(t)$ in finite time.

The second order terminal sliding variable $\sigma(t)$ is designed as follows:

$$\sigma(t) = s(t) + \delta\dot{s}^{\frac{\alpha}{\beta}}(t) \tag{4.30}$$

where $\delta > 0$ is the switching gain and α, β satisfy the following conditions,

$$\alpha,\ \beta \in \{2n+1 :\ n\, is\, an\, integer\} \tag{4.31}$$

and

$$1 < \frac{\alpha}{\beta} < 1.5 \tag{4.32}$$

After $\sigma(t)$ reaches zero in finite time, both $s(t)$ and $\dot{s}(t)$ will also reach zero in finite time. Then, the error $e(t)$ can asymptotically converge to zero. The sufficient condition for the existence of the terminal sliding mode is

$$\frac{1}{2}\frac{d\sigma^2(t)}{dt^2} < -\eta|\sigma(t)| \tag{4.33}$$

As discussed in Section 3.3 III, the switching control $u_2(t)$ is designed as

$$u_2(t) = -\int_0^t (Gg(x))^{-1}\left[\frac{\beta}{\delta\alpha}\dot{s}^{2-\frac{\alpha}{\beta}}(\tau) + G\dot{g}(x)u_2(\tau) + \eta sgn(\sigma(\tau)) + \varepsilon\sigma(\tau)\right]d\tau \tag{4.34}$$

where η, ε are design parameters chosen such that $|G\dot{\zeta}(t)| < \eta$ and $\varepsilon > 0$.

Theorem : *The sliding variable $s(t)$ converges to zero in finite time if the terminal sliding variable $\sigma(t)$ is chosen as given in (4.30) and the control law is designed as*

$$u(t) = u_1(t) + u_2(t)$$

where

$$u_1(t) = -R^{-1}g^T(x)P(x)x(t)$$

and

$$u_2(t) = -\int_0^t (Gg(x))^{-1}\left[\frac{\beta}{\delta\alpha}\dot{s}^{2-\frac{\alpha}{\beta}}(\tau) + G\dot{g}(x)u_2(t) + \eta sgn(\sigma(\tau)) + \varepsilon\sigma(\tau)\right]d\tau \tag{4.35}$$

Proof : Lyapunov function $V_2(t)$ is considered as

$$\begin{aligned}
V_2(t) &= \frac{1}{2}\sigma^2(t) \\
\dot{V}_2(t) &= \sigma(t)\dot{\sigma}(t) \\
&= \sigma(t)[\dot{s}(t) + \frac{\delta\alpha}{\beta}\dot{s}^{\frac{\alpha}{\beta}-1}(t)\ddot{s}(t)] \\
&= \sigma(t)[\dot{s}(t) + \frac{\alpha\delta}{\beta}\dot{s}^{\frac{\alpha}{\beta}-1}(t)(Gg(x)\dot{u}_2(t) + G\dot{g}(x)u_2(t) + G\dot{\zeta}(t))] \\
&= \sigma(t)[\dot{s}(t) + \frac{\alpha\delta}{\beta}\dot{s}^{\frac{\alpha}{\beta}-1}(t)(-\frac{\beta}{\delta\alpha}\dot{s}^{2-\frac{\alpha}{\beta}}(t) - G\dot{g}(x)u_2(t) - \eta sgn(\sigma(t)) - \varepsilon\sigma(t) + G\dot{g}(x)u_2(t) + G\dot{\zeta}(t))] \\
&= \sigma(t)[\frac{\alpha\delta}{\beta}\dot{s}^{\frac{\alpha}{\beta}-1}(t)(-\eta sgn(\sigma(t)) - \varepsilon\sigma(t) + G\dot{\zeta}(t))] \\
&= \frac{\alpha\delta}{\beta}\dot{s}^{\frac{\alpha}{\beta}-1}(t)[-|\sigma(t)|\eta - \varepsilon\sigma^2(t) + \sigma(t)G\dot{\zeta}(t)] \\
&\leq -\frac{\alpha\delta}{\beta}\dot{s}^{\frac{\alpha}{\beta}-1}(t)[\eta + \varepsilon|\sigma(t)| - G\dot{\zeta}(t)]|\sigma(t)| \\
&\leq -\frac{\alpha\delta}{\beta}\dot{s}^{\frac{\alpha}{\beta}-1}(t)\Omega(t)|\sigma(t)| \qquad (4.36)
\end{aligned}$$

with the condition $|G\dot{\zeta}(t)| < \eta$, $[\eta + \varepsilon|\sigma(t)| - G\dot{\zeta}(t)] = \Omega(t) > 0$. Moreover, in [66] Wang et al. showed that $\dot{s}^{\frac{\alpha}{\beta}-1}(t) > 0$ for $|s(t)| \neq 0$. Then above inequality (4.36) can be written as

$$\begin{aligned}
\dot{V}_2(t) &\leq -\widehat{\Omega}|\sigma(t)| \;\; \text{where} \;\; \widehat{\Omega} = \frac{\alpha\delta}{\beta}\dot{s}^{\frac{\alpha}{\beta}-1}(t)\Omega > 0 \;\; \text{for} \;\; |s(t)| \neq 0 \\
&\leq -\widehat{\Omega}|\sqrt{2V_2(t)}| \;\; \text{as} \;\; V_2(t) = \frac{1}{2}\sigma^2(t) \qquad (4.37)
\end{aligned}$$

Hence, finite time convergence of $\sigma(t)$ is guaranteed [101, 105].

Moreover, it can be proved that integral sliding variable $s(t)$ converges to zero in finite time. Suppose finite time t_r is required for $\sigma(t)$ to reach zero from $\sigma(0) \neq 0$ and $\sigma(t) = 0 \;\forall\; t > t_r$. So, once $\sigma(t)$ reaches zero, it remains there and based on (4.37), $s(t)$ will converge to zero in time t_s. The total time required from $\sigma(0) \neq 0$ to $s(t_s)$ can be calculated as follows.

$$
\begin{aligned}
s(t) + \delta \dot{s}^{\frac{\alpha}{\beta}}(t) &= 0 \\
\frac{1}{\delta^{\frac{\beta}{\alpha}}} s^{\frac{\beta}{\alpha}}(t) &= -\dot{s}(t) \\
\frac{1}{\delta^{\frac{\beta}{\alpha}}} dt &= -\frac{ds(t)}{s^{\frac{\beta}{\alpha}}(t)} \\
\int_{t_r}^{t_s} dt &= -\delta^{\frac{\beta}{\alpha}} \int_{s(t_r)}^{s(t_s)} \frac{ds(t)}{s^{\frac{\beta}{\alpha}}(t)} \\
t_s - t_r &= -\frac{\alpha}{\alpha - \beta} \delta^{\frac{\beta}{\alpha}} \left[s(t_s)^{\frac{\alpha-\beta}{\alpha}} - s(t_r)^{\frac{\alpha-\beta}{\alpha}} \right]
\end{aligned} \tag{4.38}
$$

At time t_s sliding variable $s(t_s) = 0$. So, above equation can be written as

$$
t_s = t_r + \frac{\alpha}{\alpha - \beta} \delta^{\frac{\beta}{\alpha}} |s(t_r)|^{\frac{\alpha-\beta}{\beta}} \tag{4.39}
$$

Hence, $s(t)$ and $\dot{s}(t)$ converge to zero in finite time.

IV. Simulation results

The proposed OSOSMC has designed for trajectory tracking of the nonlinear uncertain system but it can also be applied for stabilization of the nonlinear uncertain system. For stabilization, the desired trajectory is chosen as $x_d(t) = 0$. So, successive derivatives of $x_d(t)$ also become zero. Hence, error $e(t)$ becomes the state vector $x(t)$. To demonstrate the effectiveness of the proposed SDRE based OSOSMC, it is applied for both trajectory tracking and stabilization of the nonlinear uncertain system.

A. Tracking of the nonlinear uncertain system

Trajectory tracking problem of a nonlinear uncertain system is considered here. The proposed SDRE based OSOSMC is applied to the Van der Pol circuit described in [7]. The mathematical model of this system is given by

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -2x_1(t) + 3(1 - x_1^2(t))x_2(t) + u(t) + d_1(t) \\
y &= x_1(t)
\end{aligned} \tag{4.40}
$$

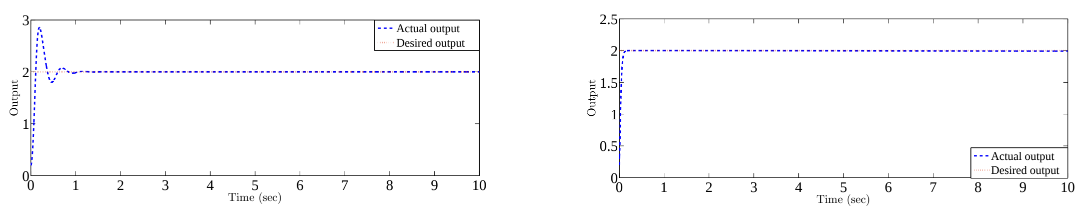
where state vector $x(t) = [x_1(t) \quad x_2(t)]^T$, $A(x) = \begin{bmatrix} 0 & 1 \\ -2 & 3\{1 - x_1^2(t)\} \end{bmatrix}$ and $g(x) = [0 \quad 1]^T$. External disturbance $d_1(t) = 2\sin(0.1\pi t) + 3\sin(0.2\sqrt{t+1})$ and the output $y = x_1$ is required to track the desired state $x_d(t) = 2$. To minimize both the tracking error and the control input, the performance index J is considered as

$$J = \int_0^{\infty} \left[e(t)^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} e(t) + u_1^T(t) 10 u_1(t) \right] dt \tag{4.41}$$

Design parameters of the proposed SDRE based optimal second order sliding mode controller (OSOSMC) (4.35) are chosen as follows:

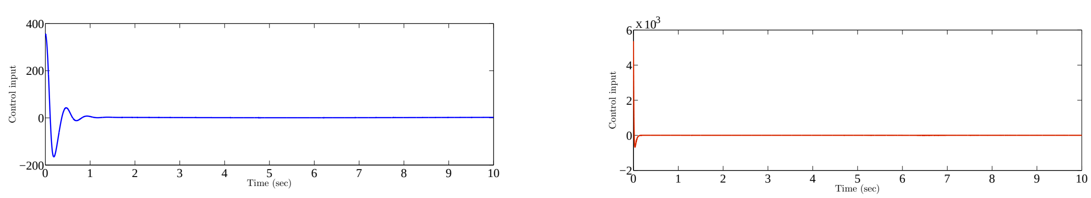
$G = [0 \quad 1]$, $\alpha = 7$, $\beta = 5$, $\delta = 0.5$, $\eta = 3$ and $\varepsilon = 0.2$.

The tracking results obtained by applying the SDRE based OSOSMC are compared with those obtained by using the terminal sliding mode controller (TSMC) proposed by Chen et al. [7] which is discussed in Appendix A.7. The output and desired states for both these cases are plotted in Figure 4.1. The control inputs required by these two methods for tracking the desired state are shown in Figure 4.2.



(a) Proposed SDRE based OSOSMC (b) TSMC proposed by Chen et al. [7]

Figure 4.1: Output tracking in Van der Pol circuit [7] for desired state $x_d = 2$



(a) Control input for proposed SDRE based OSOSMC (b) Control input for TSMC proposed by Chen et al. [7]

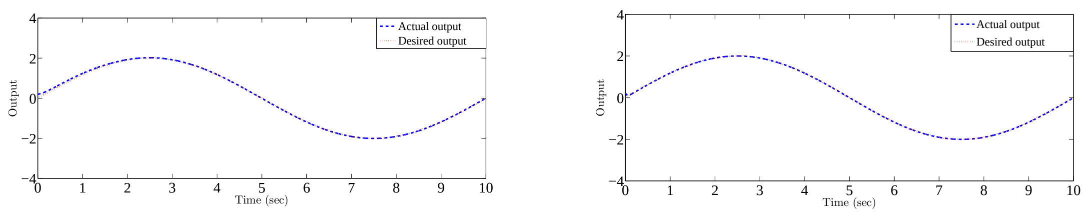
Figure 4.2: Control inputs for trajectory tracking in the Van der Pol circuit [7] for desired state $x_d = 2$

Table 4.1 shows the total variation (TV) and the second norm of the control input required for tracking $x_d = 2$, calculated for the period from 0 to 10 sec with a sampling time of 0.1 sec. It is clear from Table 4.1 that the proposed SDRE based OSOSMC is able to produce a smoother control input with substantial reduction in the control effort than that of TSMC proposed by Chen et al. [7].

Table 4.1: Comparison of control indices to track the output according to $x_d(t) = 2$

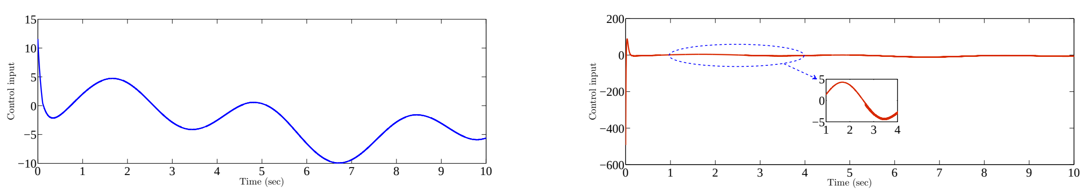
Method	Total Variation (TV)	Control Energy
Proposed SDRE based OSOSMC	805.31	404.47
TSMC [7]	5268.40	5370.40

Now the desired state is changed to $x_d = 2\sin(2\pi t)$. The output and desired states for the proposed SDRE based OSOSMC and the terminal sliding mode controller (TSMC) proposed by Chen et al. [7] are plotted in Figure 4.3. The control inputs required by these methods for tracking the desired state $x_d = 2\sin(2\pi t)$ are shown in Figure 4.4.



(a) Proposed SDRE based OSOSMC (b) TSMC proposed by Chen et al. [7]

Figure 4.3: Output tracking in Van der Pol circuit [7] for desired state $x_d = 2\sin(2\pi t)$



(a) Control input for proposed SDRE based OSOSMC (b) Control input for TSMC proposed by Chen et al. [7]

Figure 4.4: Control inputs for trajectory tracking in the Van der Pol circuit [7] for desired state $x_d = 2\sin(2\pi t)$

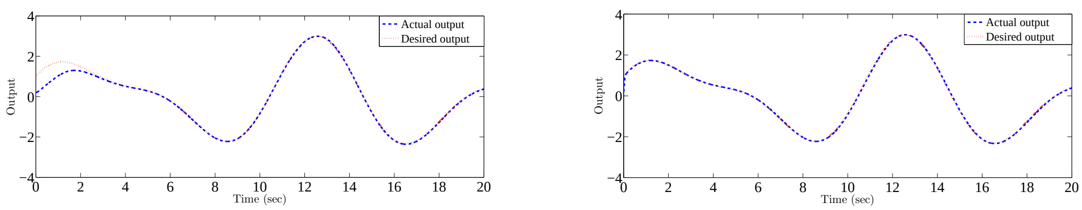
The total variation and energy in the control input $u(t)$ required for tracking the desired state $x_d = 2\sin(2\pi t)$ are computed for the time span of 0 to 10 sec with sampling time 0.1 sec. In Table 4.2 control indices of both the controllers are compared. It is clear from Table 4.2 that the proposed

SDRE based OSOSMC is able to produce a smoother control input with substantial reduction in the control effort than that of TSMC proposed by Chen et al. [7].

Table 4.2: Comparison of control indices to track the output according to $x_d(t) = 2\sin(2\pi t)$

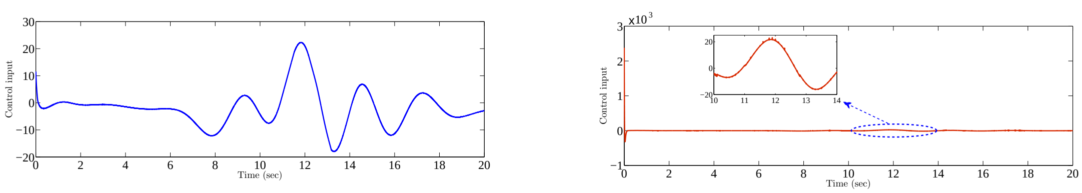
Method	Total Variation (TV)	Control Energy
Proposed SDRE based OSOSMC	57.49	46.87
TSMC [7]	556.69	494.99

Now the desired trajectory is changed to $x_d = \cos t + 2\sin(2\pi t)$. The output and desired states for the proposed SDRE based OSOSMC and the terminal sliding mode controller (TSMC) proposed by Chen et al. [7] are shown in Figure 4.5. The control inputs required by these controllers for tracking the desired state $x_d = \cos t + 2\sin(2\pi t)$ are shown in Figure 4.6.



(a) Proposed SDRE based OSOSMC (b) TSMC proposed by Chen et al. [7]

Figure 4.5: Output tracking in Van der Pol circuit [7] for desired state $x_d = \cos t + 2\sin(2\pi t)$



(a) Control input for proposed SDRE based OSOSMC (b) Control input for TSMC proposed by Chen et al. [7]

Figure 4.6: Control inputs for trajectory tracking in the Van der Pol circuit [7] for desired state $x_d = \cos t + 2\sin(2\pi t)$

Total variation (TV) and the control energy required to track the desired state $x_d = \cos t + 2\sin(2\pi t)$ are listed in Table 4.3. It is observed from Table 4.3 that the proposed SDRE based OSOSMC methodology produces a smoother control input than that of TMSC proposed by Chen et al. [7] at expense of a much lesser control effort.

Table 4.3: Comparison of control indices to track the output according to $x_d(t) = \cos t + 2\sin(2\pi t)$

Method	Total Variation (TV)	Control Energy
Proposed SDRE based OSOSMC	194.50	102.31
TSMC [7]	2650.2	2382.30

B. Stabilization of the nonlinear uncertain system

Now a third order nonlinear uncertain system [8] is considered as described below:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \\ \dot{x}_3(t) &= x_2^3(t) + u(t) + d_1(t) \end{aligned} \tag{4.42}$$

where uncertainty $d_1(t) = 0.1\sin 20t$. The SDC matrices of above third order nonlinear system (4.42) can be expressed as
$A(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & x_2^2(t) & 0 \end{bmatrix}$, $g(x) = [0 \;\; 0 \;\; 1]^T$. Initial state $x(0) = [1 \;\; -1 \;\; 0]^T$. The performance index J is defined as

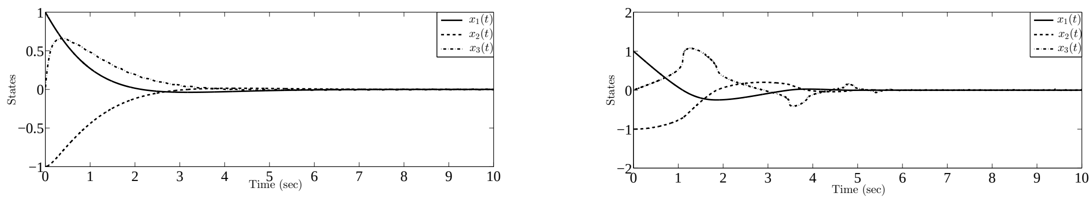
$$J = \int_0^{\infty} \left[x^T(t) \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix} x(t) + u_1^T 0.1 u_1(t) \right] dt \tag{4.43}$$

Simulation experiments are performed by varying the design parameter values and those values are selected which produce the best transient and steady sate performances by using the minimum control effort. Design parameter values of the proposed optimal second order sliding mode controller (OSOSMC) (4.35) are chosen as follows:

$G = [0 \;\; 0 \;\; 1]$, $\alpha = 7$, $\beta = 5$, $\delta = 0.15$, $\eta = 0.2$ and $\varepsilon = 0.2$.

