DESIGN OF ROBUST FUZZY CONTROLLERS FOR UNCERTAIN NONLINEAR SYSTEMS

A

Thesis submitted

for the award of the degree of

DOCTOR OF PHILOSOPHY

By

SENTHILKUMAR D

DEPARTMENT OF ELECTRONICS AND COMMUNICATION ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

GUWAHATI - 781039, ASSAM, INDIA

JULY 2010

DESIGN OF ROBUST FUZZY CONTROLLERS FOR UNCERTAIN NONLINEAR SYSTEMS

Senthilkumar D

Certificate

This is to certify that the thesis entitled "**DESIGN OF ROBUST FUZZY CONTROLLERS FOR UNCERTAIN NONLINEAR SYSTEMS**", submitted by **Senthilkumar D** (05610202), a research scholar in the *Department of Electronics and Communication Engineering, Indian Institute of Technology Guwahati*, for the award of the degree of **Doctor of Philosophy**, is a record of an original research work carried out by him under my supervision and guidance. The thesis has fulfilled all requirements as per the regulations of the institute and in my opinion has reached the standard needed for submission. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

Dated:

Guwahati.

Dr. Chitralekha Mahanta

Associate Professor

Dept. of Electronics and Communication Engg.

Indian Institute of Technology Guwahati

Guwahati - 781039, Assam, India.

Acknowledgements

First and foremost, I feel it as a great privilege in expressing my deepest and most sincere gratitude to my supervisor Dr. Chitralekha Mahanta, for her excellent guidance. Her kindness, dedication, friendly accessibility and attention to detail have been a great inspiration to me. My heartfelt thanks to my supervisor for the unlimited support and patience shown to me. I would particularly like to thank for all her help in patiently and carefully correcting all my manuscripts.

I am also very thankful to my doctoral committee members Prof. P. K. Bora, Prof. S. Majhi and Dr. H. B. Nemade for sparing their precious time to evaluate the progress of my work. Their suggestions have been valuable. I would also like to thank other faculty members for their kind help carried out during my academic studies. I am grateful to all the members of the research and technical staff of the department without whose help I could not have completed this thesis. My special thanks to Mr. Sanjib Das, Mr. Sidananda and Mrs. Jharna for maintaining an excellent computing facility and providing various resources useful for the research work.

Thanks go out to all my friends at Control and Instrumentation Laboratory. They have always been around to provide useful suggestions, companionship and created a peaceful research environment.

My friends at IITG made my life joyful and were constant source of encouragement. Among my friends, I would like to extend my special thanks to Ali, Govind, Karthikeyan, Krishnamoorthy, Mrinal, Padam Priyal, Sanjoy, Shyam and Utkal. My work in this remote place definitely would not be possible without their love and care that helped me to enjoy my new life in this IITG. Special thanks also go to Mr. Jayanna, Mr. Narasimhamurthy, Mrs. Nirmala, Dr. Padhy and Mr. Pradhan for their help during my stay.

I thank all my fellow research students for their cooperation. During these four years at IITG I have had several friends who have helped me in several ways, I would like to say a big thank you to all of them for their friendship and support.

My deepest gratitude goes to my family for their continuous love and support throughout my studies. The opportunities that they have given me and their unlimited sacrifices are the reasons where I am and what I have accomplished so far.

Finally, I believe this research experience will greatly benefit my career in the future.

Senthilkumar D

Abstract

This thesis deals with the analysis and design of robust fuzzy controllers for uncertain nonlinear systems using Takagi-Sugeno (T-S) model based approach. A T-S fuzzy model is used here to approximate the uncertain nonlinear systems where the nominal model and uncertain terms of the consequent parts of the fuzzy model are identified by a linear programming approach and then they are expressed in a form suitable for robust fuzzy controller design. With the derived T-S fuzzy model, various types of robust fuzzy controllers are designed that guarantee not only stability but also satisfy the specified performance criteria of the closed-loop control system. The first type of T-S controller is a robust fuzzy guaranteed cost controller for trajectory tracking in uncertain nonlinear systems. The fixed Lyapunov function based approach is used to develop the robust controller and the design conditions are derived as a problem of solving a set of linear matrix inequalities (LMIs). Next, this research work focuses on robust stabilization, robust H_∞ stabilization and robust H_∞ tracking control of uncertain nonlinear systems by using a richer class of Lyapunov function called parametric Lyapunov function. This parametric Lyapunov function based approach attempts to reduce the conservatism associated in the controller design for nonlinear systems with slowly varying uncertainties. The design conditions are derived as matrix inequality involving parametric uncertainties and then they are reduced to finite dimensional matrix inequalities by using the multiconvexity concept. These matrix inequalities are then solved by an iterative LMI based algorithm. Finally, the results of standard state-space T-S fuzzy system with parametric Lyapunov function based approach are extended to synthesize the robust controller for application to uncertain fuzzy descriptor systems.