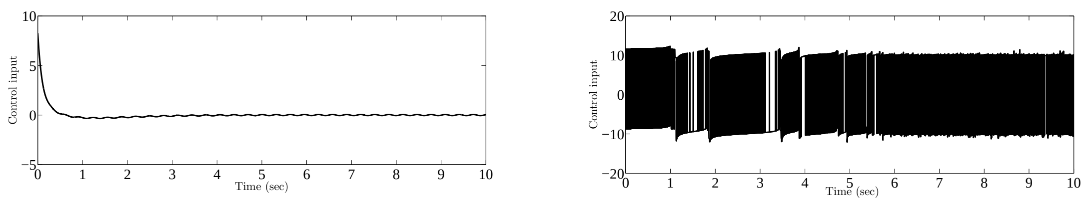
The proposed SDRE based optimal second order sliding mode controller (OSOSMC) is applied to stabilize the nonlinear uncertain system (4.42). The results obtained by applying the proposed SDRE based OSOSMC are compared with those obtained by using the terminal sliding mode control

(TSMC) proposed by Feng et al. [8] which is discussed in Appendix A.8. The states obtained by applying the proposed SDRE based OSOSMC and the TSMC by Feng et al. [8] are shown in Figure 4.7. The control inputs obtained by using these controllers are compared in Figure 4.8.



(a) States for proposed SDRE based OSOSMC (b) States for TSMC [8]

Figure 4.7: States obtained by applying proposed SDRE based OSOSMC and the TSMC [8]



(a) Control input for proposed SDRE based OSOSMC (b) Control input for TSMC [8]

Figure 4.8: Control inputs obtained by applying proposed SDRE based OSOSMC and the TSMC [8]

Total variation (TV) and the control energy required to stabilize the nonlinear triple integrator system (4.42) are listed in Table 4.4. It is observed from Table 4.4 that the proposed SDRE based OSOSMC methodology produces an exceedingly smoother control input compared to that of Feng et al. [8] at the expense of a substantially lesser control effort.

Table 4.4: Comparison of control indices to stabilize the nonlinear triple integrator system

Method	Total Variation (TV)	Control Energy
Proposed SDRE based OSOSMC	14.05	9.21
TSMC [8]	794.85	99.12

4.3 Optimal second order sliding mode controller for chaotic systems

In this section an optimal second order sliding mode controller is applied to the chaotic system [124, 125]. Chaotic systems are deterministic dynamical systems exhibiting irregular, seemingly random behavior. The chaotic system is highly sensitive to initial conditions and parametric variations. Moreover, the chaotic system is nonlinear with continuous frequency spectrum [126]. Two trajectories of the same chaotic system starting close to each other may diverge after some time. Mathematically, the chaotic system is characterized by local instability and global boundedness of its solution. Chaos control [127–129] is quite a challenging task. Idea behind chaos control is to make the system trajectory approach a desired periodic orbit embedded in the attractor. In terms of control theory it means stability of the unstable periodic orbit. The following two methods are used in chaos control.

- Calculated tiny and fast perturbations are enforced to the system once in every cycle [114].
- A continuous control signal which approaches zero as the system reaches the desire orbit, is injected into the system [130].

Methods of nonlinear control [116, 131, 132] are applicable to chaos control. The proposed OSOSMC designed for nonlinear systems is used to control chaotic systems. Certain chaotic systems can be represented in linear like structure having state dependent coefficient matrices. For these systems the optimal controller is designed based on the SDRE. The optimal controller is made robust by combining it with a second order sliding mode controller.

I. Problem statement

A nonlinear system can be defined as

$$\begin{aligned} \dot{x}(t) &= f(x_1, x_2, \cdots, x_n) \\ &= \begin{bmatrix} a_{11}f_{11}(x_1, x_2, \cdots, x_n) & a_{12}f_{12}(x_1, x_2, \cdots, x_n) & \cdots & a_{1n}f_{1n}(x_1, x_2, \cdots, x_n) \\ a_{21}f_{21}(x_1, x_2, \cdots, x_n) & a_{22}f_{22}(x_1, x_2, \cdots, x_n) & \cdots & a_{2n}f_{2n}(x_1, x_2, \cdots, x_n) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}f_{n1}(x_1, x_2, \cdots, x_n) & a_{n2}f_{n2}(x_1, x_2, \cdots, x_n) & \cdots & a_{nn}f_{nn}(x_1, x_2, \cdots, x_n) \end{bmatrix} \end{aligned} \tag{4.44}$$

where n is a an integer and $n \geq 3$.

Now for certain specific types of function $f_{11}(x_1, x_2, \cdots, x_n), \cdots, f_{nn}(x_1, x_2, \cdots, x_n)$ and particular values of $a_{11}, \cdots, a_{nn}$, the nonlinear system (4.44) exhibits a typical chaotic behavior. The peculiar features of these nonlinear systems are:

- They are strongly dependent on initial conditions.
- They are sensitive to parameter variations.
- There is existence of strong harmonics in the output.
- Dimension of state space trajectories is fractional.
- Presence of stretch direction, represented by positive Lyapunov exponent [133].

This special type of nonlinear system is classified as chaotic system. Mathematical model of some chaotic systems are shown in the Table 4.5:

From Table 4.5 it is observed that these chaotic systems can be represented as linear like structures containing state dependent coefficient (SDC) matrix. Hence, extended linearization method can be applied to these chaotic systems. Thus, chaotic systems defined in Table 4.5 can be represented as

$$\dot{x}(t) = f(x_1, x_2, \cdots, x_n) = A(x_1, x_2, \cdots, x_n) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{4.45}$$

where $A(x_1, x_2, \cdots, x_n)$ is a SDC matrix.

When the above chaotic system (4.45) is affected by uncertainty, the system can be described as

$$\dot{x}(t) = A(x)x(t) + B(u(t) + d(t)) \tag{4.46}$$

where B is the full order input distribution matrix, $u(t)$ is the control input and $d(t)$ represents matched uncertainty affecting the system.

The objective is to design a robust optimal controller for the uncertain chaotic system (4.46) based on sliding mode control. The control law $u(t)$ is divided into two parts as

$$u(t) = u_1(t) + u_2(t) \tag{4.47}$$

Table 4.5: Examples of chaotic systems

No.	Chaotic system	Mathematical model	$\dot{x}(t) = A(x)x(t)$
1.	Lorenz system	$\dot{x}(t) = \begin{bmatrix} -10x_1 + 10x_2 \\ 28x_1 - x_2 - x_1x_3 \\ -8/3x_3 + x_1x_2 \end{bmatrix}$	$\dot{x}(t) = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & -x_1 \\ x_2 & 0 & -8/3 \end{bmatrix} x(t)$
2.	Liu system	$\dot{x}(t) = \begin{bmatrix} -10x_1 + 10x_2 \\ 40x_1 - x_1x_3 \\ 4x_1^2 - 2.5x_3 \end{bmatrix}$	$\dot{x}(t) = \begin{bmatrix} -10 & 10 & 0 \\ 40 & 0 & -x_1 \\ 4x_1 & 0 & -2.5 \end{bmatrix} x(t)$
3.	Chen system	$\dot{x}(t) = \begin{bmatrix} -35x_1 + 35x_2 \\ -7x_1 + 28x_2 - x_1x_3 \\ x_1x_2 - 3x_3 \end{bmatrix}$	$\dot{x}(t) = \begin{bmatrix} -35 & 35 & 0 \\ -7 & 28 & -x_1 \\ x_2 & 0 & -3 \end{bmatrix} x(t)$
4.	Lotka Volterra system	$\dot{x}(t) = \begin{bmatrix} x_1 + 2x_1^2 - x_1x_2 - 2.9851x_1^2x_3 \\ x_1x_2 - x_2 \\ 2.9851x_1^2x_3 - 3x_3 \end{bmatrix}$	$\dot{x}(t) = \begin{bmatrix} 1 + 2x_1 & x_1 & 2.9851x_1^2 \\ x_2 & -1 & 0 \\ 2.9851x_1x_3 & 0 & -3 \end{bmatrix} x(t)$
5.	ACT attractor	$\dot{x}(t) = \begin{bmatrix} 1.8x_1 - 1.8x_2 \\ 0.02x_1^3 + x_1x_3 - 7.2x_2 \\ x_1x_2 - 2.7x_3 - 0.07x_3^2 \end{bmatrix}$	$\dot{x}(t) = \begin{bmatrix} 1.8 & -1.8 & 0 \\ 0.02x_1 & -7.2 & x_1 \\ x_2 & 0 & -2.7 - 0.07x_3 \end{bmatrix} x(t)$
6.	Simplest cubic chaotic flow	$\dot{x}(t) = \begin{bmatrix} x_2 \\ x_3 \\ -x_1 + x_1x_2^2 - 2.028x_3 \end{bmatrix}$	$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & x_1x_2 & -2.028 \end{bmatrix} x(t)$
7.	Rossler's fourth system	$\dot{x}(t) = \begin{bmatrix} -x_2 - x_3 \\ x_1 \\ 0.386x_2 - 0.386x_2^2 - 0.2x_3 \end{bmatrix}$	$\dot{x}(t) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0.386 - 0.386x_2 & -0.2 \end{bmatrix} x(t)$
8.	Rabinovich-Fabrikant attractor	$\dot{x}(t) = \begin{bmatrix} 0.87x_1 + x_1^2x_2 - x_2 + x_2x_3 \\ -x_1^3 + x_1 + 3x_1x_3 + 0.87x_2 \\ -2x_1x_2x_3 - 2.2x_3 \end{bmatrix}$	$\dot{x}(t) = \begin{bmatrix} 0.87 & x_1^2 - 1 & x_2 \\ -x_1^2 + 1 & 0.87 & 3x_1 \\ -x_2x_3 & x_1x_3 & -2.2 \end{bmatrix} x(t)$
9.	Halvarsen's cyclically symmetric attractor	$\dot{x}(t) = \begin{bmatrix} -1.27x_1 - 4x_2 - x_2^2 - 4x_3 \\ -4x_1 - 1.27x_2 - 4x_3 - x_3^2 \\ -4x - x_1^2 - 4x_2 - 1.27x_3 \end{bmatrix}$	$\dot{x}(t) = \begin{bmatrix} -1.27 & -x_2 - 4 & -4 \\ -4 & -1.27 & -x_3 - 4 \\ -x_1 - 4 & -4 & -1.27 \end{bmatrix} x(t)$
10.	Rucklidge attractor	$\dot{x}(t) = \begin{bmatrix} -2x_1 + 6.7x_2 - x_2x_3 \\ x_1 \\ x_2^2 - x_3 \end{bmatrix}$	$\dot{x}(t) = \begin{bmatrix} -2 & 6.7 & -x_2 \\ 1 & 0 & 0 \\ 0 & x_2 & -1 \end{bmatrix} x(t)$
11.	Lorenz-Stenflo	$\dot{x}(t) = \begin{bmatrix} -x_1 + x_2 + 1.5x_4 \\ 26x_1 - x_1x_3 - x_2 \\ x_1x_2 - 0.7x_3 \\ -x_1 - x_4 \end{bmatrix}$	$\dot{x}(t) = \begin{bmatrix} -1 & 1 & 0 & 1.5 \\ 26 & -1 & -x_1 & 0 \\ x_2 & 0 & -0.7 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} x(t)$

where $u_1(t)$ is the optimal controller designed to stabilize the nominal nonlinear part of the chaotic system and $u_2(t)$ is the sliding mode control used to keep the system onto the sliding surface to ensure robustness.

II. Optimal controller design

Neglecting the uncertain part, (4.46) can be written as

$$\dot{x}(t) = A(x)x(t) + Bu_1(t) \tag{4.48}$$

The performance index J chosen to optimize the control input $u_1(t)$ is considered as

$$J = \int_0^\infty [x(t)^T Q(x)x(t) + u_1(t)^T R(x)u_1(t)]dt \tag{4.49}$$

where $Q(x) \in \mathcal{R}^{n\times n}$ and $R(x) \in \mathcal{R}$ are weighing matrices which are functions of the states. Further, $Q(x)$ and $R(x)$ are a positive definite matrix. The optimal control law $u_1(t)$ is obtained as

$$u_1(t) = -R^{-1}(x)B^T P(x)x(t) = K(x)x(t) \tag{4.50}$$

where $K(x) = -R^{-1}(x)B^T P(x)$ and $P(x)$ is a positive definite symmetric matrix which is the solution of the state dependent Riccati equation

$$A(x)^T P(x) + P(x)A(x) + Q(x) - P(x)BR^{-1}(x)B^T P(x) = 0 \tag{4.51}$$

For ensuring robustness of the SDRE based optimal controller, a sliding mode controller is integrated with it.

III. Sliding mode controller design

An integral sliding variable $s(t)$ is designed as

$$s = G\left[x(t) - \int_0^t \dot{x}_{nom}(\tau)d\tau\right] \tag{4.52}$$

where $\dot{x}_{nom}(t) = A(x)x(t) + Bu_1(t)$. Knowledge of initial condition is not required for designing the integral sliding variable $s(t)$. But $s(t) \neq 0$ from the beginning. Hence for converging the sliding variable $s(t)$ in finite time, a non-singular terminal sliding surface is designed based on the integral sliding variable $s(t)$. The non-singular terminal sliding variable $\sigma(t)$ [77, 79] is given by

$$\sigma(t) = s(t) + \delta \dot{s}^{\frac{\alpha}{\beta}}(t) \tag{4.53}$$

and δ is the switching gain chosen such that

$$\delta > 0 \tag{4.54}$$

Here α, β are selected in such a way that these satisfy the following conditions

$$\alpha, \beta \in \{2n+1 : \; n \, is \, an \, integer\} \tag{4.55}$$

and

$$1 < \frac{\alpha}{\beta} < 1.5 \tag{4.56}$$

Following the procedure described in the pervious section, the sliding mode control $u_2(t)$ is designed as

$$u_2(t) = -\int_0^t (GB))^{-1} \left[\frac{\beta}{\delta\alpha} \dot{s}(\tau)^{2-\frac{\alpha}{\beta}} + \eta sgn(\sigma(\tau)) + \varepsilon\sigma(\tau) \right] d\tau \tag{4.57}$$

where the design parameter is chosen in such a way that $|GB\dot{d}(t)| < \eta$ [66].

IV. Simulation results

To demonstrate the effectiveness of the proposed optimal sliding mode controller, it is applied for stabilization of a chaotic system and simulation studies are conducted on various types of chaotic systems. The performance of the proposed SDRE based OSOSMC is found to be equally good while stabilizing chaotic systems with and without uncertainty. As such, simulation results are discussed for the case when chaotic systems get affected by uncertainties. Simulation results obtained by using the proposed optimal second order sliding mode controller (OSOSMC) are compared with results of the adaptive sliding mode controller (ASMC) proposed by Roopaei et al. [9].

A. Lorenz system

Lorenz system [9] is described as

$$\begin{aligned}
\dot{x}_1(t) &= -ax_1(t) + ax_2(t) \\
\dot{x}_2(t) &= rx_1(t) - x_2(t) - x_1(t)x_3(t) \\
\dot{x}_3(t) &= -bx_3(t) + x_1(t)x_2(t)
\end{aligned} \tag{4.58}$$

where $x_1(t), \; x_2(t), \; x_3(t)$ are state variables and $a, \; r, \; b$ are known non-negative constants. In this example $a = 10, \; b = \frac{8}{3}, \; r = 28$. The chaotic behavior of the Lorenz system is shown in Figure 4.9.

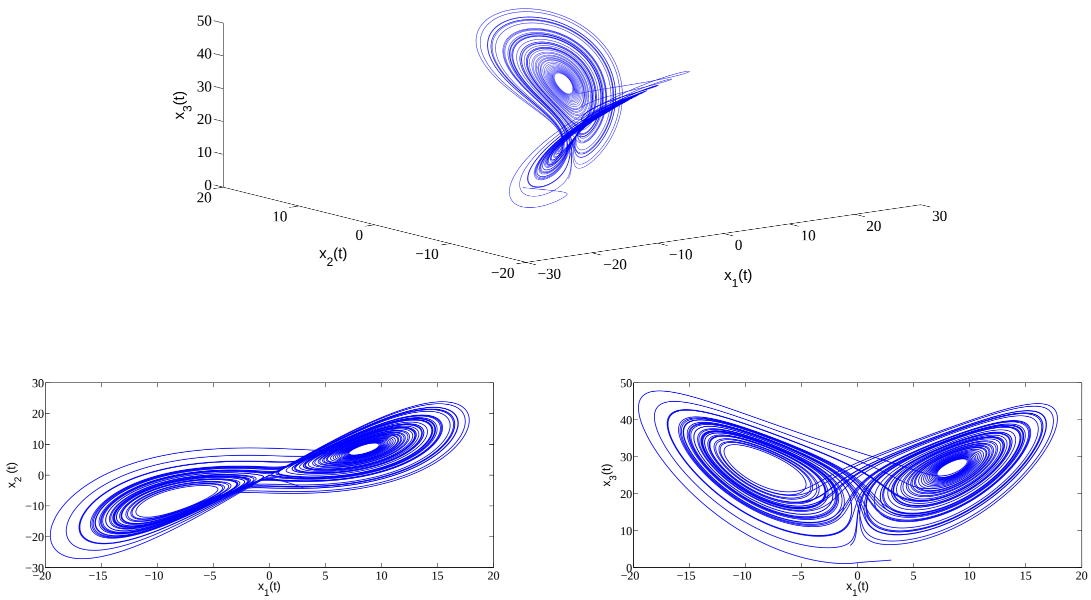


Figure 4.9: State space trajectories of Lorenz system

It is considered that the chaotic system (4.58) is affected by disturbance

$\Delta f(x) = 0.5 - sin(\pi x_1(t))\sin(2\pi x_2(t))\sin(3\ pix_3(t))$ and a control input $u(t)$ is applied to the state $x_1(t)$. Now (4.58) is given by

$$\dot{x}_1(t) = -ax_1(t) + ax_2(t) + u(t) + \Delta f(x)$$
$$\dot{x}_2(t) = rx_1(t) - x_2(t) - x_1(t)x_3(t)$$
$$\dot{x}_3(t) = -bx_3(t) + x_1(t)x_2(t) \tag{4.59}$$

The performance index J is chosen as

$$J = \int_0^{\infty} \left(x^T(t) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t) + u_1^T(t)u_1(t) \right) dt \tag{4.60}$$

Using the SDRE, the optimal feedback control is found as $u_1(t) = -R^{-1}B^T P(x)x$

where $R = 1,\ B = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ and $P(x)$ is the solution of the SDRE defined in (4.51). Design parameters of the integral and terminal sliding mode controller are chosen as follows:

$G = [1\ \ 0\ \ 0],\ p = 7,\ q = 5,\ \delta = 0.2,\ \eta = .7,\ \varepsilon = 0.1.$

The proposed controller is applied to the Lorenz system for the stabilization. The state space trajectories of the stable lorenz system are shown in Figure 4.10

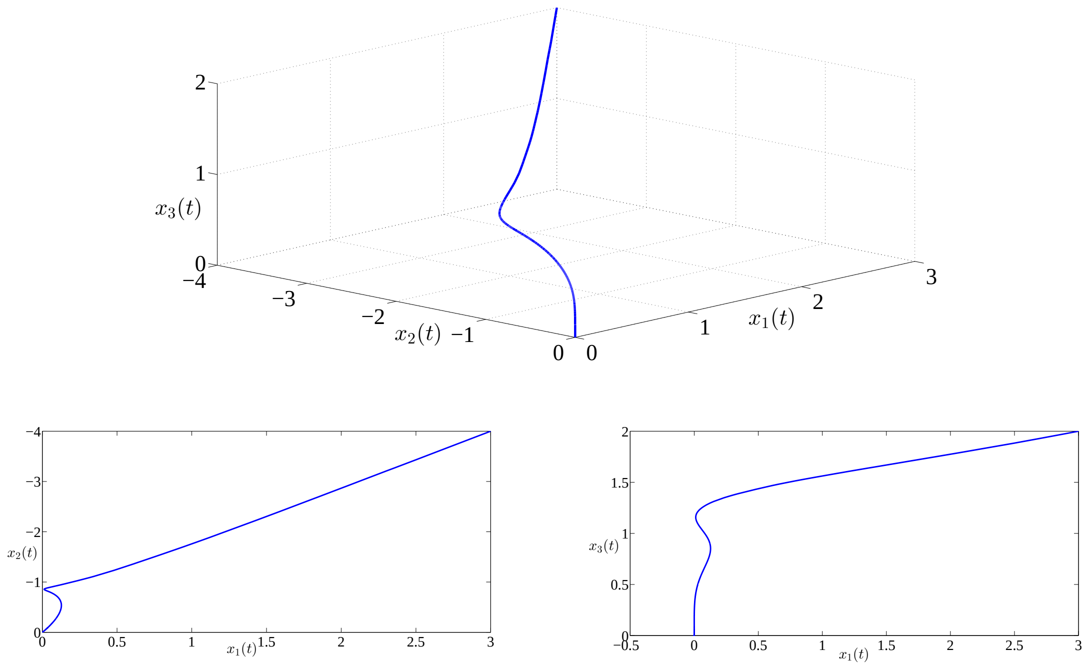


Figure 4.10: State space trajectories of Lorenz system after applying the proposed SDRE based OSOSMC

The states $x_1(t)$, $x_2(t)$ and $x_3(t)$ obtained by applying the proposed OSOSMC are compared with those obtained by using the ASMC proposed by Roopaei et al. [9] in Figures 4.11-4.13. ASMC proposed by Roopaei et al. is discussed in Appendix A.9.