Contents

List of Figures

List of Tables

List of Acronyms

AFSMC	Adaptive fuzzy sliding mode controller
BMI	Bilinear matrix inequality
CFS	continuous fuzzy system
FLC	Fuzzy logic controller
ILMI	Iterative linear matrix inequality
LMI	Linear matrix inequality
PDC	Parallel distributed compensation
PID	Proportional integral derivative
PMI	Polytnomial matrix inequality
SOS	Sum of squares
T-S	Takagi-Sugeno
TORA	Translational oscillator with an eccentric rotational proof mass actuator

List of Symbols

$\boldsymbol{A}^{-1}$	Inverse of square matrix $\boldsymbol{A}$
$\boldsymbol{A}^{T}$	Transpose of matrix $\boldsymbol{A}$
$\boldsymbol{A}^{-T}$	Transpose of the inverse of square matrix $\boldsymbol{A}$
$\boldsymbol{A} > \mathbf{0}$	Matrix $\boldsymbol{A}$ is a positive definite matrix
$\boldsymbol{A} < \mathbf{0}$	Matrix $\boldsymbol{A}$ is a negative definite matrix
$\boldsymbol{A} \geq \mathbf{0}$	Matrix $\boldsymbol{A}$ is a positive semi-definite matrix
$\boldsymbol{A} \leq \mathbf{0}$	Matrix $\boldsymbol{A}$ is a negative semi-definite matrix
$\boldsymbol{A} > \boldsymbol{B}$	Matrix $(\boldsymbol{A} - \boldsymbol{B})$ is a positive definite matrix
$\|a\|$	Absolute value of a
$\underline{\overline{a}}$	a is a interval valued variable
$\overline{a}$	Upper bound of interval valued variable $\underline{\overline{a}}$
$\underline{a}$	Lower bound of interval valued variable $\underline{\overline{a}}$
$\underline{\overline{g}}(\cdot)$	Function $g(\cdot)$ is a interval valued function
$\overline{g}(\cdot)$	Upper bound of interval valued function $\underline{\overline{g}}(\cdot)$
$\underline{g}(\cdot)$	Lower bound of interval valued function $\underline{\overline{g}}(\cdot)$
$\boldsymbol{I}$	Unit matrix
$\inf(\cdot)$	Infimum of $(\cdot)$
$\boldsymbol{K}$	Feedback gain matrix
$\min(\cdot)$	Minimum value of $(\cdot)$
$\max(\cdot)$	Maximum value of $(\cdot)$
μ	Fuzzy membership value
$\mathbb{R}^{n}$	Real vector space of dimension n
$\mathbb{R}^{n\times m}$	Real vector space of dimension $n \times m$
$\boldsymbol{r}(t)$	Reference input
$\sup(\cdot)$	Supremum of $(\cdot)$

t	Time
$\boldsymbol{u}(t)$	Input vector
$\boldsymbol{w}(t)$	External disturbance input vector
$\boldsymbol{x}(t)$	State vector of a system
$\boldsymbol{x}_r(t)$	Reference state vector
$\boldsymbol{y}(t)$	Output vector of a system
$\boldsymbol{Z}$	Zero matrix
$*$	Transposed elements (matrices) for symmetric positions

Chapter 1

Introduction

1.1 Research Background

Many real physical systems are nonlinear in nature. Controlling nonlinear systems is a difficult problem due to their complex nature. This problem becomes more acute when the system's parameters are uncertain, for example the plant, the sensor and the actuator in a control system. The most fundamental aspect of this connection is that the uncertainty involved in any problem solving situation is a result of some information deficiency, which may be incomplete, imprecise, fragmentary, not fully reliable, vague, contradictory, or deficient in some other way [3]. Uncertainty affects decision-making and appears in a number of different forms. It is an inherent part of real world systems and the controllers designed for such uncertain systems are required to act in an appropriate manner and eliminate the effect of imprecise information.

Traditional control is an accurate and cost efficient solution for control problems involving simple or linear systems. But if the system to be controlled is very complex and includes uncertainties, fuzzy logic may be the appropriate technique [4]. Fuzzy control systems have proven to be superior in performance when compared with conventional control systems especially in controlling nonlinear, ill-defined and complex system [5–12]. Earlier, fuzzy controllers were designed by mimicking the knowledge of human operator for generating the control action [13–16]. These controllers are designed using the extensive experience of the system designers and performance of such control schemes is generally satisfactory. However, such designs are often criticized for lack of generalized method for stability analysis of the closed loop fuzzy control systems [17]. Therefore, many researchers have worked to improve the performance of the fuzzy logic controllers (FLCs) by systematic design and at the same time to ensure stability. Malki et al. [18, 19] proposed a method for design of fuzzy PID controllers ensuring stability. Tao et al. [20, 21] proposed adaptive fuzzy sliding mode controllers (AFSMC) for linear systems with mismatched time-varying uncertainties. In recent years, a large number of researchers are involved in analysis and design of fuzzy controllers using the Takagi-Sugeno (T–S) model [22,23]. Fuzzy controllers developed using the T–S fuzzy model have been applied to a number of control problems; e.g. [2,8,24–32].

The T–S fuzzy model, which utilizes the local linear description, provides a basis for development of a systematic framework for stability analysis and controller design by taking the advantage of powerful modern control theory and techniques. Recently linear matrix inequality (LMI) based approach has received significant attention for designing T–S fuzzy model based controller [2,8,12, 26,29,33–35]. The basic stability conditions in terms of linear matrix inequalities (LMIs) are derived based on a Lyapunov approach. The fuzzy model based system is guaranteed to be asymptotically stable if there exists a solution to Lyapunov inequalities. These LMI stability conditions can be solved numerically and efficiently using convex programming techniques [36–38]. Therefore formulating the control problem as an LMI problem is equivalent to finding a "solution" to the original problem [8].

1.2 Preliminaries and Overview of Previous Work

1.2.1 T–S Fuzzy System

Fuzzy system theory enables us to utilize qualitative, linguistic information about a highly complex nonlinear system to construct a mathematical model for it. The T–S fuzzy model consists of a number of rule-based linear models and membership functions which determine the degrees of confidence of the rule. The T–S fuzzy model can be used to approximate the global behavior of a highly complex nonlinear system [39]. In the T–S fuzzy model, local dynamics in different state space regions are described by fuzzy *IF-THEN* rules, each of which represents the local linear subsystem in different state-space regions. The overall model of the system is obtained by fuzzily "blending" these linear models through nonlinear fuzzy membership functions. Unlike conventional modeling, which uses a single model to describe the global behavior of a system, fuzzy modeling is essentially a multimodel approach in which simple linear submodels are fuzzily combined to describe the global behavior of the system.

A continuous-time T–S fuzzy model is of the following form [40]:

Plant rule i:

IF $z_1(t)$ is N_{i1} and $\cdots$ $z_p(t)$ is N_{ip}, *THEN*

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= (\boldsymbol{A}_i + \Delta\boldsymbol{A}_i(t))\boldsymbol{x}(t) + (\boldsymbol{B}_i + \Delta\boldsymbol{B}_i(t))\boldsymbol{u}(t) + (\boldsymbol{B}_{wi} + \Delta\boldsymbol{B}_{wi}(t))\boldsymbol{w}(t)\\ \boldsymbol{y}(t) &= (\boldsymbol{C}_i + \Delta\boldsymbol{C}_i(t))\boldsymbol{x}(t) + (\boldsymbol{D}_i + \Delta\boldsymbol{D}_i(t))\boldsymbol{u}(t) + (\boldsymbol{D}_{wi} + \Delta\boldsymbol{D}_{wi}(t))\boldsymbol{w}(t), \quad i = 1, 2, ..., r\end{aligned} \tag{1.1}$$

where N_{ij} is the fuzzy set, $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state vector, $\boldsymbol{u}(t) \in \mathbb{R}^m$ is the control input, $\boldsymbol{w}(t) \in \mathbb{R}^{m_w}$ is the disturbance input and $\boldsymbol{y}(t) \in \mathbb{R}^q$ is the output vector. Here, $\boldsymbol{A}_i \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}_i \in \mathbb{R}^{n\times m}$, $\boldsymbol{B}_{wi} \in \mathbb{R}^{n\times m_w}$, $\boldsymbol{C}_i \in \mathbb{R}^{q\times n}$, $\boldsymbol{D}_i \in \mathbb{R}^{q\times m}$ and $\boldsymbol{D}_{wi} \in \mathbb{R}^{q\times m_w}$; $z_1(t), z_2(t), ..., z_p(t)$ are premise variables and r is the number of *IF-THEN* rules. $\Delta\boldsymbol{A}_i(t)$, $\Delta\boldsymbol{B}_i(t)$, $\Delta\boldsymbol{B}_{wi}(t)$, $\Delta\boldsymbol{C}_i(t)$, $\Delta\boldsymbol{D}_i(t)$ are $\Delta\boldsymbol{D}_{wi}(t)$ uncertain matrices of appropriate dimension that represent the uncertainties and modeling error.

By fuzzy blending, the overall fuzzy model is inferred as follows [8]:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \sum_{i=1}^{r} \mu_i(z(t))(\boldsymbol{A}_i + \Delta\boldsymbol{A}_i(t))\boldsymbol{x}(t) + (\boldsymbol{B}_i + \Delta\boldsymbol{B}_i(t))\boldsymbol{u}(t) + (\boldsymbol{B}_{wi} + \Delta\boldsymbol{B}_{wi}(t))\boldsymbol{w}(t))\\ \boldsymbol{y}(t) &= \sum_{i=1}^{r} \mu_i(z(t))((\boldsymbol{C}_i + \Delta\boldsymbol{C}_i(t))\boldsymbol{x}(t) + (\boldsymbol{D}_i + \Delta\boldsymbol{D}_i(t))\boldsymbol{u}(t) + (\boldsymbol{D}_{wi} + \Delta\boldsymbol{D}_{wi}(t))\boldsymbol{w}(t))\end{aligned} \tag{1.2}$$