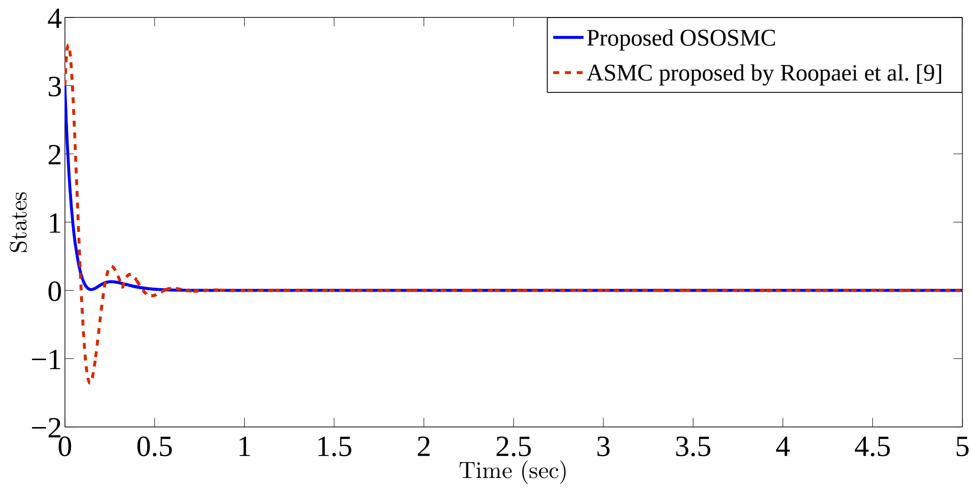


Figure 4.11: $x_1(t)$ obtained by using the proposed OSOSMC and ASMC proposed by Roopaei et al. [9]

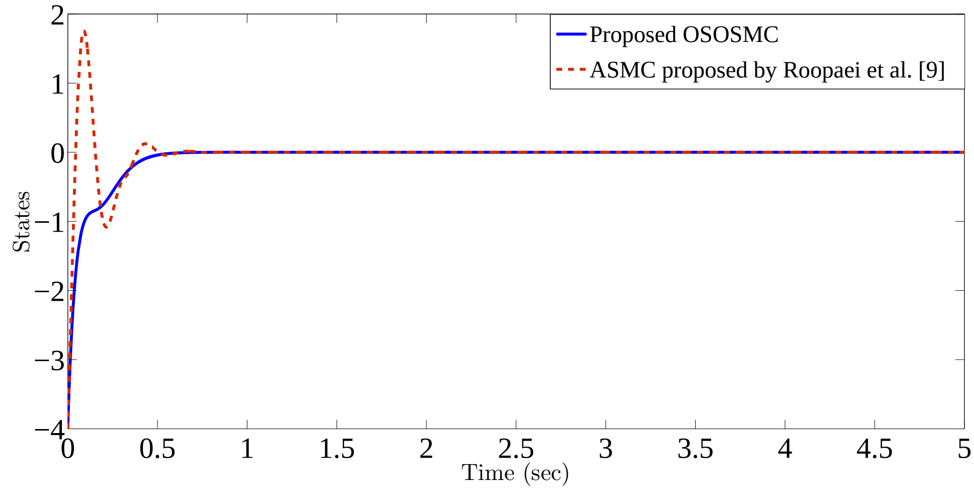


Figure 4.12: $x_2(t)$ obtained by using the proposed OSOSMC and ASMC proposed by Roopaei et al. [9]

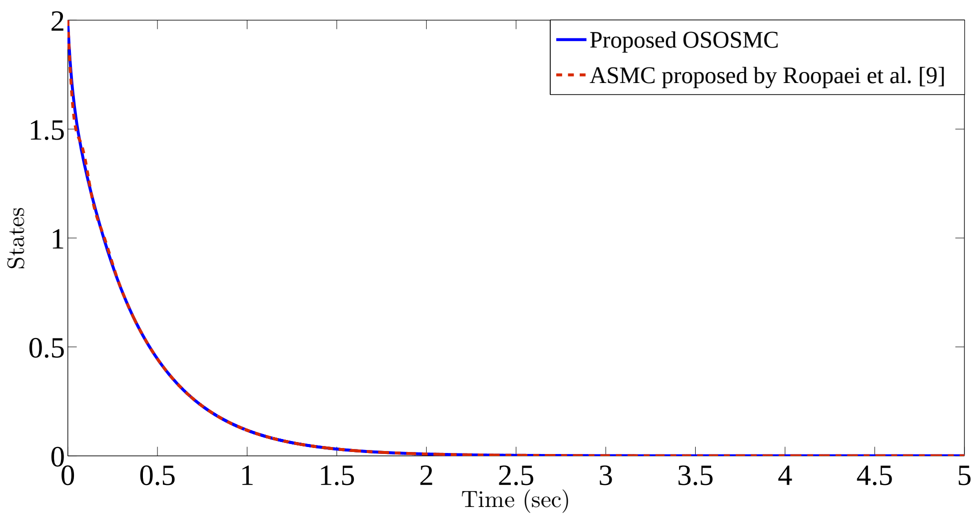


Figure 4.13: $x_3(t)$ obtained by using the proposed OSOSMC and ASMC proposed by Roopaei et al. [9]

The control inputs obtained by using the proposed OSOSMC and ASMC proposed by Roopaei et al. [9] are shown in Figure 4.14.

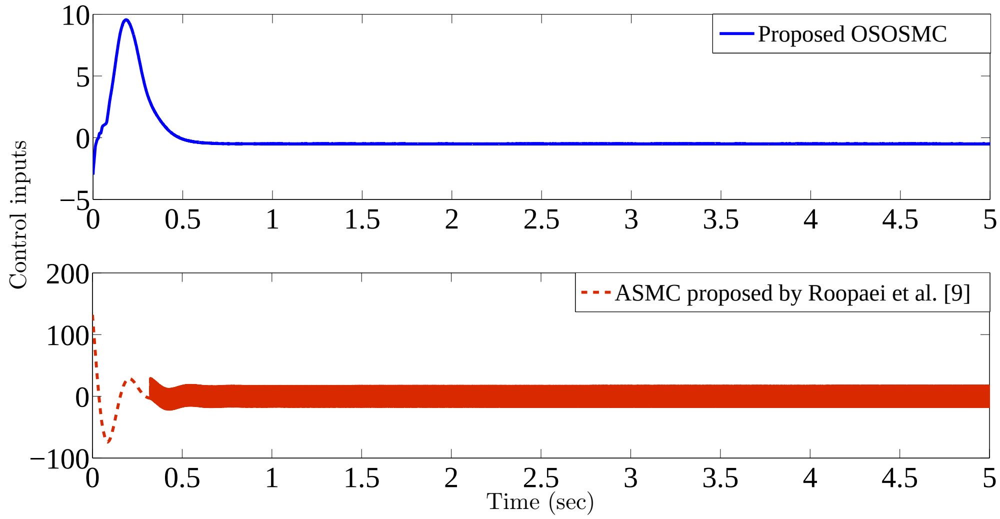


Figure 4.14: Control inputs obtained by using the proposed OSOSMC and ASMC proposed by Roopaei et al. [9]

It is observed from above figures that the proposed OSOSMC is able to stabilize the highly unstable Lorenz system by spending a substantially lower control input in comparison to the ASMC proposed by Roopaei et al. [9] but with the same stabilization speed. Moreover, the control input in the case of the proposed OSOSMC is smooth without any chattering. In order to evaluate the controller performance, the total variation (TV) [106] and control energy of the control input $u(t)$ are computed. Table 4.6 shows the TV and the 2-norm of the control input calculated for the period from 0 to 5sec with a sampling time of 0.01sec. It is clear from Table 4.6 that the proposed OSOSMC is able to produce a smoother control input with substantial reduction in the control effort than the ASMC proposed by Roopaei et al. [9].

Table 4.6: Comparison of Control Indices

Method	Total Variation (TV)	Control Energy
ASMC [9]	6074.00	438.00
Proposed OSOSMC	25.29	36.00

B. Liu system

The Liu system [9] is described as

$$\begin{aligned}
\dot{x}_1(t) &= -ax_1(t) + ax_2(t) \\
\dot{x}_2(t) &= bx_1(t) - kx_1(t)x_3(t) \\
\dot{x}_3(t) &= -cx_3(t) + hx_1^2(t)
\end{aligned} \tag{4.61}$$

where $x_1(t),\ x_2(t),\ x_3(t)$ are state variables and $a,\ b,\ k,\ c,\ h$ are known non-negative constants. In this example the constants are $a = 10,\ b = 40,\ k = 1,\ c = 2.5,\ h = 4$. The chaotic behavior of the Liu system is shown in Figure 4.15.

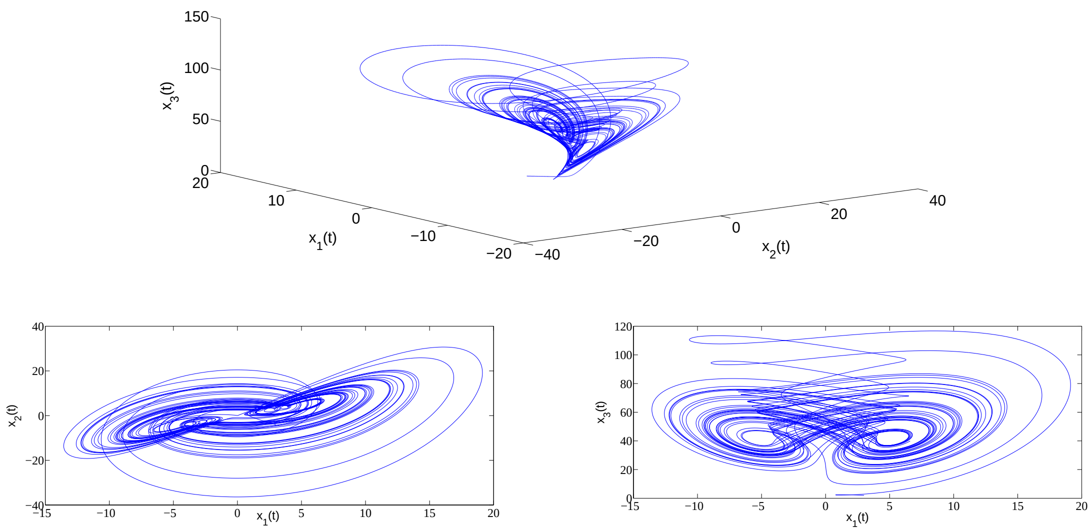


Figure 4.15: State space trajectories of Liu system

After adding the control input $u(t)$ and disturbance $\Delta f(x) = 0.5 - sin(\pi x_1(t)) \sin(2\pi x_2(t)) \sin(3\, pix_3(t))$, Liu system (4.61) is obtained as

$$\begin{aligned}
\dot{x}_1(t) &= -ax_1(t) + ax_2(t) + u(t) + \Delta f(x) \\
\dot{x}_2(t) &= bx_1(t) - kx_1(t)x_3(t) \\
\dot{x}_3(t) &= -cx_3(t) + hx_1^2(t)
\end{aligned} \tag{4.62}$$

The same performance index and sliding surface as considered in Lorenz system are chosen to stabilize the Liu system. The proposed controller is applied to the Liu system for stabilization. The state space trajectories of the stable Liu system are shown in Figure 4.16.

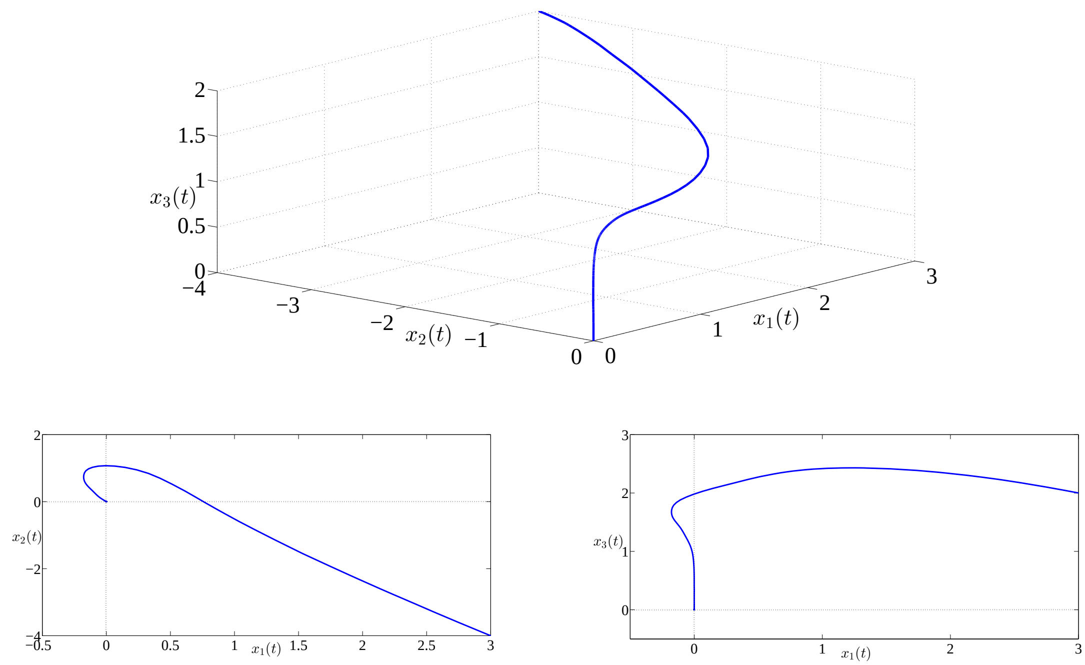


Figure 4.16: State space trajectories of Liu system after applying the proposed SDRE based OSOSMC

Simulation results are compared with those of the ASMC proposed by Roopaei et al. [9]. The states $x_1(t)$, $x_2(t)$ and $x_3(t)$ obtained by applying the proposed OSOSMC are compared with those obtained by using the adaptive sliding mode controller proposed by Roopaei et al. [9] and are shown in Figures 4.17-4.19.

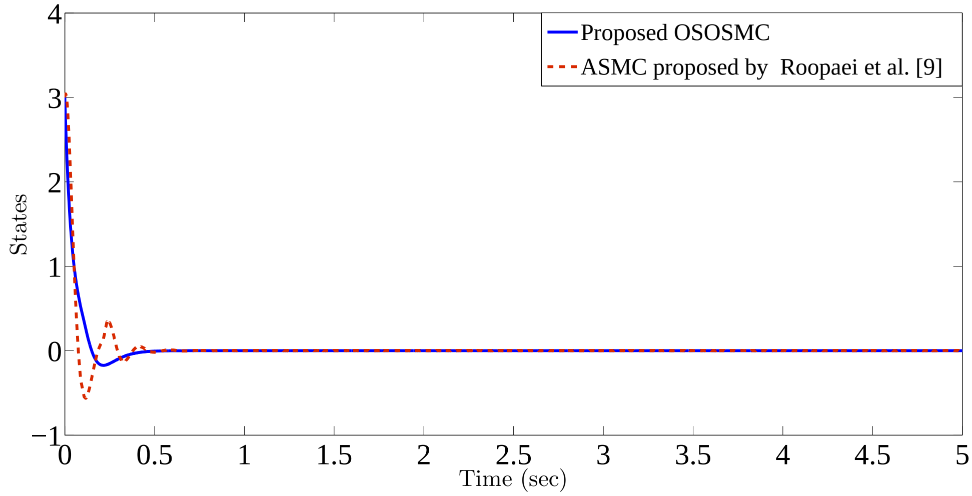


Figure 4.17: State $x_1(t)$ of Liu system obtained by using the proposed OSOSMC and ASMC proposed by Roopaei et al. [9]

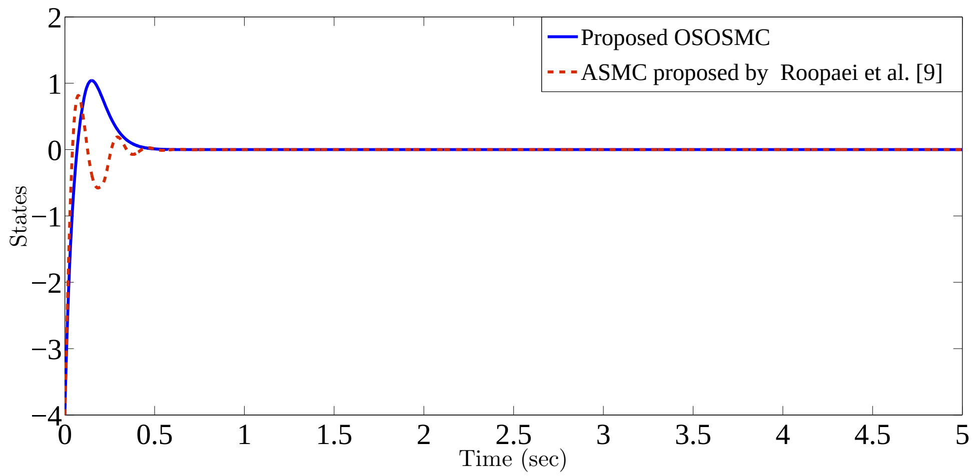


Figure 4.18: State $x_2(t)$ of Liu system obtained by using the proposed OSOSMC and ASMC proposed by Roopaei et al. [9]

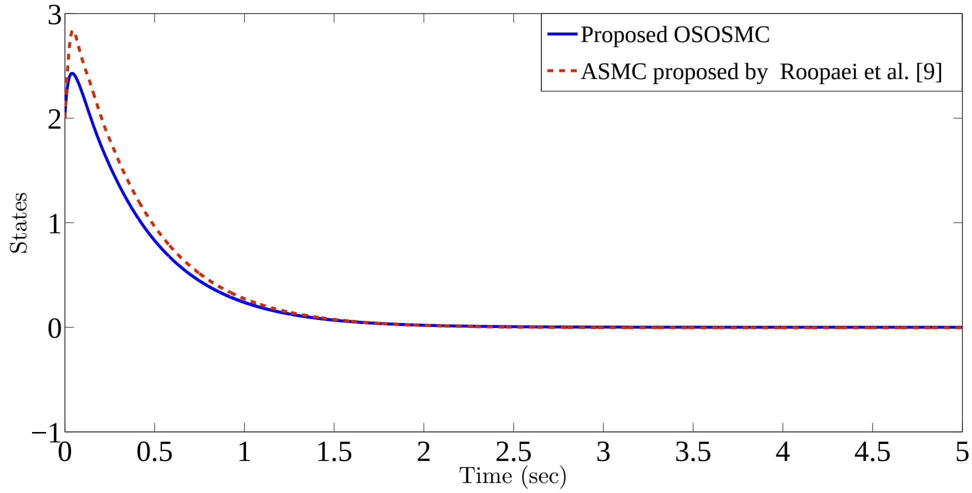


Figure 4.19: State $x_3(t)$ of Liu system obtained by using the proposed OSOSMC and ASMC proposed by Roopaei et al. [9]

The control inputs obtained by using the proposed OSOSMC and ASMC proposed by Roopaei et al. [9] are shown in Figures 4.20.