where

$$\mu_i(z(t)) = \frac{\omega_i(z(t))}{\sum_{i=1}^{r}\omega_i(z(t))}, \quad \omega_i(z(t)) = \prod_{j=1}^{p} \mathcal{N}_{ij}(z_j(t))$$

and $\mathcal{N}_{ij}(z_j(t))$ is the degree of the membership of $z_j(t)$ in N_{ij}. Here $\omega_i(z(t)) \geq 0$, for $i = 1, 2, ..., r$ and $\sum_{i=1}^{r}\omega_i(z(t)) > 0$ for all t. Therefore, $\mu_i(z(t)) \geq 0$ for $i = 1, 2, ..., r$ and $\sum_{i=1}^{r}\mu_i(z(t)) = 1$.

1.2.2 Construction of the Fuzzy Model

A number of methods are discussed in literature for identification of T–S fuzzy model [1, 41–44]. The T–S model based fuzzy control design [8] is illustrated in Fig. 1.1. In general, there are two approaches for obtaining the fuzzy model. One is by deriving the model using the nonlinear system equations and the other is based on the input-output data generated using the original nonlinear system.

In the first case, the fuzzy model identification involves deriving the structure and parameters of the fuzzy model by linearizing the nonlinear dynamical equation about a number of operating points. This method is suitable for system or plants that can be represented by analytical or physical models and it is generally suitable for mechanical systems since the nonlinear dynamical equation can be derived by Lagrange method or Newton-Euler method [42,45]. The authors in [8,42,46] explained this

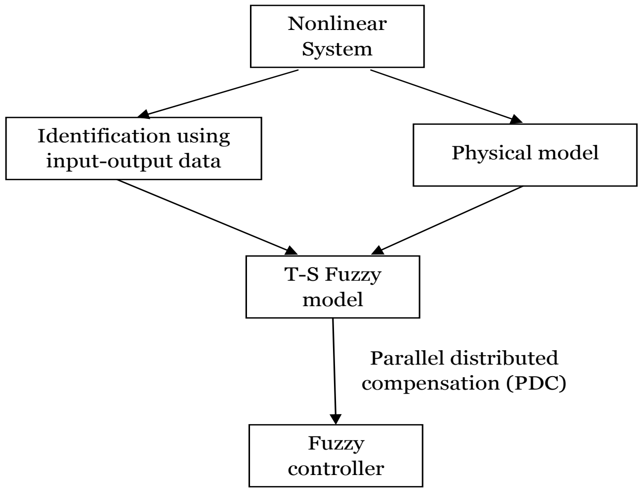


Fig. 1.1: T–S Model-based fuzzy control design.

method of deriving the structure and parameters of the fuzzy model by utilizing the idea of "sector nonlinearity", "local approximation" or a combination of them. The sector nonlinearity approach is effective in global or semiglobal fuzzy modeling which can exactly represent the dynamics of a nonlinear system.

In the latter approach, the fuzzy model identification determines the structure and parameters of fuzzy models from input-output data for those cases where it is not possible or too difficult to represent the physical (or mathematical) model of the nonlinear system. One of the most popular methods for fuzzy model identification is by using a fuzzy clustering technique for identifying number of rules and parameters of fuzzy membership functions and using a least squares method for the identification of local linear models. A number of methods has been proposed for identification of fuzzy model using the input-output data [10, 41, 47].

Fuzzy modeling by Local Linearization

In this method, the fuzzy model of the nonlinear system is derived from the system dynamic equations. The main aspect of this approach is the approximation of the nonlinear system by choosing suitable linear terms. The approximation capability of the fuzzy system depends on the interpolation properties of the neighboring subsystems.

The steps involved in deriving the fuzzy model of a nonlinear system are reviewed using the procedure explained in [45, 48]. Let us consider a nonlinear system with its model in the following form

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \tag{1.3}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^m$ and $\boldsymbol{f} \in \mathbb{R}^n$.

Expanding $\boldsymbol{f}$ by means of Taylor series around $(\boldsymbol{x}_e, \boldsymbol{u}_e)$ yields

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}_e, \boldsymbol{u}_e) + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_e \\ \boldsymbol{u}=\boldsymbol{u}_e}} (\boldsymbol{x} - \boldsymbol{x}_e) + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_e \\ \boldsymbol{u}=\boldsymbol{u}_e}} (\boldsymbol{u} - \boldsymbol{u}_e) \\ &\quad + \text{higher order terms.}\end{aligned} \tag{1.4}$$