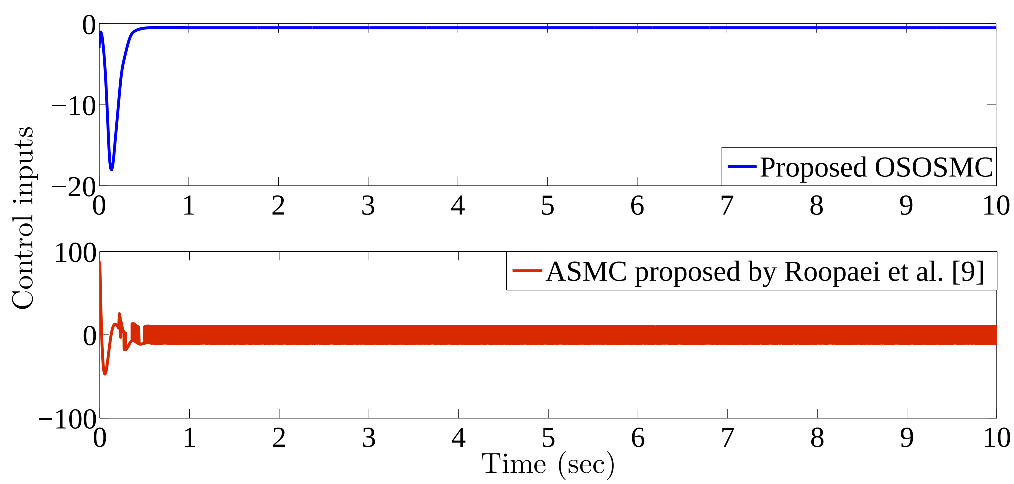


Figure 4.20: Control inputs obtained by using the proposed OSOSMC and ASMC proposed by Roopaei et al. [9] for Liu system

Similar to the previous example, for the Liu system also it is observed that the proposed OSOSMC is able to achieve stabilization performance at par with the ASMC proposed by Roopaei et al. [9] but at the cost of significantly lesser control input. Moreover, the control input in the case of the proposed

OSOSMC is chattering free. Table 4.7 compares the total variation (TV) and the 2-norm of the control input for the proposed OSOSMC and ASMC proposed by Roopaei et al. [9] in stabilizing the Liu system. The control indices are computed for the period from 0 to 10 sec with a sampling time of 0.01 sec. From Table 4.7 it is evident that the proposed OSOSMC method produces a substantially smoother control input spending a significantly lower control energy than the ASMC proposed by Roopaei et al. [9].

Table 4.7: Comparison of Control Indices

Method	Total Variation (TV)	Control Energy
ASMC [9]	2401.90	267.32
Proposed OSOSMC	37.90	58.36

C. Lorenz-Stenflo

The Lorenz-Stenflo [9] system is described as

$$\begin{aligned}
\dot{x}_1(t) &= -ax_1(t) + ax_2(t) + cx_4(t) \\
\dot{x}_2(t) &= x_1(t)(r - x_3(t)) - x_2(t) \\
\dot{x}_3(t) &= x_1(t)x_2(t) - bx_3(t) \\
\dot{x}_4(t) &= -x_1(t) - ax_4(t)
\end{aligned} \tag{4.63}$$

where $x_1(t),\ x_2(t),\ x_3(t),\ x_4(t)$ are state variables and $a,\ b,\ c,\ r$ are known non-negative constants. In this example the constants are $a = 1,\ b = 0.7,\ c = 1.5,\ r = 26$. The chaotic behavior of the Lorenz-Stenflo system is shown in Figure 4.21.

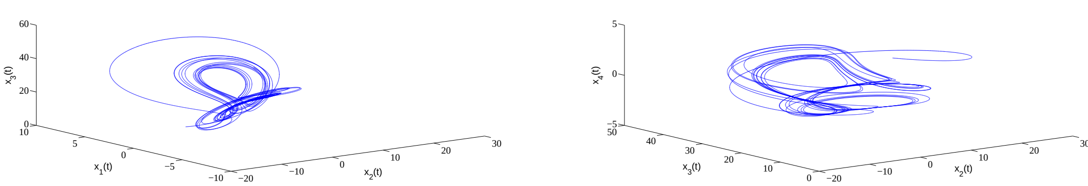


Figure 4.21: State space trajectories of Lorenz-Stenflo system

After adding the control input $u(t)$ and disturbance

$\Delta f(x) = 0.5 - sin(\pi x_1(t))\sin(2\pi x_2(t))\sin(3\ pix_3(t))\sin(4\ pix_4(t))$, Lorenz-Stenflo system (4.63) can be rewritten as

$$\begin{aligned}
\dot{x}_1(t) &= -ax_1(t) + ax_2(t) + cx_4(t) + u(t) + \Delta f(x)\\
\dot{x}_2(t) &= x_1(t)(r - x_3(t)) - x_2(t)\\
\dot{x}_3(t) &= x_1(t)x_2(t) - bx_3(t)\\
\dot{x}_4(t) &= -x_1(t) - ax_4(t)
\end{aligned} \tag{4.64}$$

The performance index J is chosen as

$$J = \int_0^\infty \left(x^T(t) \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} x(t) + u^T(t)u(t) \right) dt \tag{4.65}$$

Using the SDRE based method, the optimal control is found as $u_1(t) = -R^{-1}B^TP(x)x$ where $R = 1,\ B = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$ and $P(x)$ is the solution of the SDRE defined in (4.51). Design parameters of the integral and terminal sliding mode controller are chosen as $G = [1\ \ 0\ \ 0\ \ 0],\ p = 7,\ q = 5,\ \delta = 0.2,$ $\eta = 0.7$, $\varepsilon = 0.1$. The proposed controller is applied to the Lorenz-Stenflo system for the stabilization. The state space trajectories of the stable Lorenz-Stenflo system are shown in Figure 4.22

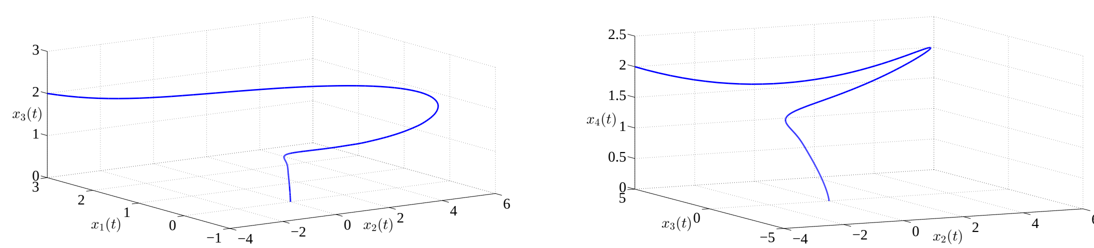


Figure 4.22: State space trajectories of Lorenz-Stenflo system after applying the proposed SDRE based OSOSMC

The states $x_1(t)$, $x_2(t)$, $x_3(t)$ and $x_4(t)$ obtained by applying the proposed OSOSMC are compared with those obtained by using the adaptive sliding mode controller proposed by Roopaei et al. [9] and are shown in Figures 4.23-4.26.

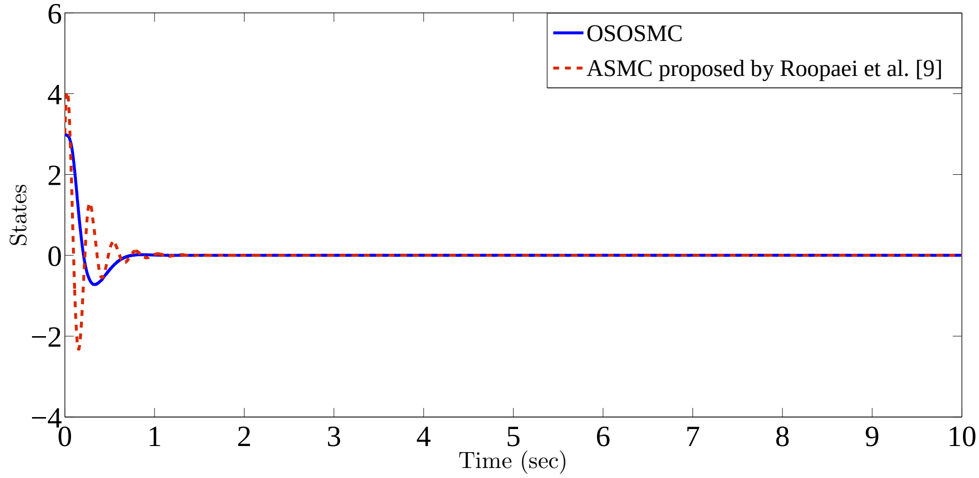


Figure 4.23: State $x_1(t)$ of Lorenz-Stenflo system obtained by using the proposed OSOSMC and ASMC proposed by Roopaei et al. [9]

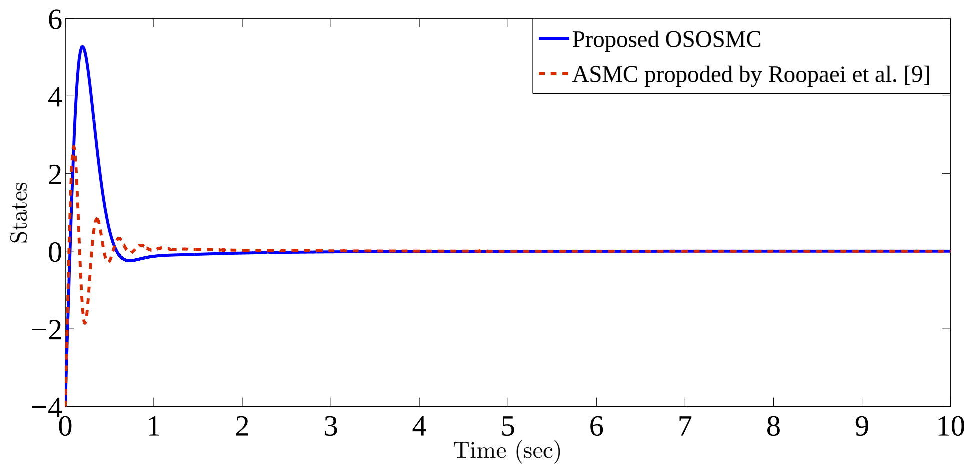


Figure 4.24: State $x_2(t)$ of Lorenz-Stenflo system obtained by using the proposed OSOSMC and ASMC proposed by Roopaei et al. [9]

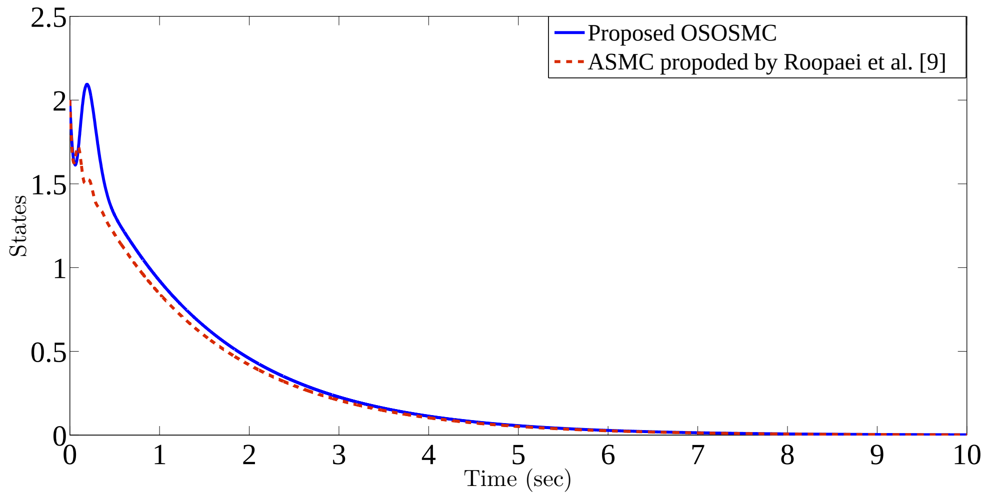


Figure 4.25: State $x_3(t)$ of Lorenz-Stenflo system obtained by using the proposed OSOSMC and ASMC proposed by Roopaei et al. [9]

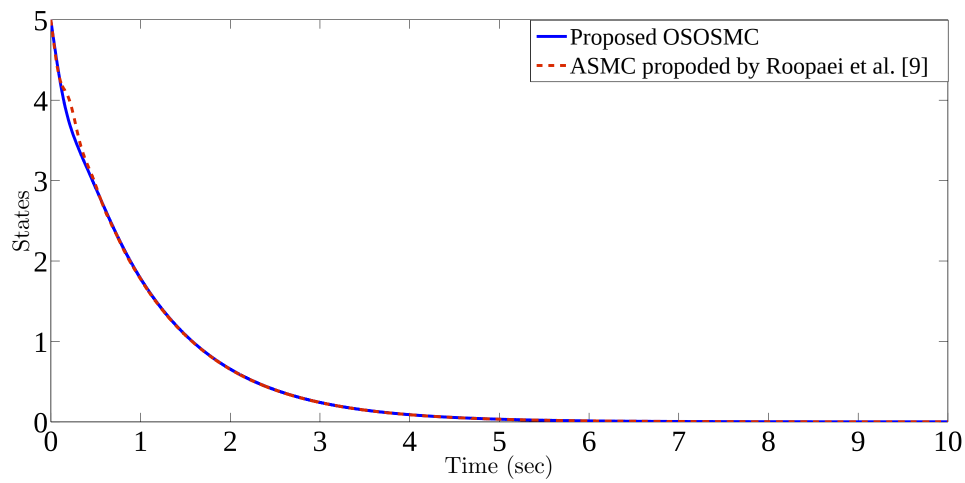


Figure 4.26: State $x_4(t)$ of Lorenz-Stenflo system obtained by using the proposed OSOSMC and ASMC proposed by Roopaei et al. [9]

The control inputs obtained by using the proposed OSOSMC and ASMC proposed by Roopaei et al. [9] are shown in Figures 4.27.

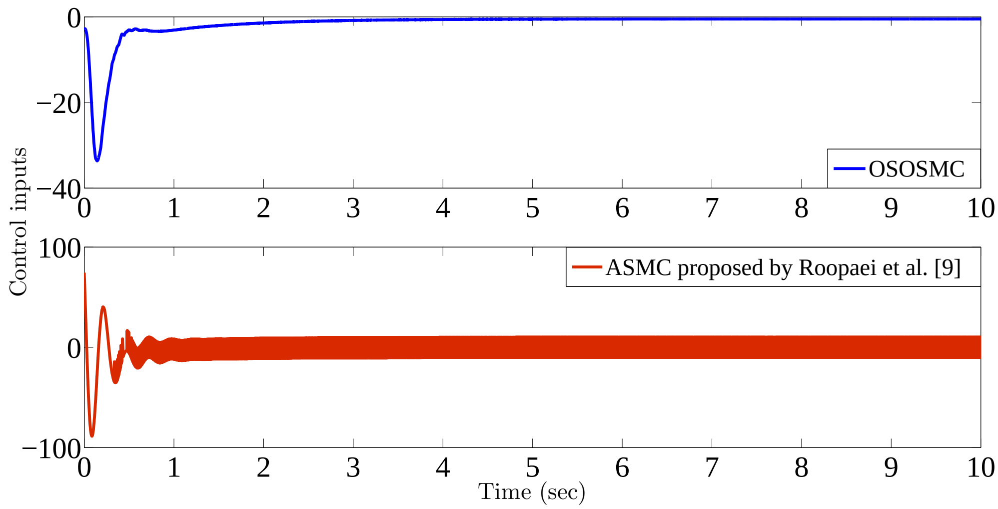


Figure 4.27: Control inputs obtained by using the proposed OSOSMC and ASMC proposed by Roopaei et al. [9] for Lorenz-Stenflo system

The proposed OSOSMC demonstrates stabilization performance and control input usage in accordance with results obtained in earlier chaotic system examples. Table 4.8 compares the total variation (TV) and the 2-norm of the control input obtained for the proposed OSOSMC and ASMC proposed by Roopaei et al. [9] in stabilizing the Lorenz-Stenflo system. The control indices are computed for the period from 0 to 10 sec with a sampling time of 0.01 sec. It is observed from Table 4.8 that the proposed OSOSMC method produces a far smoother control input at a cost of much lower control effort than the ASMC proposed by Roopaei et al. [9].

Table 4.8: Comparison of Control Indices

Method	Total Variation (TV)	Control Energy
ASMC [9]	4530.8	429.99
Proposed OSOSMC	69.53	134.22

4.4 Summary

In this chapter a state dependent Riccati equation (SDRE) based optimal second order sliding mode controller (OSOSMC) is proposed for nonlinear uncertain systems. The nonlinear system is converted into a linear like structure by using extended linearization where the system matrix and the input distribution matrix are state dependent. The optimal control law is designed by solving the state dependent Riccati equation arising in the nominal nonlinear system. As the optimal controller is highly sensitive to the uncertainty and disturbance present in the system, an integral sliding mode controller is integrated with the optimal controller for imparting robustness. To reduce chattering, a second order sliding mode methodology is proposed by designing a nonsingular terminal sliding mode controller based on the integral sliding variable. Simulation results establish effectiveness of the proposed SDRE based optimal second order sliding mode controller (OSOSMC). The proposed SDRE based OSOSMC is applied for stabilization of the chaotic system which is a special case of highly unstable nonlinear system. Chaotic systems which can be represented as linear like structures having state dependent coefficient (SDC) matrices are successfully stabilized by using the proposed SDRE based OSOSMC by spending a significantly lower control effort. Simulation results confirm the affectiveness of the proposed SDRE based OSOSMC which is found to exhibit superior performance compared to some already existing controller designed for controlling chaotic systems.

5

Control Lyapunov function based optimal second order sliding mode controller for nonlinear uncertain systems

Contents

5.1 Introduction

In Chapter 4 a state dependent Riccati equation based OSOSMC was proposed for the nonlinear system affected by the matched type of uncertainty. The state dependent Riccati equation based optimal controller can only be designed for nonlinear systems which can be described as linear like structures by using state dependent coefficient (SDC) matrices. However, it may not be possible to use extended linearization through SDC matrices for all nonlinear systems. For nonlinear systems which cannot be easily defined as linear like structures, control Lyapunov function (CLF) based optimal controller [21–24] has been developed. In [24] Sontag first proposed the control Lyapunov function (CLF) based optimal controller for nonlinear systems. The CLF is defined for systems with inputs having no specified feedback law. If the CLF can be found for the nonlinear system, there would exist a feedback controller to make the system asymptotically stable. Moreover, every CLF solves the HJB equation associated with a meaningful cost. So, if it is possible to find the CLF for a nonlinear system, it is also possible to find the optimal control law without actually solving the HJB equation. To make the optimal controller insensitive to uncertainties and external disturbance, an established way is to integrate the optimal controller with a sliding mode control (SMC) strategy by using an integral sliding surface. However, conventional integral sliding mode controller cannot tackle mismatched uncertainties and its control input contains high frequency chattering. Active research is going on to design sliding mode controllers which are able to tackle mismatched uncertainties. In [82–84] Choi proposed a linear matrix inequality (LMI) based sliding surface to stabilize linear systems affected by mismatched uncertainties. But in the case of nonlinear systems, this method is not applicable. Observer based SMC [11,89] has been proposed to stabilize nonlinear systems affected by mismatched uncertainties. To reduce chattering in the control input, higher order sliding mode controllers (HOSMC) [1,3,65,66] have been successfully used. The HOSMC retains all key qualities of the conventional SMC while eliminating the chattering at the same time.

In this chapter an optimal second order sliding mode controller (OSOSMC) is proposed for controlling nonlinear uncertain systems affected by both matched and mismatched types of uncertainties. For this purpose a control Lyapunov function (CLF) based optimal controller [21–24] is designed. To tackle the uncertainty, the optimal controller is integrated with the sliding mode control (SMC) by utilizing an integral sliding surface. A second order sliding mode control strategy is proposed by designing a nonsingular terminal sliding mode control based on the integral sliding variable. To estimate the mismatched uncertainty affecting the system, a disturbance observer is used. Simulation study is conducted to investigate the performance of the proposed controller.

Outline of the chapter as follows. In Section 5.2 an optimal second order sliding mode controller is designed for the nonlinear system affected by the matched uncertainty. The optimal controller is designed by defining a control Lyapunov function (CLF) and the second order sliding mode controller is realized by designing a non-singular terminal sliding mode controller based on an integral sliding variable. The proposed controller is applied for both stabilization and tracking problems. In Section 5.3 an optimal second order sliding mode controller is designed for the nonlinear system affected by the mismatched uncertainty. The optimal controller is designed based on the CLF defined for the nominal nonlinear system. A disturbance observer is utilized to estimate the mismatched uncertainty and the second order sliding mode controller is designed using an integral sliding variable based non-singular terminal sliding surface. Simulations are performed to study effectiveness of the proposed controller. Summary of this chapter is presented in Section 5.4.

5.2 Optimal second order sliding mode controller for nonlinear systems affected by matched uncertainties

In this section an optimal second order sliding mode controller is proposed for nonlinear systems affected by matched uncertainties. The nominal nonlinear system is stabilized by using a CLF based optimal controller. A Lyapunov function is defined for the open loop system and the optimal controller is designed to satisfy the Lyapunov stability criterion. In [24] Sontag has shown that the CLF based feedback controller which stabilizes the nominal nonlinear system also minimizes certain performance index. For a particular nonlinear system more than one control Lyapunov function can be defined to design the optimal controller. For the proposed controller a suitable control Lyapunov function is chosen. The CLF based optimal controller is then combined with a second order sliding mode controller for ensuring robustness against matched uncertainties. The second order sliding mode controller is implemented by using a non-singular terminal sliding surface based on an integral sliding variable.

I. Problem statement

The following nonlinear uncertain system is considered:

$$\dot{x}(t) = f(x) + \Delta f(x) + (g(x) + \Delta g(x))u(t) + d(t) \tag{5.1}$$

where $x(t) \in \mathcal{R}^n$ is the state vector and $u(t) \in \mathcal{R}$ is the control input. Further, $f(x)$, $g(x)$ are the nominal parts of the system and $g(x) \neq 0 \; \forall x$ in $t \in [0, \infty)$. System uncertainties are represented by $\Delta f(x)$, $\Delta g(x)$ and external disturbance is denoted by $d(t)$.

Assumption:

It is assumed that system uncertainties $\Delta f(x)$, $\Delta g(x)$ and external disturbance $d(t)$ satisfy the matching condition which means that these are in the range space of the input matrix. Hence, it can be written that,

$$\Delta f(x) + \Delta g(x)u(t) + d(t) = g(x)d_t(x, u, t) \tag{5.2}$$

where $d_t(x, u, t)$ is total uncertainty of system (5.1). Using (5.2), (5.1) can be expressed as

$$\dot{x}(t) = f(x) + g(x)u(t) + g(x)d_t(x, u, t) \tag{5.3}$$

The objective of the proposed control scheme is to design a chattering free optimal sliding mode controller for the nonlinear uncertain system (5.3). The design of the optimal sliding mode controller is followed in two steps, viz. (i) designing the optimal controller for the nominal nonlinear system and (ii) designing a sliding mode controller to tackle the uncertainty affecting the system. So the control input $u(t)$ can be expressed as,

$$u(t) = u_1(t) + u_2(t) \tag{5.4}$$

where $u_1(t)$ is the optimal control law to stabilize the nominal system and $u_2(t)$ is the sliding mode control used to keep the system onto the sliding surface to ensure robustness in presence of uncertainties and external disturbance.

Using (5.4), (5.3) can be defined as

$$\dot{x}(t) = f(x) + g(x)u_1(t) + g(x)u_2(t) + g(x)d_t(x, u, t) \tag{5.5}$$

II. Optimal control for the nominal system

Neglecting the uncertainties and external disturbance, the system defined in (5.5) can be written as,

$$\dot{x}(t) = f(x) + g(x)u_1(t) \tag{5.6}$$

and the performance index J chosen to optimize the control input $u_1(t)$ is defined as

$$J = \int_0^\infty (l(x) + u_1^T(t)Ru_1(t))dt \tag{5.7}$$

where $l(x)$ is a continuously differentiable, positive semidefinite function, $R \in \mathcal{R}^1$ is positive definite and $[f, l]$ is zero state detectable, with the desired solution being a state feedback control law. It is to be noted that existence of a Lyapunov function for the nonlinear system (5.6) is a necessary and sufficient condition for determining its stability. One way to stabilize a nonlinear system is to select a Lyapunov function $V(x)$ first and then try to find a feedback control $u_1(t)$ that makes $\dot{V}(x)$ negative definite. Lyapunov stability criterion finds stability of dynamic systems without inputs and it has been typically applied to closed loop control systems. But the idea of the control Lyapunov function (CLF) [22] based controller is to define a Lyapunov candidate for the open loop system and then design a feedback loop that makes the Lyapunov function's derivative negative. Hence, if it is possible to find the CLF, then it is also possible to find a stabilizing feedback control law $u_1(t)$.

The Lyapunov function considered for the system (5.6) is a positive definite, radially unbounded function $V(x)$ and its derivative is given by

$$\dot{V}(x) = L_f V(x) + L_g V(x) u_1(t) \tag{5.8}$$

where L represents the Lie derivative operator.

Now, $V(x)$ is a CLF if $\forall x(t) \neq 0$,

$$L_g V(x) = 0 \Longrightarrow L_f V(x) < 0. \tag{5.9}$$

By using standard convergence theorem [134] it is inferred that if (5.6) is stabilizable, then there exists a CLF. On the other hand, if there exists a CLF for the system (5.6), then there also exists an asymptotically stabilizing controller which stabilizes the system (5.6). Sontag [24] proposed a CLF based controller as given below,

$$u_1(t) = \begin{cases} -[\frac{a(x)+\sqrt{a(x)^2+l(x)b(x)^T R^{-1} b(x)}}{b(x)b(x)^T}]b(x)^T & for \quad b(x) \neq 0 \\ 0 & for \quad b(x) = 0 \end{cases} \tag{5.10}$$

where

$$a(x) = L_f V(x), \qquad b(x) = L_g V(x) \tag{5.11}$$

The control law $u_1(t)$ defined in (5.10) stabilizes the nominal nonlinear system defined in (5.6) by minimizing the performance index (5.7).

Minimization of the performance index [135]

In order to find the optimal stabilizing controller for a nonlinear system, the Hamilton Jacobi Bellman (HJB) equation [14] needs to be solved. The HJB equation [14] is given by,

$$l(x) + L_f V^* - \frac{1}{4} L_g V^* R^{-1}(x)(L_g V^*)^T = 0 \tag{5.12}$$

where V^* is the solution of the HJB equation and is commonly referred to as a value function defined as

$$V^* = \inf_{u_1(t)} \int_t^{\infty} \left(l(x) + u_1^T(t) R u_1(t)\right) d\tau \tag{5.13}$$

If there exists a continuously differentiable, positive definite solution of the HJB equation (5.12), then the optimal controller is defined as [22]

$$u^*(t) = -\frac{1}{2} R^{-1} L_g V^* \tag{5.14}$$

If the level curve of $V(x)$ agrees with the shape of V^*, then Sontag's formula (5.10) produces the optimal controller. However,in general, V^* is not the same as $V(x)$. So, a scaler function λ is considered such that $V^* = \lambda V(x)$. Now the optimal controller is given by

$$u^*(t) = -\frac{1}{2} R^{-1} (\lambda L_g V(x))^T \tag{5.15}$$

Moreover, λ can be determined by substituting $V^* = \lambda V(x)$ in the HJB equation defined in (5.12) as,

$$l(x) + \lambda L_f(V(x)) - \frac{\lambda^2}{4} L_g V(x) R^{-1} (L_g(V(x)))^T = 0 \tag{5.16}$$

Now, by solving the above equation (5.16) and using (5.11), λ is found as

$$\lambda = 2\left(\frac{a(x) + \sqrt{a(x)^2 + l(x)b(x)^T R^{-1} b(x)}}{R^{-1} b(x) b(x)^T}\right) \tag{5.17}$$

Substituting the value of λ in (5.15), controller $u^*(t)$ is obtained as

$$u^*(t) = -\left[\frac{a(x) + \sqrt{a(x)^2 + l(x)b(x)^T R^{-1} b(x)}}{b(x)b(x)^T}\right] b(x)^T \tag{5.18}$$

So, the control input $u^*(t)$ is defined as follows,

$$u^*(t) = \begin{cases} -\left[\frac{a(x)+\sqrt{a(x)^2+l(x)b(x)^T R^{-1} b(x)}}{b(x)b(x)^T}\right] b(x)^T & for \quad b(x) \neq 0 \\ 0 & for \quad b(x) = 0 \end{cases} \tag{5.19}$$

The optimal controller $u^*(t)$ is exactly the same as the controller $u_1(t)$ found by using Sontag's formula (5.10). So, it is proved that the control effort $u_1(t)$ minimizes the performance index (5.7).

Stabilization of the nominal system [24]

Let us consider (5.8) again and replace $u_1(t)$ there by using (5.10). Utilizing the property of the CLF i.e. $L_f V(x) < 0$ when $L_g V(x) = 0$, it can be resolved that $\dot{V}(x) < 0$ for $u_1(t) = 0$ and for $u_1(t) \neq 0$, (5.8) is obtained by using (5.11) as,

$$\begin{aligned} \dot{V}(x) &= -\sqrt{a(x)^2 + l(x)b(x)^T R^{-1} b(x)} \\ &< -|\sqrt{a(x)^2 + l(x)b(x)^T R^{-1} b(x)}| \\ &< 0 \end{aligned} \tag{5.20}$$

Hence, it is proved that the control input defined in (5.10) minimizes the performance index (5.7) and stabilizes the nominal nonlinear system (5.6).

Remark 1. In (5.7), the weighing matrix R requires to be chosen suitably to design the optimal control law ensuring the desired performance.

Remark 2. It is easy to find the CLF for two dimensional systems by using analytical methods. However, if the system dimension is higher, it is difficult to choose the CLF analytically.

Remark 3. For applying the CLF based optimal controller, exact knowledge of the system being considered is a necessary prerequisite. But if the system is affected by uncertainty in the neighborhood of the equilibrium state, performance of the optimal controller degrades and it may even fail. An efficient way to overcome this limitation is to integrate the optimal controller with the sliding mode control (SMC) scheme which is an established method to ensure robustness.

III. Second order sliding mode control

An optimal controller is highly sensitive to uncertainties and disturbances. To make the optimal controller robust, it is now integrated with a sliding mode controller. For this purpose, an integral sliding variable s is designed as

$$s = G\left[x(t) - \int_0^t \dot{x}_{nom}(\tau)d\tau\right] \tag{5.21}$$

where $\dot{x}_{nom}(t) = f(x) + g(x)u_1(t)$. Knowledge about the initial condition $x(0)$ is not required at the time of designing the integral sliding surface $s(t) = 0$. However, $s(t) \neq 0$ from the very beginning. Hence, to converge the integral sliding variable $s(t)$ in finite time, a non-singular terminal sliding surface is designed based on the integral sliding variable $s(t)$. The non-singular terminal sliding variable is designed as σ [77, 79] where

$$\sigma(t) = s(t) + \delta\dot{s}^{\frac{\alpha}{\beta}}(t) \tag{5.22}$$

and δ is the switching gain chosen such that

$$\delta > 0 \tag{5.23}$$

Here α, β are selected in such a way that these satisfy the following conditions:

$$\alpha, \beta \in \{2n+1:\ n\, is\, an\, integer\} \tag{5.24}$$

and

$$1 < \frac{\alpha}{\beta} < 1.5 \tag{5.25}$$

Following the procedure described in Section 3.3 III, the sliding mode control $u_2(t)$ is designed as

$$u_2(t) = -\int_0^t (Gg(x))^{-1}\left[\frac{\beta}{\delta\alpha}\dot{s(\tau)}^{2-\frac{\alpha}{\beta}} + G\dot{g}(x)u_2(\tau) + \eta sgn(\sigma(\tau)) + \varepsilon\sigma(\tau)\right]d\tau \tag{5.26}$$

where the design parameters η, ε are chosen in such a way that $|Gg(x)\dot{d}_t(x,u,t)| < \eta$ [66] and $\varepsilon > 0$. The convergence of the sliding variables has already been proved in Chapter 4.

IV. Simulation Results

The proposed optimal second order sliding mode controller (OSOSMC) is applied for stabilization and tracking problems involving nonlinear systems affected by matched uncertainties. The simulation experiments are explained below.

Example 1: Stabilization of a nonlinear uncertain system

A second order nonlinear uncertain system [10] is considered as

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -(x_1^2(t)+1.5x_2(t))+u-(0.3\sin(t)x_1^2(t)+0.2\cos(t)x_2(t))+d(t)\end{aligned} \tag{5.27}$$

where $d(t)$ is a random noise of 0 mean and 0.5 variance. The above second order nonlinear system (5.27) can be expressed as (5.1) where $f(x) = [x_2(t) \quad -x_1^2(t) - 1.5x_2(t)]^T$, $g(x) = [0 \quad 1]^T$, $\Delta f(x) = [0 \quad -0.3\sin(t)x_1^2 - 0.2\cos(t)x_2]^T$ and $\Delta g(x) = 0$. The initial state $x(0) = [1.5 \quad 0]^T$. The performance index J is defined as

$$J = \int_0^\infty \left[l(x)x + u_1^T(t)10u_1(t)\right] dt \tag{5.28}$$

where $l(x)$ is chosen as $x_1^2(t) + x_2^2(t)$.

After preliminary simulation it was found that the following $V(x)$ (5.29) gives rises to the best optimization and stabilization. Hence this value of $V(x)$ is chosen.

$$V(x) = x^T(t)\begin{bmatrix} 5.8 & 5.51 \\ 5.51 & 5.51 \end{bmatrix} x(t). \tag{5.29}$$

Design parameters of the proposed optimal second order sliding mode controller (OSOSMC) are chosen as follows:

$G = [0 \quad 1]$, $\alpha = 7$, $\beta = 5$, $\delta = 0.5$, $\eta = 0.6$ and $\varepsilon = 0.2$.

The proposed optimal second order sliding mode controller (OSOSMC) is applied to stabilize the system (5.27). The results obtained by applying the proposed optimal second order sliding mode controller are compared with those obtained by using the adaptive sliding mode controller designed by Kuo et al. [10] which is discussed in Appendix A.10. The states and the control inputs obtained by applying the proposed optimal second order sliding mode control (OSOSMC) and the adaptive sliding mode control (ASMC) proposed by Kuo et al. [10] are shown in Figure 5.1 and Figure 5.2 respectively. It is observed from Figures 5.1-5.2 that both the controllers stabilize the considered nonlinear uncertain system to the equilibrium state at the same rate although the control input in the case of the proposed OSOSMC contains lesser chattering than that of Kuo et al. [10].

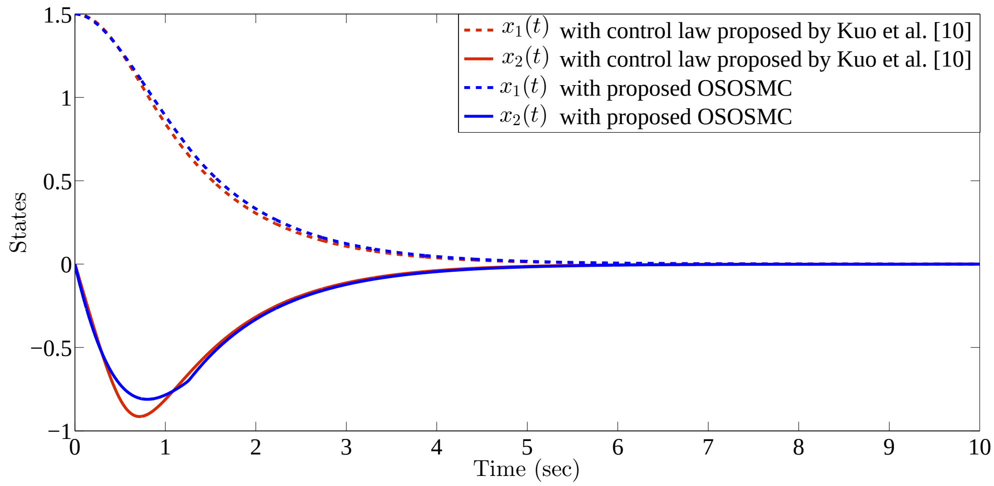


Figure 5.1: States obtained by applying proposed OSOSMC and ASMC proposed by Kuo et al. [10]

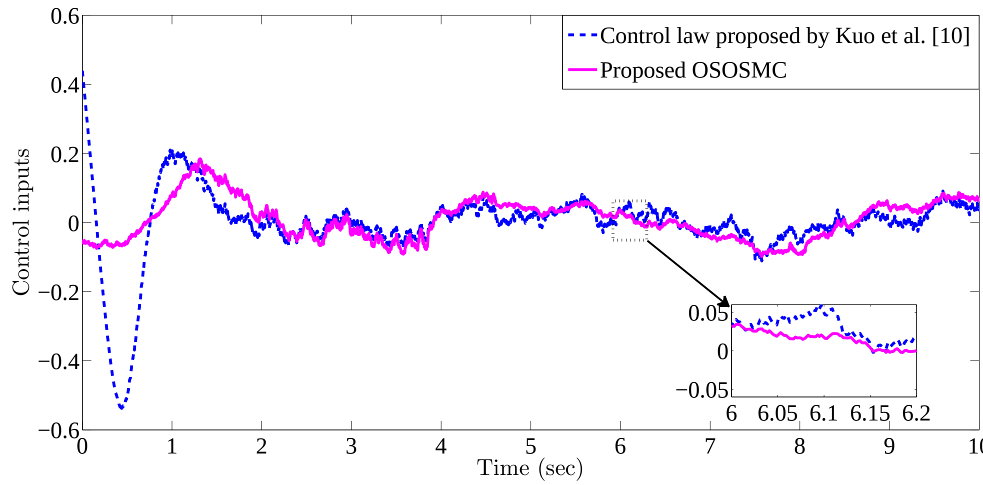


Figure 5.2: Control inputs obtained by applying proposed OSOSMC and ASMC proposed by Kuo et al. [10]

To measure the smoothness of the control input, the total variation (TV) [106] is computed for both controllers. The control energy spent by the controller is found by calculating its second norm.

Table 5.1 shows the TV and the 2-norm of the control input calculated for the period from 0 to 10 sec with a sampling time of 0.1 sec. It is clear from Table 5.1 that the proposed OSOSMC is able to produce a smoother control input with substantial reduction in the control effort than that of the ASMC proposed by Kuo et al. [10].