A point $(\boldsymbol{x}_e^T, \boldsymbol{u}_e^T)^T \in \mathbb{R}^{(n+m)}$ is an equilibrium point of (1.3) if $\boldsymbol{f}(\boldsymbol{x}_e, \boldsymbol{u}_e) = \boldsymbol{0}$; that is at $(\boldsymbol{x}_e, \boldsymbol{u}_e)$, $\dot{\boldsymbol{x}} = \boldsymbol{0}$. Let $\delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_e$ and $\delta\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{u}_e$. The linearized model about the equilibrium point $(\boldsymbol{x}_e, \boldsymbol{u}_e)$ is obtained by neglecting the higher order terms and observing that for the equilibrium point $\boldsymbol{f}(\boldsymbol{x}_e, \boldsymbol{u}_e) = \boldsymbol{0}$, the linearized model has the form

$$\frac{d}{dt}\delta\boldsymbol{x} = \boldsymbol{A}\delta\boldsymbol{x} + \boldsymbol{B}\delta\boldsymbol{u} \tag{1.5}$$

where

$$\boldsymbol{A} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_e \\ \boldsymbol{u}=\boldsymbol{u}_e}} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_e \\ \boldsymbol{u}=\boldsymbol{u}_e}}$$

and

$$\boldsymbol{B} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_e \\ \boldsymbol{u}=\boldsymbol{u}_e}} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_e \\ \boldsymbol{u}=\boldsymbol{u}_e}}.$$

The local linear model at $(\boldsymbol{x}_e^T, \boldsymbol{u}_e^T)^T = \boldsymbol{0}$ can be obtained by using the above linearization technique. Next, the local linear models describing the plant's behavior at the remaining operating points are considered. For example, if the local linear model with constant matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are to be found, then in the neighborhood of $(\boldsymbol{x}_d, \boldsymbol{u}_d)$,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t). \tag{1.6}$$

Let $\boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)) = \boldsymbol{F}(\boldsymbol{x}(t)) + \boldsymbol{G}(\boldsymbol{x}(t), \boldsymbol{u}(t))$. Then the following condition should be satisfied in the neighborhood of $(\boldsymbol{x}_d, \boldsymbol{u}_d)$:

$$\boldsymbol{F}(\boldsymbol{x}(t)) + \boldsymbol{G}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \approxeq \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{1.7}$$

Let $\boldsymbol{a}_i^T$ be the i^{th} row of matrix $\boldsymbol{A}$ and $\boldsymbol{b}_i^T$ be the i^{th} row of matrix $\boldsymbol{B}$. Then for the purpose of further analysis, the following relation can be written:

$$f_i(\boldsymbol{x}(t)) + g_i(\boldsymbol{x}(t), \boldsymbol{u}(t)) \approxeq \boldsymbol{a}_i^T \boldsymbol{x}(t) + \boldsymbol{b}_i^T \boldsymbol{u}(t) \tag{1.8}$$

At the operating point, (1.8) becomes

$$f_i(\boldsymbol{x}_d) + g_i(\boldsymbol{x}_d, \boldsymbol{u}_d) \approxeq \boldsymbol{a}_i^T \boldsymbol{x}_d + \boldsymbol{b}_i^T \boldsymbol{u}_d. \tag{1.9}$$

Expanding the left hand side of (1.8) about $(\boldsymbol{x}_d, \boldsymbol{u}_d)$ by Taylor series and neglecting the higher order terms, (1.8) becomes

$$\begin{aligned} &f_i(\boldsymbol{x}_d) + \nabla_{\boldsymbol{x}}^T f_i(\boldsymbol{x}_d)(\boldsymbol{x} - \boldsymbol{x}_d) + g_i(\boldsymbol{x}_d, \boldsymbol{u}_d) \\ &\quad + \nabla_{\boldsymbol{x}}^T g_i(\boldsymbol{x}_d, \boldsymbol{u}_d)(\boldsymbol{x} - \boldsymbol{x}_d) + \nabla_{\boldsymbol{u}}^T g_i(\boldsymbol{x}_d, \boldsymbol{u}_d)(\boldsymbol{u} - \boldsymbol{u}_d) \;\approxeq\; \boldsymbol{a}_i^T \boldsymbol{x}_d + \boldsymbol{b}_i^T \boldsymbol{u}_d \end{aligned} \tag{1.10}$$

where $\nabla_{\boldsymbol{x}}^T$ and $\nabla_{\boldsymbol{u}}^T$ are gradients with respect to $\boldsymbol{x}(t)$ and $\boldsymbol{u}(t)$, respectively. From (1.9) and (1.10), the following relations can be written:

$$\nabla_{\boldsymbol{x}}^T f_i(\boldsymbol{x}_d) + \nabla_{\boldsymbol{x}}^T g_i(\boldsymbol{x}_d, \boldsymbol{u}_d) \;\approxeq\; \boldsymbol{a}_i^T \tag{1.11}$$

$$\nabla_{\boldsymbol{u}}^T g_i(\boldsymbol{x}_d, \boldsymbol{u}_d) \;\approxeq\; \boldsymbol{b}_i^T. \tag{1.12}$$

With the above relations, a convex constrained optimization is formulated [45] and the constant matrices in the model are obtained by solving that optimization problem. □

Stability is one important aspect in the traditional knowledge of Automatic Control. The parametric uncertainties are principal factors responsible for degraded stability and performance of an uncertain nonlinear closed-loop control system. In fact, in many cases it is very difficult, if not impossible, to obtain the accurate values of some system parameters. This is due to the inaccurate measurement, inaccessibility to the system parameters or on-line variation of the parameters. Therefore, this has promoted active research in the area of robust control for uncertain nonlinear systems [19, 25, 27, 32]. Clearly, it is very important to study robust stability against parametric uncertainties in the T-S fuzzy-model-based control systems.