Table 5.1: Comparison of control indices of the ASMC [10] and the proposed OSOSMC

Method	Total Variation (TV)	Control Energy
ASMC [10]	3.65	1.13
Proposed OSOSMC	1.52	0.60

Example 2: Tracking by a nonlinear uncertain system

A single inverted pendulum [4] is now considered. The state space model of the single inverted pendulum [4] is given as

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= \frac{g\sin x_1(t) - [mlx_2^2(t)\cos x_1(t)\sin\frac{x_1(t)}{m_c} + m]}{l[\frac{4}{3} - (m\cos^2\frac{x_1(t)}{m_c} + m)]} + \frac{\cos\frac{x_1(t)}{m_c} + m}{l[\frac{4}{3} - (m\cos^2\frac{x_1(t)}{m_c} + m)]}u(t) + d(t) \\
y(t) &= x_1(t)
\end{aligned} \tag{5.30}$$

where $x_1(t)$ is the swing angle and $x_2(t)$ is the swing speed. Parameters of the single inverted pendulum are tabulated in Table 5.2. The disturbance $d(t)$ is considered as $7\sin(10x_1(t)) + \cos x_2(t)$. The output of the system is swing angle which is defined as $y(t) = x_1(t)$

Table 5.2: Parameters of the single inverted pendulum [4]

Parameters	Description	values
g	Gravitational constant	$9.8\ m.sec^{-2}$
m_c	Mass of the cart	$1\ kg$
m	Mass of the pendulum	$0.1\ kg$
l	Effective length of the pendulum	$0.5\ m$

The proposed controller is used such that output $y(t) = x_1(t)$ tracks the desired trajectory $x_d(t) = \sin(0.5\pi t)$ using minimum control effort. Tracking error is defined as

$$e(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - \begin{bmatrix} x_d(t) \\ \dot{x}_d(t) \end{bmatrix} \tag{5.31}$$

To design the CLF based OSOSMC, the performance index J is considered as

$$J = \int_0^\infty \left[l(e) + u_1^2(t)\right] dt \tag{5.32}$$

where $l(e)$ is chosen as $e_1^2(t) + e_2^2(t)$. The control Lyapunov function V is selected as

After preliminary simulation it was found that $V(x)$ in 5.33 gives rises to the best optimization and stabilization. Hence this value of $V(x)$ is chosen.

$$V = e^T(t)\begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.4 \end{bmatrix} e(t). \tag{5.33}$$

The integral sliding surface is designed as

$$s(t) = e(t) - \int_0^t \dot{e}_{nom}(\tau)d\tau = 0 \tag{5.34}$$

where $e_{nom}(t)$ is the nominal part of the system defined in error domain. Design parameters of the proposed optimal second order sliding mode controller (OSOSMC) (5.26) are chosen as follows:

$G = [0 \;\; 1]$, $\alpha = 7$, $\beta = 5$, $\delta = 0.5$, $\eta = 9$ and $\varepsilon = 0.2$.

Simulation results are compared with those obtained by using the integral sliding mode controller proposed by Mondal and Mahanta [4] which is discussed in Appendix A.4. The sliding surface designed in [4] is given by

$$s_1(t) = e(t) - e(0) - \int \upsilon_{nom} = 0 \tag{5.35}$$

where υ_{nom} is defined as $\upsilon_{nom} = -3sign(e_1(t))|e_1(t)|^{\frac{3}{4}} - 2.5sign(e_2(t))|e_2(t)|^{\frac{3}{5}}$

The controller designed in [4] is given by

$$u(t) = \overline{b}^{-1}[\overline{a} - \ddot{x}_d(t) + \upsilon_{nom} - 10sign(s_1(t))] \tag{5.36}$$

where $\overline{a} = \frac{g\sin x_1(t) - [mlx_2^2(t)\cos x_1(t)\sin\frac{x_1(t)}{m_c} + m]}{l[\frac{4}{3} - (m\cos^2\frac{x_1(t)}{m_c} + m)]}$ and $\overline{b} = \frac{\cos\frac{x_1(t)}{m_c} + m}{l[\frac{4}{3} - (m\cos^2\frac{x_1(t)}{m_c} + m)]}$.

The states obtained by applying the proposed CLF based optimal second order sliding mode control (OSOSMC) and the integral sliding mode control (SMC) [4] are shown in Figure 5.3 and Figure 5.4 respectively. The control inputs for these cases are compared in Figures 5.5- 5.6. It is observed from Figures 5.5-5.6 that the proposed CLF base OSOSMC requires reduced control energy and contains lesser chattering than the integral SMC [4] .

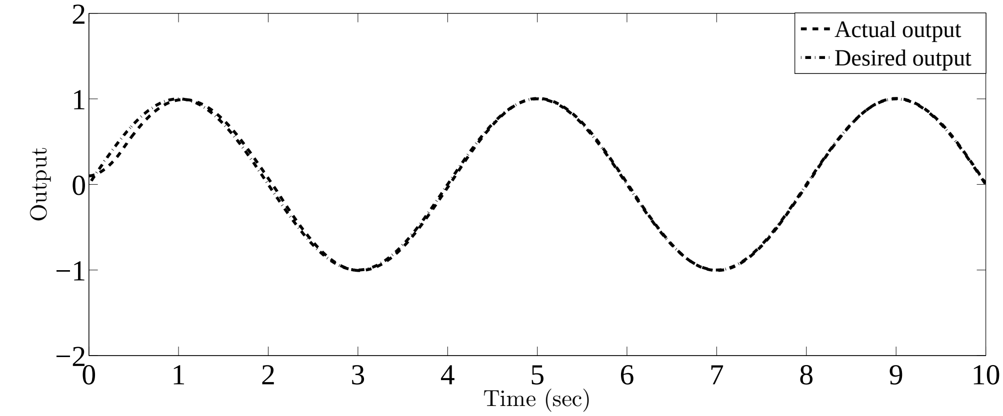


Figure 5.3: States obtained by applying the proposed CLF based OSOSMC

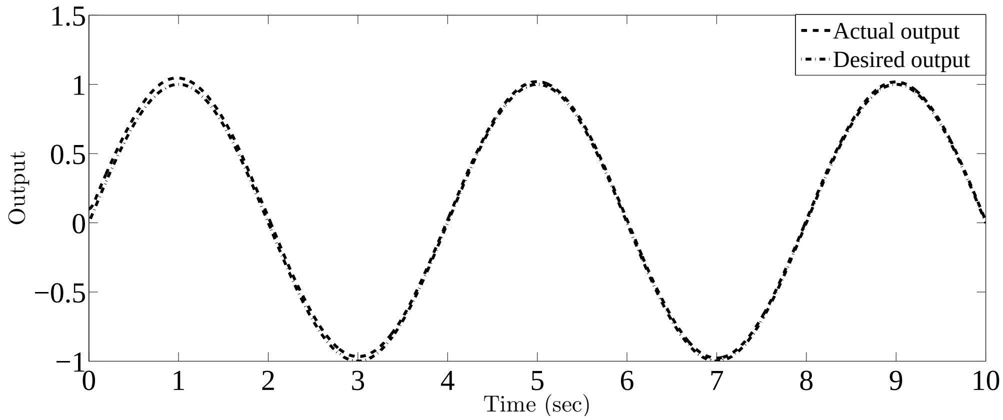


Figure 5.4: States obtained by applying the integral SMC [4]

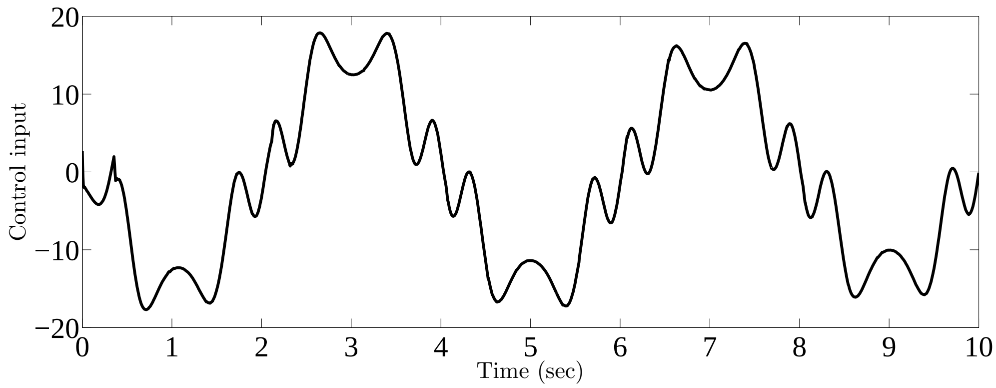


Figure 5.5: Control input obtained by applying the proposed CLF based OSOSMC

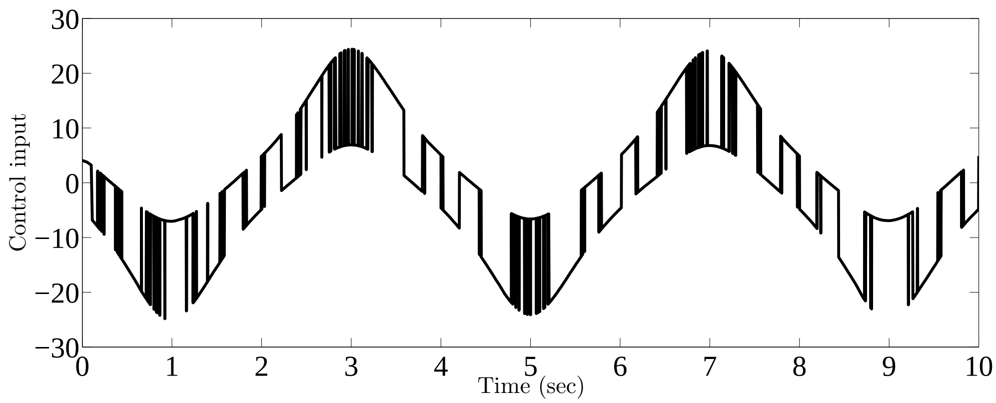


Figure 5.6: Control input obtained by applying the integral SMC [4]

To measure the smoothness of the control input, the total variation (TV) [106] is computed for both the controllers. Further, 2-norm of the control input is calculated to assess the control energy spent by the controller.

Table 5.3 shows the TV and the 2-norm of the control inputs calculated for the period from 0 to 10 sec with a sampling time of 0.1 sec. It is clear from Table 5.3 that the proposed CLF based OSOSMC is able to produce a smoother control input with reduction in the control effort than that of the integral SMC [4].

Table 5.3: Comparison of control indices of proposed CLF based OSOSMC and integral SMC [4]

Method	Total Variation (TV)	Control Energy
Proposed CLF based OSOSMC	320.31	106.92
Integral SMC [4]	640.17	121.54

5.3 Optimal second order sliding mode controller for nonlinear systems affected by mismatched uncertainties

Now an optimal second order sliding mode controller is proposed for the nonlinear system affected by mismatched type of uncertainty. The optimal controller is designed for the nominal nonlinear system using the control Lyapunov function (CLF) as discussed in the previous section. As the sliding mode controller (SMC) cannot tackle the mismatched uncertainty, it is estimated by using a disturbance observer. A second order sliding mode methodology is proposed by designing a non-singular terminal sliding mode based on an integral sliding variable.

I. Problem statement

A nonlinear system with mismatched uncertainty is considered as given below:

$$\dot{x}(t) = f(x) + g(x)u(t) + g_1 d(t) \tag{5.37}$$

where $x(t) \in \mathcal{R}^n$ is the state vector and $u(t) \in \mathcal{R}$ is the control input. Further, $f(x)$, $g(x)$ are the nominal parts of the nonlinear system and $g(x) \neq 0$ $\forall x$ in $t \in [0, \infty)$. Moreover, $d(t)$ represents the uncertainty affecting the system and g_1 is not in the range space of $g(x)$. As such, $d(t)$ does not satisfy the matching condition.

Assumption: The disturbance $d(t)$ is unknown but bounded and $\dot{d}(t) = 0$.

The objective of the proposed control method is to design an optimal second order sliding mode controller for the above nonlinear system affected by the mismatched uncertainty. The design of the proposed observer based optimal second order sliding mode controller is divided into three steps, (i) designing the optimal controller for the nominal nonlinear system, (ii) estimating the disturbance by using a disturbance observer, (iii) designing the observer based second order sliding mode controller to tackle the mismatched uncertainty. So, the control input $u(t)$ is divided into two parts and can be expressed as

$$u(t) = u_1(t) + u_2(t) \tag{5.38}$$

where $u_1(t)$ is the optimal control law to stabilize the nominal nonlinear system and $u_2(t)$ is the sliding mode control used to keep the system onto the sliding surface to ensure robustness in presence of mismatched uncertainties.

II. Optimal controller design

Neglecting the uncertainty, nominal part of the system defined in (5.37) can be written as,

$$\dot{x}(t) = f(x) + g(x)u_1(t) \tag{5.39}$$

and the performance index J chosen to optimize the control input $u_1(t)$ is defined as

$$J = \int_0^{\infty} [l(x) + u_1(t)^T R u_1(t)]dt \tag{5.40}$$

where $l(x)$ is a continuously differentiable, positive semi-definite function, $R \in \mathcal{R}$ is positive definite and $[f, l]$ is zero state detectable, with the desired solution being a state feedback control law. The optimal controller is designed for the nominal nonlinear system using the CLF as discussed in the previous section. For the system defined in (5.39), the Lyapunov function is chosen as a positive definite, radially unbounded function $V(x)$ and the derivative of the Lyapunov function is given by

$$\dot{V}(x) = L_f V(x) + L_g V(x) u_1(t) \tag{5.41}$$

where L represents the Lie derivative operator.

Now, $V(x)$ is a control Lyapunov function (CLF) if $\forall x(t) \neq 0$,

$$L_g V(x) = 0 \Longrightarrow L_f V(x) < 0. \tag{5.42}$$

By standard converge theorems [134], if (5.39) is stabilizable, then there exists a CLF. On the other hand, if there exists a CLF for the system (5.39), then there also exists a feedback controller which stabilizes the system (5.39). The CLF based optimal controller is designed as follows

$$u_1(t) = \begin{cases} -[\frac{a(x)+\sqrt{a(x)^2+l(x)b(x)^T R^{-1} b(x)}}{b(x)b(x)^T}]b(x)^T & for \quad b(x) \neq 0 \\ 0 & for \quad b(x) = 0 \end{cases} \tag{5.43}$$

where

$$a(x) = L_f V(x), \qquad b(x) = L_g V(x) \tag{5.44}$$

Though the designed optimal controller minimizes the performance index (5.40), the major constraint for its application is the necessary prerequisite of exact knowledge about the system. If the system is affected by uncertainty, the controller may fail. In order to overcome this limitation, a sliding mode control strategy is embedded with the optimal controller.

III. Sliding mode controller based on disturbance observer

An integral sliding mode controller (ISMC) is combined with the optimal controller designed above to impart robustness. However, a conventional ISMC cannot tackle the mismatched uncertainty. So, the integral sliding surface is designed based on disturbance estimation by using a nonlinear disturbance observer (DOB) [11] defined as follows:

$$\begin{aligned} \dot{p}(t) &= -\rho g_1 p(t) - \rho[g_1 \rho x(t) + f(x) + g(x)u(t)] \\ \widehat{d}(t) &= p(t) + \rho x(t) \end{aligned} \tag{5.45}$$

where $\widehat{d}(t)$ denotes the estimation of the disturbance $d(t)$. Further, $p(t)$, ρ represent the internal state of the nonlinear disturbance observer and the observer gain respectively. Here ρ is chosen such that ρg_1 becomes positive definite. It implies that

$$\dot{e}_d(t) + \rho g_1 e_d(t) = 0 \tag{5.46}$$

is asymptotically stable, where $e_d(t) = d(t) - \widehat{d}(t)$. So, it can be written that

$$\lim_{t\to\infty} e_d(t) = 0 \tag{5.47}$$

The integral sliding variable s is chosen as follows:

$$s(t) = G[x(t) - \int_0^t \dot{\Phi}(\tau)d\tau + g_1\widehat{d}(t)] \tag{5.48}$$

where G is a design parameter which is so chosen such that $Gg(x)$ is invertible. Moreover, $\dot{\Phi}(t)$ is defined as

$$\dot{\Phi}(t) = f(x) + g(x)u_1(t) \tag{5.49}$$

First time derivative of the integral sliding variable $s(t)$ in (5.48) is obtained as

$$\dot{s}(t) = G[\dot{x}(t) - \dot{\Phi}(t) + g_1\dot{\widehat{d}}(t)] \tag{5.50}$$

Substituting the values of $\dot{x}(t)$ and $\dot{\Phi}(t)$ from (5.37) and (5.49) into (5.50) yields

$$\dot{s}(t) = G[g(x)u_2(t) + g_1 d(t) + g_1\dot{\widehat{d}}(t)] \tag{5.51}$$

Further, using the assumption that $\dot{d}(t) = 0$, second derivative of the sliding variable $s(t)$ is found as

$$\ddot{s}(t) = G[\dot{g}(x)u_2(t) + g(x)\dot{u}_2(t) + g_1\ddot{\widehat{d}}(t)] \tag{5.52}$$

To overcome chattering, a second order sliding mode is proposed by using a non-singular terminal sliding variable $\sigma(t)$ as follows:

$$\sigma(t) = s(t) + \delta\dot{s}(t)^{\frac{\alpha}{\beta}} \tag{5.53}$$

where δ is the switching gain chosen such that

$$\delta > 0 \tag{5.54}$$

Here α, β are selected in such a way that these satisfy the conditions

$$\alpha, \beta \in \{2n + 1 : \ n \, is \, an \, integer\} \tag{5.55}$$

and

$$1 < \frac{\alpha}{\beta} < 1.5 \tag{5.56}$$

The motivation behind choosing a non-singular terminal sliding variable $\sigma(t)$ in the second order sliding mode control scheme is for achieving finite time convergence of the sliding variables. Taking the first time derivative of the terminal sliding variable (5.53) yields

$$\begin{aligned}\dot{\sigma}(t) &= \dot{s}(t) + \delta\frac{\alpha}{\beta}\dot{s}(t)^{\frac{\alpha}{\beta}-1}\ddot{s}(t) \\ &= \delta\frac{\alpha}{\beta}\dot{s}(t)^{\frac{\alpha}{\beta}-1}(\frac{\beta}{\delta\alpha}\dot{s}(t)^{2-\frac{\alpha}{\beta}} + \ddot{s}(t))\end{aligned} \tag{5.57}$$

For the design parameters α, β satisfying (5.55) and (5.56), it can be shown [79] that

$$\begin{aligned}\dot{s}(t)^{\frac{\alpha}{\beta}-1} > 0 \quad for \quad \dot{s}(t) \neq 0 \\ \dot{s}(t)^{\frac{\alpha}{\beta}-1} = 0 \quad only\,for \quad \dot{s}(t) = 0\end{aligned} \tag{5.58}$$

Further, from (5.54), (5.55) and (5.58), $\delta\frac{\alpha}{\beta}\dot{s}(t)^{\frac{\alpha}{\beta}-1}$ in (5.57) can be replaced by $\eta_2 > 0$ for $\dot{s}(t) \neq 0$. Hence (5.57) can be written as

$$\dot{\sigma}(t) = \eta_2(\frac{\beta}{\delta\alpha}\dot{s}(t)^{2-\frac{\alpha}{\beta}} + \ddot{s}(t)) \tag{5.59}$$

The above strategy of using a terminal sliding mode based on an integral sliding variable gives rise to a second order SMC. Using the constant plus proportional reaching law [104] for the terminal sliding variable $\sigma(t)$ gives rise to

$$\dot{\sigma}(t) = -\eta_1 sgn(\sigma(t)) - \varepsilon_1\sigma(t) \tag{5.60}$$

where $\eta_1 > 0$ and $\varepsilon_1 > 0$.