When the interest aims at modeling an uncertain nonlinear system, a typical approach is to first define the nominal system by an ordinary T–S fuzzy model. Next the uncertainties are introduced into the T–S fuzzy model using norm bounded uncertain matrices [12,25,43,49,50]. These uncertain terms usually include unmodeled dynamics. The approximation capability of a fuzzy system is discussed in [51] and it concludes that any continuous nonlinear function can be approximated within an error bound if sufficient number of rules are used. This approximation error influences the control performance and this band of error is usually added to the uncertainty block of the fuzzy model.

Taniguchi et al. [42] presented a systematic method for fuzzy control that includes fuzzy identification, rule reduction and robust compensation for nonlinear systems. Here, the uncertain blocks compensate for modeling error. Lo and Lin [43] presented robust H_∞ nonlinear modeling and control via uncertain fuzzy systems. Here, the uncertainties are expressed using non-fuzzy uncertain bounding matrices. Škrjanc et al. [1, 52, 53] proposed a methodology for interval fuzzy model identification to approximate functions from a finite set of input-output measurements. This identification method uses the concepts from linear programming and it provides a lower and upper fuzzy model which enclose the whole band of uncertainties. The approximation capability of this method is explained with a first order (*affine*) fuzzy model and another singleton fuzzy model.

1.2.3 T–S Fuzzy Controller Design

T–S fuzzy model has been recognized as a popular and powerful tool in approximating a complex nonlinear system and many corresponding control techniques have been developed in recent years. The T–S model based controller design is carried out with the so-called parallel distributed compensation (PDC) method proposed in [24, 26, 54, 55]. The controller shares the same fuzzy sets with the fuzzy model in the premise parts and each control rule is designed from the corresponding rule of the T-S fuzzy system, respectively. The controller design is carried out using a quadratic Lyapunov function $V(\boldsymbol{x}(t)) = \boldsymbol{x}^T(t)\boldsymbol{P}\boldsymbol{x}(t)$, where the control problem is then reformulated into another problem of solving LMIs. Then the local feedback gains can be easily obtained by solving these LMIs using the powerful computational tools, such as MATLAB LMI solver [37] and SeDuMi [56].

In the domain of controller design, it is often of primary interest to synthesize a controller to satisfy certain performance function in addition to ensuring stability. Several control schemes were proposed based on this idea of performance function minimization [57–60]. A large number of controllers have evolved that minimize the worst-case ratio of output energy to disturbance energy which is known as an H_∞ control problem [2, 27, 61, 62]. Other performance criteria which are considered for designing a controller are quadratic cost minimization, norm minimization and pole placement. Among them guaranteed cost control aims at stabilizing the system while maintaining an adequate level of performance represented by a quadratic performance function and it is an important problem for systems with uncertain parameters [63–65].

Chang and Peng [57] introduced the idea of guaranteed cost control for systems with uncertain parameters. This approach provides a control law that guarantees an upper bound for the system performance function. The recent results related to fuzzy guaranteed cost control are based on

LMI optimization method which can be handled efficiently with interior point methods [36]. Tong et.al [66] derived the sufficient conditions for fuzzy based robust tracking control for uncertain non-linear systems. The conditions to satisfy a matrix inequality are derived in bilinear form and are solved by a two-step procedure to satisfy the prescribed disturbance attenuation level. There is no efficient algorithm for this kind of control problems involving bilinear matrix inequality (BMI). The two-step procedure used for solving the inequality in [66] cannot be applied for a minimization problem involving bilinear matrix inequality. Chen and Liu [64] considered the fuzzy guaranteed cost control design problem for nonlinear systems having time-varying delay. They derived the sufficient conditions for construction of guaranteed cost controller for both state feedback and observer-based output feedback cases.

In most of the above cited literature, the controller design and stability analysis are carried out using common Lyapunov function. It is required to find a common positive definite matrix satisfying the LMIs corresponding to all the local models. Generally, conservatism exists in the results with constant Lyapunov function based design and finding the common positive definite matrix might not be possible especially for highly nonlinear complex systems. During the last few years, a number of controller design methods was proposed based on some special class of Lyapunov functions such as piecewise quadratic Lyapunov function [67–73], fuzzy Lyapunov function [74–79] and polynomial Lyapunov function [80–82]. These types of Lyapunov function are much richer class of Lyapunov function candidates than the common Lyapunov function and thus, these are able to deal with a larger class of fuzzy dynamic systems. Moreover, the common or fixed Lyapunov function is a special case of this richer class of Lyapunov function.

Design with Piecewise Quadratic Lyapunov Function

The authors in [67–73, 83] presented stability analysis and controller design for fuzzy systems with piecewise quadratic Lyapunov function. The approach considers several Lyapunov functions across different regions and each component of the multiple function is required to satisfy the Lyapunov inequality for the T–S model only inside a subset of the state-space. For example, the author in [83] considered the following state space partition in his design.

$$\bar{S}_i = S_i \cup \partial S_i, \quad i = 1, 2, \ldots, r \tag{1.13}$$

where

$$S_i = \{z, \quad \mu_i(z) > \mu_l(z), \quad l = 1, 2, \ldots, r, i \neq l\}, \quad i \in \{1, 2, \ldots, r\} \tag{1.14}$$

and its boundary

$$\partial S_i = \{z, \quad \mu_i(z) = \mu_l(z), \quad l = 1, 2, \ldots, r, i \neq l\}, \quad i \in \{1, 2, \ldots, r\}. \tag{1.15}$$

With such partitions, the global model of the fuzzy system is represented with suitable subsystems in each region and the stability analysis or the controller design is presented with the following piecewise Lyapunov function candidate,

$$V(\boldsymbol{x}(t)) = \boldsymbol{x}^T(t)\boldsymbol{P}_i\boldsymbol{x}(t), \quad z \in \bar{S}_i, \quad i \in \{1, 2, \ldots, r\} \tag{1.16}$$

The results for stability analysis or the controller design problem are then obtained by solving a set of LMIs or bilinear matrix inequalities (BMIs).

Design with Fuzzy Lyapunov Function

The stability analysis and the controller design using the fuzzy Lyapunov function is presented in [74–79, 84]. Here a fuzzy Lyapunov function is obtained by fuzzy blending of multiple Lyapunov functions. Fuzzy Lyapunov function shares the same membership function with the T–S model of the system and it is defined as

$$V(\boldsymbol{x}(t)) = \boldsymbol{x}^T(t)\sum_{i=1}^{r}\mu_i(z)\boldsymbol{P}_i\boldsymbol{x}(t). \tag{1.17}$$

Unlike the piecewise Lyapunov function, the fuzzy Lyapunov function is smooth and therefore the boundary condition problem does not exist. The design using the fuzzy Lyapunov function is much more complicated than the constant Lyapunov function based approach since it involves derivatives of the premise membership functions. Recently, the authors in [76] proposed a method for designing the fuzzy controller using fuzzy Lyapunov function which does not require the bounds of the derivatives of the membership functions.

Design with Polynomial Lyapunov Function

In recent years, another class of Lyapunov function which received attention in fuzzy controller design is the polynomial Lyapunov function [80–82]. This approach is suitable for fuzzy systems with polynomial rule consequence. Polynomial fuzzy model representation is a generalization of the

T–S fuzzy model and it is more effective in representing a nonlinear system. The stability analysis and the stabilizability conditions for the polynomial fuzzy systems are derived using the polynomial Lyapunov function which is defined as

$$V(\boldsymbol{x}(t)) = \boldsymbol{x}^T(\boldsymbol{x}(t))\boldsymbol{P}(\boldsymbol{x}(t))\boldsymbol{x}(\boldsymbol{x}(t)) \tag{1.18}$$

The conditions are derived in terms of sum of squares (SOS) which can be numerically solved using the recently developed SOSTOOLS [85]. The conditions presented are more general and less conservative than the LMI based approach for general T–S fuzzy system.

1.2.4 T–S Fuzzy Descriptor System and Control

The descriptor system, which differs from a state-space representation describes a wider class of systems and it can be found in certain mechanical and electrical systems [8]. The advantage of choosing the descriptor representation over the state-space model is that the number of LMI conditions for designing the controller can be reduced for certain problems [86, 87].

A descriptor fuzzy system is defined by extending the T–S state-space model. The ordinary T–S fuzzy model is a special case of descriptor fuzzy system. The descriptor fuzzy system is defined in the following form [88]:

Plant rule i:

IF $z_1^e(t)$ is N_1^e, ... , $z_{p^k}^e(t)$ is $N_{p^k}^e$ and $z_1(t)$ is N_1, ... ,$z_p(t)$ is N_p *THEN*

$$\boldsymbol{E}_k\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t) \tag{1.19}$$

$z_1(t), \cdots, z_p(t)$ are premise variables, p is the number of premise variables, N_{ij} $(j = 1...p)$ is the fuzzy set and r is the number of rules. Here, $\boldsymbol{x}(t) \in \mathbb{R}^{n\times 1}$ is the state vector and $\boldsymbol{u}(t) \in \mathbb{R}^{m\times 1}$ is the input vector. $\boldsymbol{A}_i \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}_i \in \mathbb{R}^{n\times m}$ and $\boldsymbol{E}_k \in \mathbb{R}^{n\times n}$ are constant real matrices.