Substituting the value of $\dot{\sigma}(t)$ from (5.59), (5.60) can be expressed as

$$\begin{aligned}\eta_2(\frac{\beta}{\delta\alpha}\dot{s}(t)^{2-\frac{\alpha}{\beta}} + \ddot{s}(t)) = -\eta_1 sgn(\sigma(t)) - \varepsilon_1\sigma(t) \\ or, \frac{\beta}{\delta\alpha}\dot{s}(t)^{2-\frac{\alpha}{\beta}} + \ddot{s}(t) = -\eta sgn(\sigma(t)) - \varepsilon\sigma(t)\end{aligned} \tag{5.61}$$

where $\eta = \frac{\eta_1}{\eta_2} > 0$ and $\varepsilon = \frac{\varepsilon_1}{\eta_2} > 0$. Then (5.61) can be rewritten as

$$\ddot{s}(t) = -\eta sgn(\sigma(t)) - \varepsilon\sigma(t) - \frac{\beta}{\delta\alpha}\dot{s}(t)^{2-\frac{\alpha}{\beta}} \tag{5.62}$$

Hence, from (5.62) and (5.52), the switching control law is designed as

$$\begin{aligned} u_2(t) \quad &= -\int_0^t (Gg(x))^{-1}\left[\tfrac{\beta}{\delta\alpha}\dot{s}(\tau)^{2-\frac{\alpha}{\beta}} + G\dot{g}(x)u_2(\tau)\right. \\ &\left. + Gg_1\ddot{\hat{d}}(\tau) + \eta sgn(\sigma(\tau)) + \varepsilon\sigma(\tau)\right] d\tau \end{aligned} \tag{5.63}$$

Stability analysis of the sliding surfaces

Let us consider the Lyapunov function as $V_1(t)$ given by

$$\begin{aligned} V_1(t) &= \frac{1}{2}\sigma^2(t) \\ \dot{V}_1(t) &= \sigma(t)\dot{\sigma}(t) \\ &= \sigma(t)[\dot{s}(t) + \frac{\alpha\delta}{\beta}\dot{s}(t)^{\frac{\alpha}{\beta}-1}\ddot{s}(t)] \\ &= \sigma(t)[\dot{s}(t) + \frac{\alpha\delta}{\beta}\dot{s}(t)^{\frac{\alpha}{\beta}-1}(Gg(x)\dot{u}_2(t) + G\dot{g}(x)u_2(t) + Gg_1\ddot{\hat{d}}(t))] \end{aligned}$$

$$\begin{aligned} &= \sigma(t)[\dot{s}(t) + \frac{\alpha\delta}{\beta}\dot{s}(t)^{\frac{\alpha}{\beta}-1}(-\frac{\beta}{\delta\alpha}\dot{s}(t)^{2-\frac{\alpha}{\beta}} - G\dot{g}(x)u_2(t) - Gg_1\ddot{\hat{d}}(t) - \eta sgn(\sigma(t)) \\ &- \varepsilon\sigma(t) + G\dot{g}(x)u_2(t) + Gg_1\ddot{\hat{d}}(t))] \\ &= \sigma(t)[\frac{\alpha\delta}{\beta}\dot{s}(t)^{\frac{\alpha}{\beta}-1}(-\eta sgn(\sigma(t)) - \varepsilon\sigma(t))] \\ &= \frac{\alpha\delta}{\beta}\dot{s}(t)^{\frac{\alpha}{\beta}-1}[-|\sigma(t)|\eta - \varepsilon\sigma(t)^2] \\ &\leq -\frac{\alpha\delta}{\beta}\dot{s}(t)^{\frac{\alpha}{\beta}-1}[\eta + \varepsilon|\sigma(t)|]|\sigma(t)| \\ &\leq -\frac{\alpha\delta}{\beta}\dot{s}(t)^{\frac{\alpha}{\beta}-1}\kappa|\sigma(t)| \end{aligned} \tag{5.64}$$

where $\kappa = \eta + \varepsilon|\sigma(t)| > 0$.Moreover, in [66] Wang et al. showed that $\dot{s}^{\frac{\alpha}{\beta}-1}(t) > 0$ for $|s(t)| \neq 0$.

So, (5.64) can be written as

$$\begin{aligned} \dot{V}_1(t) \quad &\leq \quad -\widehat{\kappa}|\sigma(t)| \;\text{ where }\; \widehat{\kappa} = \frac{\alpha\delta}{\beta}\dot{s}(t)^{\frac{\alpha}{\beta}-1}\kappa > 0 \;\text{ for }\; |s(t)| \neq 0 \\ &\leq \quad -\widehat{\kappa}|\sqrt{2V_1(t)}| \;\text{ as }\; V_1(t) = \frac{1}{2}\sigma^2(t) \end{aligned} \tag{5.65}$$

Hence, finite time stability of the sliding variable $\sigma(t)$ is guaranteed [101, 105].

Moreover, it can be proved that integral sliding variable $s(t)$ converges to zero in finite time. Suppose in time t_r, $\sigma(t)$ reaches zero from $\sigma(0) \neq 0$ and $\sigma(t) = 0 \;\forall\; t > t_r$. So, once $\sigma(t)$ reaches zero, it remains at zero and based on (5.64), $s(t)$ will converge to zero in finite time t_s. The total time required from $\sigma(0) \neq 0$ to $s(t_s)$ is defined as follows:

$$t_s = t_r + \frac{\alpha}{\alpha - \beta}\delta^{\frac{\beta}{\alpha}} s(t_r)^{\frac{\alpha-\beta}{\alpha}} \tag{5.66}$$

So, the sliding variables satisfy finite time convergence.

IV. Simulation results

The proposed disturbance observer based optimal second order sliding mode controller (DOB-OSOSMC) is applied for stabilization of the following system [11]:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + d(t) \\ \dot{x}_2(t) &= -2x_1(t) - x_2(t) + e^{x_1(t)} + u(t) \end{aligned} \tag{5.67}$$

where initial state $x(0) = [1 \quad -1]^T$ and disturbance $d(t) = 0.5$ is applied after 6 sec. The system (5.67) can be expressed as (5.37) where $f(x) = [x_2(t) \quad -2x_1(t) - x_2(t) + e^{x_1(t)}]^T$, $g(x) = [0 \quad 1]^T$, $g_1 = [1 \quad 0]^T$. The performance index J is defined as

$$J = \int_0^\infty \left[x(t)^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(t) + u(t)^T 1 u(t) \right] dt \tag{5.68}$$

After preliminary simulation it was found that $V(x)$ in 5.69 yields the best optimization and stabilization results. Hence this value of $V(x)$ is chosen.

$$V(x) = x(t)^T \begin{bmatrix} 13 & 5 \\ 5 & 5 \end{bmatrix} x(t). \tag{5.69}$$

Design parameters of the proposed controller are tabulated in Table 5.4.

Table 5.4: Design parameters

Observer and Controllers	Parameters
DOB	$\rho = [6 \;\; 0]$
ISMC	$G = [3.2 \;\; 1]$
TSMC	$\alpha = 5$, $\beta = 3$, $\delta = 2$, $\eta = 0.1$ and $\varepsilon = 0.1$

Simulation results obtained by applying the proposed disturbance observer based optimal second order sliding mode controller (DOB-OSOSMC) are compared with those obtained by using the disturbance observer based sliding mode controller (DOB-SMC) designed by Yang et al. [11] which is discussed in Appendix A.11. States $x_1(t)$ and $x_2(t)$ obtained by applying the proposed DOB-OSOSMC are shown in Figure5.7. In Figure 5.8, the states $x_1(t)$ and $x_2(t)$ obtained by using the DOB-SMC [11] are shown. The control inputs obtained by applying the proposed DOB-OSOSMC and the DOB-SMC [11] are compared in Fig. 5.9. Actual disturbance and its estimated value obtained by using the proposed DOB-OSOSMC are shown in Figure 5.10.

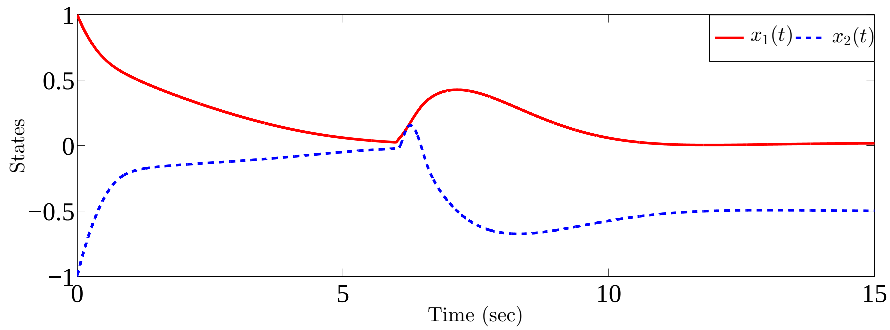


Figure 5.7: States $x_1(t)$ and $x_2(t)$ obtained by applying the proposed DOB-OSOSMC

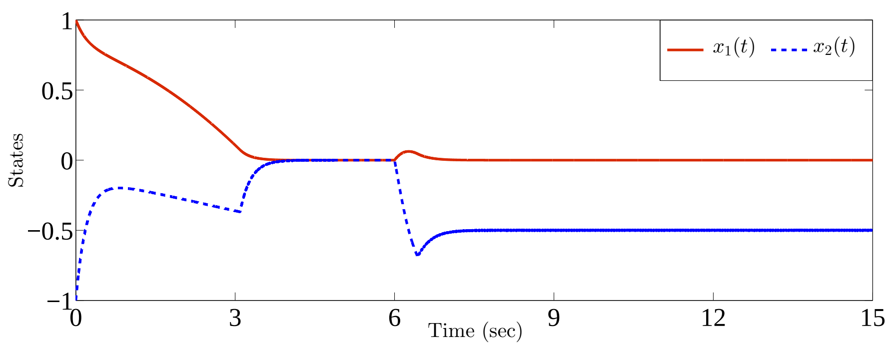


Figure 5.8: States $x_1(t)$ and $x_2(t)$ obtained by applying the DOB-SMC proposed by Yang et al. [11]

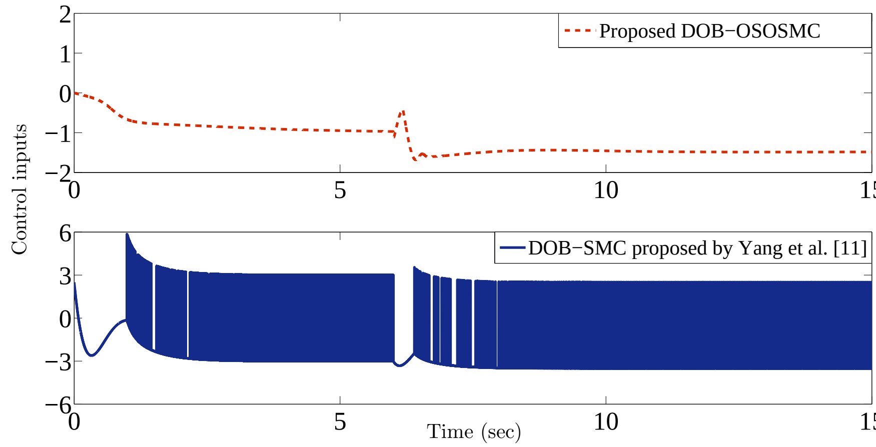


Figure 5.9: Control inputs obtained by applying the proposed DOB-OSOSMC and the DOB-SMC proposed by Yang et al. [11]

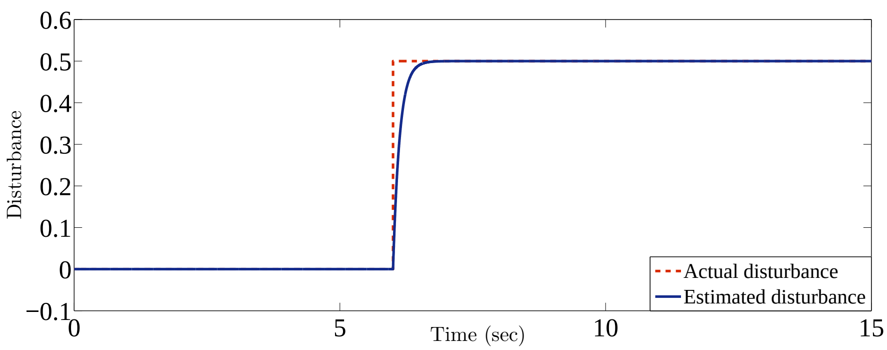


Figure 5.10: Actual and estimated disturbance obtained by applying the proposed DOB-OSOSMC

From above figures it is evident that the proposed DOB-OSOSMC achieves similar performance as that of the DOB-SMC designed by Yang et al. [11] but at the cost of a much reduced control input. Moreover, the control input in the case of the DOB-SMC [11] contains excessive chattering whereas the proposed DOB-OSOSMC offers a smooth chattering free control input.

To measure the smoothness of the control input, its total variation (TV) [106] is computed. For getting knowledge about the energy spent by the control input, its 2-norm is calculated. Table 5.5 shows the TV and the 2-norm of the control input calculated for the period from 0 to 15 sec with a sampling time of 0.01 sec. It is clear from Table 5.5 that the proposed DOB-OSOSMC is able to produce a smoother control input with substantial reduction in the control effort than the DOB-SMC proposed by Yang et al. [11]

Table 5.5: Comparison of control indices

Method	Total Variation (TV)	Control Energy
Proposed DOB-OSOSMC	2.85	15.32
DOB-SMC proposed by Yang et al. [11]	1533.60	112.09

5.4 Summary

In this chapter an optimal second order sliding mode controller is proposed for two types of nonlinear uncertain systems. In the first case, the nonlinear system is affected by the matched uncertainty and in the second case, the nonlinear system is affected by the mismatched uncertainty. For both these cases, the optimal controller is designed by using the control Lyapunov function (CLF). An integral sliding mode controller is combined with the optimal controller to tackle the matched uncertainty. In the case of mismatched uncertainty, a disturbance observer is used for uncertainty estimation based on which the integral sliding surface is designed. To avoid the high frequency chattering inherent in conventional first order sliding mode controllers, a second order sliding mode methodology is proposed here and the same is realized by using an integral sliding variable based non-singular terminal sliding mode. The terminal sliding mode converges the sliding variables in finite time. Simulation results confirm that the proposed controller requires significantly lesser control effort than conventional sliding mode controllers in stabilizing an uncertain nonlinear system. In addition, the proposed controller reduces chattering to a large extent.

6

Conclusions and Scope for future work

Contents

6.1 Conclusions

This thesis is an attempt to design a robust optimal control methodology to minimize the control effort required for controlling an uncertain system. Specifically, this thesis aims to develop chattering free optimal sliding mode controllers (OSMCs) for both linear and nonlinear systems which are affected by matched and mismatched types of uncertainty. The common methodology adopted in the research work is to use classical optimal control technique and utilize sliding mode control method to impart robustness to the optimal controller for its unabated performance in the face of disturbances. The thesis work yielded robust optimal controllers which are discussed briefly as follows.

An optimal adaptive sliding mode controller (OASMC) is designed to control the linear system affected by matched uncertainty with unknown upper bound. The optimal controller is designed for the nominal linear system based on the LQR technique and is combined with the SMC by designing an integral sliding surface. As the upper bound of the uncertainty is unknown, an adaptive method is used to design the switching control in the SMC. The proposed OASMC shows satisfactory performance in stabilizing and tracking problems. Compared to conventional SMCs, the proposed optimal sliding mode controller spends a lower control energy but maintaining similar performance standard. For controlling linear systems affected by the mismatched uncertainty using minimum control input, a disturbance observer based optimal sliding mode controller is proposed. The optimal controller is designed for the nominal linear system based on the linear quadratic regulator (LQR) method and an integral sliding surface is combined with the optimal control for making it immune to uncertainties. The sliding surface is designed using estimated value of the mismatched uncertainty. A disturbance observer is used for the disturbance estimation.

The main disadvantage of the OSMC is the presence of undesired high frequency chattering in the control input which is detrimental for the controller. In order to overcome this inherent difficulty of the OSMC, an optimal second order sliding mode controller (OSOSMC) is proposed. The optimal controller is designed for the nominal linear system using the LQR technique and integrated with a SOSMC. The second order sliding mode methodology is realized by designing a non-singular terminal sliding surface based on an integral sliding variable. The proposed controller is applied for both stabilization and tracking of linear uncertain SISO systems affected by matched uncertainty. The proposed OSOSMC is also applied for stabilization of linear uncertain decoupled MIMO systems affected by the matched uncertainty. The proposed optimal second order sliding mode controller uses a substantially lower control effort than some existing sliding mode controllers while offering the same performance level.

For controlling nonlinear uncertain systems affected by matched uncertainty, an optimal second order sliding mode controller (OSOSMC) is proposed using the state dependent Riccati equation (SDRE). To design the SDRE based optimal controller, the nonlinear system needs to be represented as a linear like structure. Using extended linearization, the nonlinear system is represented as a linear like structure having state dependent coefficient (SDC) matrices. After designing the optimal controller, it is integrated with the second order sliding mode controller (SOSMC) which is implemented by designing an integral sliding variable based non-singular terminal sliding mode controller. The proposed controller is applied for stabilization and tracking problems and it was found that the control energy used in the proposed SDRE based optimal second order sliding mode controller is significantly reduced compared to some existing sliding mode controllers but maintaining comparable performance level. The proposed SDRE based OSOSMC is also successfully applied for stabilization of the chaotic system which is a special case of highly unstable nonlinear systems. The proposed control strategy can successfully stabilize those chaotic systems which can be represented as linear like structures.

The proposed SDRE based OSOSMC is not applicable for those nonlinear systems which cannot be represented as linear like structures. For such nonlinear uncertain systems, a control Lyapunov function (CLF) based optimal second order sliding mode controller (OSOSMC) is proposed. The CLF is chosen for the open loop system and then a feedback controller is designed to optimize the desired performance index. After designing the optimal controller for the nominal nonlinear system, it is integrated with the SOSMC designed by using integral sliding variable based terminal sliding mode. The proposed CLF based OSOSMC is applied for stabilization and tracking and in accordance with earlier results, it was observed that the control energy used in the proposed CLF based OSOSMC is significantly lower in comparison to some existing sliding mode controllers but without compromising on the performance standard. The proposed CLF based OSOSMC cannot tackle mismatched uncertainty. In order to handle mismatched uncertainty in nonlinear systems, a disturbance observer based optimal second order sliding mode controller (DOB-OSOSMC) is proposed. The optimal controller is designed for the nominal system based on the CLF and a disturbance observer is used to estimate the mismatched uncertainty. Based on the estimated value of the uncertainty, an integral sliding variable is designed to combine the optimal controller with the SMC. To mitigate chattering in the control input, the SMC is made second order by using a non-singular terminal sliding surface based on an integral sliding variable. The proposed DOB-OSOSMC is applied for stabilization of nonlinear systems affected by mismatched uncertainty and its performance is found to be superior than existing disturbance observer based SMCs as regards control input usage and smoothness.

6.2 Scope for future work

There are several ways in which the work in this thesis can be extended and further investigated. Some of them are listed as follows.

- The thesis is aimed at minimization of the control energy. In future, time optimization can also be investigated.
- In this work infinite horizon optimal control problem is considered. In future, optimal second order sliding mode controller may be designed by considering finite horizon optimal control problem.
- The work presented in this thesis can be extended to design in discrete domain.
- The performance of the controller designed for nonlinear systems affected by the mismatched uncertainty leaves scope for improvement.
- The proposed controller does not guarantee finite time convergence of the system states though the sliding variables reach zero in finite time. In future attempt can be made to achieve finite time stability of the system.

A

Appendix

Contents

A.1 Higher order sliding mode control based on integral sliding mode proposed by Laghrouche et al.

In [1] Laghrouche et al. proposed a higher order sliding mode control methodology based on integral sliding mode. Here a nonlinear system is considered as

$$\begin{aligned} \dot{x} &= f(x,t) + g(x,t)u \\ y &= s(x,t) \end{aligned} \tag{A.1}$$

where x is the state variable, u is the control input and $s(x,t)$ is a measured smooth output function. $f(x,t)$ and $g(x,t)$ are uncertain smooth functions. By defining a discontinuous control function, the r-th order SMC approach allows finite time stabilization of the sliding variable s and its $r-1$ first time derivatives to zero. The r-th order SMC of (A.1) with respect to the sliding variable s is equivalent to the finite time stabilization of

$$\begin{aligned} \dot{z}_i &= z_{i+1} \\ \dot{z}_r &= \phi(.) + \gamma(.)u \end{aligned} \tag{A.2}$$

with $1 \leq i \leq r-1$ and $z = [z_1 z_2 \cdots z_r]^T = [s \ \ddot{s} \ s^{(r-1)}]^T$ and functions $\phi()$ and $\gamma()$ are bounded uncertain functions. System (A.2) is trivially rewritten as

$$\begin{aligned} \dot{z}_i &= z_{i+1} \\ \dot{z}_r &= \phi(.) + (\gamma(.) - 1)u + u \\ \dot{z}_r &= \beta(.) + u \end{aligned} \tag{A.3}$$

where $\beta(.) = \phi(.) + (\gamma(.) - 1)u$. The control input u is divided into two parts as $u = u_0 + u_1$, with u_0 being the ideal control and u_1 being the integral sliding mode control. The ideal control u_0 is found optimally to he following performance criterion:

$$J = \frac{1}{2}\int_0^{t_F} \left[z^T Q z + u_1^2\right] dt \tag{A.4}$$

where time t_F is finite and Q is a symmetric positive definite matrix. The nominal system is defined as

$$\dot{z} = Az + Bu_0 \tag{A.5}$$

where A and B are given by $A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \ddots & \cdots & 1 \\ 0 & \ddots & \ddots & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$.

The Control law u_0 is defined as

$$u_0 = \begin{cases} -B^T Pz + B^T \chi, & for \;\; 0 \leq t \leq t_F \\ -B^T Pz, & for \;\; t \geq t_F. \end{cases} \tag{A.6}$$

where $\dot{\chi} = -(A^T - PBB^T)\chi$ and $0 = PA + A^T P - PBB^T P + Q$. To control system (A.3), the sliding variable σ is chosen as

$$\sigma = z_r - \int \dot{u}_0 \tag{A.7}$$

The discontinuous control law u_1 is designed as

$$u_1 = \eta sign(\sigma) \tag{A.8}$$

where $\eta > 0$. In the example of the mass-spring-damper system (2.25) in Chapter 2, simulation was conducted for Laghrouche et al.'s method [1] by choosing weighing matrix Q as an identity matrix and $\eta = 0.6$.

A.2 Dynamic compensator-based second-order sliding mode controller design for mechanical systems proposed by Chang

In [2] Chang proposed a compensator based second order sliding mode controller. Here a dynamic system is defined as

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) + B_2(u(t) + d(t)) \end{aligned} \tag{A.9}$$

where the vector $u(t)$ represents the control forces and the vector $d(t)$ denotes unknown disturbances with the upper bound $||d(t)|| \leq a$. Here the system matrix is defined as $A = \begin{bmatrix} 0 & I \\ A_{21} & A_{22} \end{bmatrix}$

and control distribution matrix is defined as $B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$.

To design the sliding surface, a matrix G is chosen such that the matrix $[A - B(GB)^{-1}GA]$ is stable. Matrix G is decomposed as $G = [G_1 \;\; G_2]$.

The sliding surface is proposed as

$$\dot{s}(t) = G_2 x_1(t) + z(t) = 0 \tag{A.10}$$

where $z(t)$ is defined as

$$\dot{z}(t) = -L_1 z(t) - L_1 G_2 x_1(t) + G_1 x_1(t)$$

Here L_1 is the positive definite diagonal matrix. The control law $u(t)$ is proposed as

$$u(t) = -(G_2 B_2)^{-1}(G_2 A_{21} x_1(t) - v(t) + L_2 s(t) + K sign(s(t))) \tag{A.11}$$

Further, L_2 is a diagonal matrix and $v(t)$ is found from

$$\dot{v}(t) = -\beta(G_2 A_{22} + G_1) x_2(t) - \beta v(t) \quad \text{gain}$$

where $\beta > 0$. In the example of the inverted pendulum (3.39) in Chapter 3, simulation was performed for Chang's method [2] with the following parameters:

$G = [-2.0203 \quad -8.6720 \quad -2.2320 \quad -1.6992]$, $L_1 = 3$, $L_2 = -1$, $K = -4$, $\beta = 30$.

A.3 A novel higher order sliding mode control scheme proposed by Defoort et al.

In [3] Defoort et al. proposed a higher order sliding mode controller for uncertain systems. In [3] m single input single output independent integrator chains were defined as follows:

$$\begin{aligned} \dot{z}_{1,i} &= z_{2,i} \\ &\vdots \\ \dot{z}_{r_i-1,i} &= z_{r_i,1} \\ \dot{z}_{r_i,i} &= \omega_{nom,i} \end{aligned} \tag{A.12}$$

$\forall\ i \in \{1,\cdots,m\}$ and $\omega_{nom,i}(z_{1,i}) = -k_{1,i}sgn(z_{1,i})|z_{1,i}|^{\alpha_{1,i}} - \cdots - k_{r_i,i}sgn(z_{r_1,i})|z_{r_i,i}|^{\alpha_{r_i,i}}$ with $\alpha_{1,i},\cdots,\alpha_{r_i,i}$ satisfying $\alpha_{j-1,i} = \frac{\alpha_{j,i}\alpha_{j+1,i}}{2\alpha_{j+1,i}-\alpha_{j,i}}, j = 2,\cdots,r_i$ and $\alpha_{r_i+1,i} = 1,\ \alpha_{r_i,i} = \alpha_i$.

The sliding surface was chosen as

$$\sigma = [z_{r_1,1}, z_{r_2,2}, \cdots, z_{r_m,m}]^T + z_{aux} = 0 \tag{A.13}$$

where $\dot{z}_{aux} = -\omega_{nom}(z)$

Then a control law was chosen as

$$u = \omega_{nom}(z) + \omega_{disc}(z, z_{aux}) \tag{A.14}$$

where $\omega_{disc}(z, z_{aux})$ is the discontinuous control which was defined as

$$\omega_{disc}(z, z_{aux}) = -G(z)sign(\sigma) \tag{A.15}$$

where gain $G(z)$ satisfies $G(z) \geq \frac{(1-\nu)||\omega_{nom}(z)||+\rho+\eta}{\nu}$ with $1 \geq \nu > 0,\quad \eta > 0$ and ρ is the upper bound of uncertainty.

For the simulation of triple integrator system (3.43) in chapter 3, the parameters are chosen as $r_i = 3,\ k_{1,i} = 1,\ k_{2,i} = 1.5,\ k_{3,i} = 1.5,\ \alpha_i = \frac{3}{4},\ G = 1.5.$

A.4 Adaptive integral higher order sliding mode controller for uncertain systems proposed by Mondal et al.

In [4] Mondal and Mahanta proposed an adaptive integral higher order sliding mode controller for uncertain systems. The uncertain system is defined as

$$\begin{aligned} \dot{z}_i &= z_{i+1} \\ \dot{z}_n &= a(z) + b(z)u + \Delta F_n(z,t) \end{aligned} \tag{A.16}$$

where $1 \leq i \leq n-1$ and an integral sliding surface is chosen as

$$\sigma = z_n - z_n(0) - \int \omega_{nom}(z)dt = 0 \tag{A.17}$$

where $\omega_{nom}(z) = -k_1 sgnz_1|z_1|^{\alpha_1} - k_2 sgnz_2|z_2|^{\alpha_2} - \cdots - k_n sgnz_n|z_n|^{\alpha_n}$ with $\alpha_1,\cdots,\alpha_n$ satisfying $\alpha_{i-1} = \frac{\alpha_i\alpha_{i+1}}{2\alpha_{i+1}-\alpha_i}, i = 2,\cdots,n$ and $\alpha_{n+1} = 1$.

A conventional sliding surface is chosen as

$$\sigma_1 = \dot{\sigma} + k\sigma = 0 \tag{A.18}$$

Then the control law is obtained as

$$\dot{u} = -b(z)^{-1}\{\dot{a}(z) + \dot{b}(z)u - \omega_{nom} + k(\dot{z}_n - \omega_{nom}) + \rho_1\sigma_1 + \widehat{T}sgn\sigma_1\} \tag{A.19}$$

where $\rho_1 \geq 0$ and $\widehat{T}$ is the estimated value of the upper bound of uncertainty.

The adaptive tuning law is given by

$$\dot{\widehat{T}} = \nu||\sigma_1|| \tag{A.20}$$

where ν is a positive constant.

For simulation of the triple integrator system (3.43) in Chapter 3, the parameters are chosen as $k = 2, \rho_1 = 3, \nu = 0.8, \widehat{T}(0) = 0.5$.

A.5 Robust output tracking control of an uncertain linear system via a modified optimal linear-quadratic method proposed by Shieh et al.

In [5] Shieh et al. proposed a robust output tracking problem for a class of uncertain linear systems. The linear system is defined as

$$\dot{x}(t) = [A + \Delta A]x(t) + [B + \Delta B]u(t) \tag{A.21}$$

where $x(t)$ is the system state and $u(t)$ is the control law. The system matrix and control distribution matrix are denoted as A and B respectively and $\Delta A,\ \Delta B$ are system uncertainties.

For tracking control, an augmented system is proposed as

$$\dot{x}(t) = [A + \Delta A]x(t) + [B + \Delta B]u(t) \tag{A.22}$$

$$\dot{q}(t) = Cx(t) - y_r \tag{A.23}$$

where $q(t)$ is an auxiliary state and y_r is the desired trajectory.

Then the augmented state equation is found as [5]

$$\dot{z}(t) = [A_z + \Delta A_z]z(t) + [B_z + \Delta B_z]u(t) + \xi \tag{A.24}$$

where $z(t) = [x(t) \;\; q(t)]^T$, $A_z = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}$, $\Delta A_z = \begin{pmatrix} \Delta A & 0 \\ 0 & 0 \end{pmatrix}$, $B_z = [B \;\; 0]^T$, $\Delta B_z = [\Delta B \;\; 0]^T$, $\xi = [0 \quad -y_r]^T$.

The performance index is defined by

$$J = \int_0^\infty \exp(2\epsilon)[\zeta z(t)^T z(t) + \rho u(t)^T u(t)]dt \tag{A.25}$$

where $\zeta > 0$, $\rho > 0$ and parameter ϵ is also positive denoting the prescribed degree of stability.

The control input $u(t)$ is obtained as

$$u(t) = -(1+\alpha)\rho^{-1}B_z^T P z(t) = -(1+\alpha)kz(t) \tag{A.26}$$

where P is the solution of the following Riccati equation and designed parameter $\alpha \geq 0$.

$$(A_z + \epsilon I_z)^T P + P(A_z + \epsilon I_z) - \rho^{-1} P B_z B_z^T P + \zeta = 0 I_z \tag{A.27}$$

with I_z the $(n+r)$-dimensional identity matrix.

For simulation of the maglev vehicle model (3.59), design parameters are chosen as $\epsilon = 2$, $\rho = 1$, $\zeta = 4000000$, $\alpha = 1$.

A.6 Dynamic sliding mode controller design for chattering reduction proposed by Chang

In [6] Chang proposed a dynamic sliding mode control methodology which could successfully eliminate chattering in the control input. The uncertain system is defined as

$$\dot{x}(t) = Ax(t) + B(u(t) + d(t,x)) \tag{A.28}$$

where $x(t)$ is the state vector, $u(t)$ is the control input vector and $d(t)$ is the unknown matched disturbance vector with the known upper bounds.

The sliding surface is chosen as

$$\sigma(t) = Gx(t) = 0 \tag{A.29}$$

The chattering free sliding mode control law is proposed as

$$\dot{u}(t) = -(GB)^{-1}[(GAB + L_1GB)u(t) - (GA^2 + L_1GA)x(t) - L_2s(t) - ksign(\sigma(t))] \tag{A.30}$$

where L_1, L_2 are positive definite diagonal matrices.

For the simulation of a batch reactor (3.87), the design parameters are chosen as

$$G = \begin{pmatrix} -0.0106 & -0.5565 & -0.02 & 0.0934 \\ -0.6031 & -0.3576 & 1.1415 & -1.2726 \end{pmatrix}, \ L_1 = 15, \ L_2 = 100, k = 20.$$

A.7 Terminal sliding mode tracking control for a class of SISO uncertain nonlinear systems proposed by Chen et al.

Chen et al. proposed [7] a terminal sliding mode tracking control for the single input and single output (SISO) uncertain nonlinear system. The nonlinear system considered is given by

$$\begin{aligned} \dot{x}_i &= x_{i+1}, \quad i = 1, 2, \ldots, n-1 \\ x_n &= f(x) + g(x)u + d \\ y &= x_1 \end{aligned} \tag{A.31}$$

where $x = [x_1, x_2, \ldots, x_n]^T$ is the system's measurable state vector, $f(x)$ and $g(x)$ are known nonlinear functions, u is the system control input, y is the system output and d is the external disturbance. To design a sliding mode disturbance observer with finite time convergence, the following auxiliary variable is introduced,

$$s = z - x_n \tag{A.32}$$

where z is defined as

$$\dot{z} = -ks - \beta sign(s) - \varepsilon s^{p_0/q_0} - |f(x)|sign(s) + g(x)u \tag{A.33}$$

where p_0 and q_0 are odd positive integers with $p_0 < q_0$. Design parameters k, β and ε are positive and $\beta > |d|$.

The terminal sliding mode disturbance estimate $\widehat{d}$ is given by

$$\widehat{d} = -ks - \beta sign(s) - \varepsilon s^{p_0/q_0} - |f(x)|sign(s) - f(x) \tag{A.34}$$

Terminal sliding mode is proposed as

$$\sigma_1 = y - y_d = 0 \tag{A.35}$$

where y_d is the desired trajectory. A recursive procedure for terminal sliding mode control of uncertain nonlinear systems is proposed as

$$\begin{aligned} s_2 &= \dot{s}_1 + \alpha_1 s_1 + \beta_1 s_1^{p_1/q_1} \\ s_3 &= \dot{s}_2 + \alpha_2 s_2 + \beta_2 s_2^{p_2/q_2} \\ &\vdots \\ s_n &= \dot{s}_{n-1} + \alpha_{n-1} s_{n-1} + \beta_{n-1} s_{n-1}^{p_{n-1}/q_{n-1}} + s \end{aligned} \tag{A.36}$$

$$\tag{A.37}$$

where $\alpha_i > 0$ and $\beta_i > 0$. Further, p_i and q_i for $i = 1, 2, ..., n+1$ are positive odd integers with $p_i < q_i$.

The control input u is obtained as

$$u = -\frac{1}{g(x)}(f(x) - y_d^{(n)} + \sum_{j=1}^{n-1} \alpha_j s_j^{n-j} + \sum_{j=1}^{n-1} \beta_j \frac{d^(n-j)}{dt^{(n-j)}} s_j^{p_j/q_j} + \widehat{d} + \delta s_n + \mu s_n^{p_n/q_n}) \tag{A.38}$$

where $y_d^{(n)}$ is the n-th time derivative of y.

To simulate the Van der Pol circuit (4.40) defined in Chapter 4, design parameters are chosen as $k = 2500,\ \ \beta = 4,\ \varepsilon = 0.5,\ p_0 = 5,\ q_0 = 9,\ \alpha_1 = 50,\ \beta_1 = 0.5,\ p_1 = p_2 = 5,\ q_1 = q_2 = 7,\ \delta = 60,\ \mu = 0.8$.

A.8 Nonsingular terminal sliding mode control of nonlinear systems proposed by Feng et al.

A global nonsingular terminal sliding mode control strategy for nonlinear systems was developed by Feng et al. [8]. Here a third order nonlinear system is considered as given below,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= f(x) + d(x) + b(x)u \end{aligned} \tag{A.39}$$

where $x = [x_1,\ x_2,\ x_3]^T$ represents the system state vector, u is the control input, $f(x) \neq 0$ and $b(x) \neq 0$ are two smooth nonlinear functions of x, $d(x)$ represents the uncertainties.

The terminal sliding mode manifold is defined as

$$\sigma = x_3 + c_1 x_1^{\alpha_1} + c_2 x_2^{\alpha_2} = 0 \tag{A.40}$$

where α_1 and α_2 are found as

$$\begin{aligned} \alpha_1 &= q/(2p - q) \\ \alpha_2 &= q/p \end{aligned}$$

Here design parameters p and q are positive odd integers satisfying $1 < p/q < 2$.

The control law u is proposed as

$$u = b^{-1}(x)(-f(x) + sat(u_f, u_s) - ksgn(\sigma)) \tag{A.41}$$

where $u_f = -c_1\alpha_1 x_1^{\alpha_1 - 1} x_2 - c_2\alpha_2 x_2^{\alpha_2 - 1} x_3$ and $u_s > 0$.

To simulate the third order nonlinear uncertain system (4.42) in Chapter 4, the design parameters are chosen as $c_1 = c_2 = 1,\ k = 10.1,\ \alpha_1 = 3/5,\ \alpha_2 = 3/7,\ u_s = 2$.

A.9 Adaptive sliding mode control in a novel class of chaotic systems proposed by Roopaei et al.

Roopaei et al. [9] proposed a robust adaptive sliding mode control strategy for an uncertain chaotic system. A time varying sliding surface is designed based on adaptive gain tuning.

The chaotic Lorenz system [9] is described as

$$\begin{aligned} \dot{x} &= f(x, y, z) - \alpha x + \Delta f(x, y, z) + u(t) \\ \dot{y} &= xg(x, y, z) + z\psi(x, y, z) - \beta y \\ \dot{z} &= xh(x, y, z) - y\psi(x, y, z) - \gamma z \end{aligned} \tag{A.42}$$

where x, y and z are state variables and α, β, γ are non-negative known constants. All the four functions

$f(.), g(.), h(.), \psi(.)$ are considered as smooth. The sliding surface is designed as

$$\sigma(t) = x(t) + \varphi(t) = 0 \tag{A.43}$$

where $\varphi(t)$ is an adaptive function given by

$$\dot{\varphi}(t) = yg(x, y, z) + zh(x, y, z) + \alpha x + \rho x \tag{A.44}$$

The control input $u(t)$ is found as

$$u(t) = -yg(x, y, z) - zh(x, y, z) - \rho x - f(x, y, z) + k_a sgn(\sigma) \tag{A.45}$$

where k_a is the reaching gain obtained by using the following adaptive law,

$$\dot{k}_a = -\gamma|\sigma(t)|. \tag{A.46}$$

To simulate the chaotic systems defined in (4.58), (4.61) and (4.63) the parameters are chosen as $\rho = 7,\ \gamma| = 0.1,\ k_a(0) = 10$

A.10 Sliding mode control with self-tuning law for uncertain non-linear systems proposed by Kuo et al.

In [10] Kuo et al. proposed an adaptive sliding mode controller for tracking of a second order nonlinear uncertain system. Desired states was chosen as $[0 \quad 0]^T$. Hence, it becomes a stabilization problem.

Here a second order nonlinear uncertain system [10] is considered as

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -(x_1^2(t) + 1.5x_2(t)) + u - (0.3sin(t)x_1^2(t) + 0.2cos(t)x_2(t)) + d(t) \end{aligned} \tag{A.47}$$

where $d(t)$ is a random noise of 0 mean and 0.5 variance. The sliding surface is designed as

$$\sigma(t) = c[x_1(t) \quad x_2(t)] = 0 \tag{A.48}$$

where c is a design parameter which is chosen as $[1 \;\; 1]$. The control input $u(t$ is proposed as

$$u(t) = (cg(x))^{-1}(cf(x) + cg(x) + \widehat{\beta}\phi(\widehat{\lambda}, \sigma)) \tag{A.49}$$

where $\phi(\widehat{\lambda}, \sigma)$ is a bipolar sigmoid function. Two tuning parameters $\widehat{\beta}$ and $\widehat{\lambda}$ are introduced to approximate the terminal gain and the terminal boundary layer.

To simulate the nonlinear uncertain system (5.27) in Chapter 5, design parameters are chosen as $c = [1 \;\; 1]$, $\widehat{\beta}(0) = 2$, $\widehat{\lambda}(0) = 2$.

A.11 Sliding mode control for systems with mismatched uncertainties via a disturbance observer proposed by Yang et al.

The proposed disturbance observer based optimal second order sliding mode controller (DOB-OSOSMC) is compared with the disturbance observer based sliding mode controller (DOB-SMC) proposed by Yang et al. [11].

The numerical system considered is defined as [11]:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + d(t) \\ \dot{x}_2(t) &= -2x_1(t) - x_2(t) + e^{x_1(t)} + u(t) \end{aligned} \tag{A.50}$$

where initial state $x(0) = [1 \quad -1]^T$ and disturbance $d(t) = 0.5$ is applied after 6 sec. The sliding surface is designed as

$$\sigma(t) = x_2(t) + cx_1(t) = 0 \tag{A.51}$$

where c is the design parameter. The control input $u(t)$ is proposed as

$$u(t) = -(-2x_1(t) - x_2(t) + e^{x_1(t)} + c(x_2(t) + \widehat{d}(t)) + ksign(\sigma(t))) \tag{A.52}$$

Here $\widehat{d}(t)$ is the estimated value of the mismatched disturbance and k is the switching gain. To simulate the nonlinear uncertain system (5.67) in Chapter 5 design parameters are chosen as $c = 5, \quad k = 3$.

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