In recent years, research has progressed in the area of stability analysis, stabilization control, H_∞ stabilization and model following control for fuzzy descriptor systems [86, 87, 89]. Stability analysis and controller design are carried out by defining $\boldsymbol{x}^*(t) = [\boldsymbol{x}^T(t) \quad \dot{\boldsymbol{x}}^T(t)]^T$ and rewriting the fuzzy system in the following augmented form

$$\boldsymbol{E}^*\dot{\boldsymbol{x}}^*(t) = \sum_{i=1}^{r}\sum_{k=1}^{r^e}\mu_i\mu_k^e(\boldsymbol{A}_{ik}^*\boldsymbol{x}^*(t) + \boldsymbol{B}_i^*\boldsymbol{u}(t)) \tag{1.20}$$

where

$$\boldsymbol{E}^* = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{A}_{ik}^* = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{A}_i & -\boldsymbol{E}_k \end{bmatrix}, \quad \boldsymbol{B}_i^* = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{B}_i \end{bmatrix}.$$

A candidate of quadratic Lyapunov function of the form

$$V(t) = \boldsymbol{x}^{*T}(t)\boldsymbol{E}^{*T}\boldsymbol{X}\boldsymbol{x}^{*}(t) \tag{1.21}$$

is considered and conditions for stability or stabilization are derived in terms of LMIs.

Today adequate literature is available that discusses about the use of linear matrix inequality (LMI) approach for design of T–S model based fuzzy controller using the constant Lyapunov functions (e.g., [2, 62]).

1.2.5 Observer based output feedback control

In real-world control problems, however, it is often the case that the complete information of the states of a system is not always available. In such cases, one needs to resort to output feedback design methods such as observer-based designs. Observer designs, and output feedback control designs with or without various performance indexes have also been well developed for TS fuzzy systems based on common quadratic Lyapunov functions and LMIs [31, 66, 84, 90, 91].

A typical structure of observer for TS fuzzy systems is given below [8]:

Observer rule i:

IF $z_1(t)$ is N_{i1} and $\cdots$ $z_p(t)$ is N_{ip}, *THEN*

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}}(t) &= \boldsymbol{A}_i\hat{\boldsymbol{x}}(t) + \boldsymbol{B}_i\boldsymbol{u}(t) + \boldsymbol{L}_i(\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t)) \\ \hat{\boldsymbol{y}}(t) &= \boldsymbol{C}_i\hat{\boldsymbol{x}}(t), \qquad i = 1, 2, ..., r \end{aligned} \tag{1.22}$$

The dependence of the premise variables on the state variables makes it necessary to consider two cases for fuzzy observer design: (i) $z_1(t), \ldots, z_p(t)$ do not depend on the state variables estimated by a fuzzy observer, (ii) $z_1(t), \ldots, z_p(t)$ depend on the state variables estimated by a fuzzy observer.

The fuzzy observer for case (i) is represented as follows:

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}}(t) &= \sum_{i=1}^{r} \mu_i(z(t))(\boldsymbol{A}_i\hat{\boldsymbol{x}}(t) + \boldsymbol{B}_i\boldsymbol{u}(t) + \boldsymbol{L}_i(\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t))) \\ \boldsymbol{y}(t) &= \sum_{i=1}^{r} \mu_i(z(t))(\boldsymbol{C}_i\boldsymbol{x}(t)) \end{aligned} \tag{1.23}$$

For case (ii) the fuzzy observer is represented as follows:

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}}(t) &= \sum_{i=1}^{r} \mu_i(\hat{z}(t))(\boldsymbol{A}_i\hat{\boldsymbol{x}}(t) + \boldsymbol{B}_i\boldsymbol{u}(t) + \boldsymbol{L}_i(\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t))) \\ \boldsymbol{y}(t) &= \sum_{i=1}^{r} \mu_i(\hat{z}(t))(\boldsymbol{C}_i\boldsymbol{x}(t)) \end{aligned} \tag{1.24}$$

In the presence of the fuzzy observer for case (i), the PDC fuzzy controller takes the following form:

$$\boldsymbol{u}(t) = \sum_{i=1}^{r} \mu_i(z(t))(\boldsymbol{K}_i\hat{\boldsymbol{x}}(t)) \tag{1.25}$$

Similarly for case (ii), the PDC fuzzy controller takes the following form:

$$\boldsymbol{u}(t) = \sum_{i=1}^{r} \mu_i(\hat{z}(t))(\boldsymbol{K}_i\hat{\boldsymbol{x}}(t)) \tag{1.26}$$

The stability analysis and design of the augmented system for case (i) are more straightforward, whereas the stability analysis and design for case (ii) is complicated since the premise variables depend on the state variables, which have to estimated by a fuzzy observer. The controller design and observer design can be independently carried out based on common quadratic Lyapunov functions, and the resulting closed-loop fuzzy control system, with estimated state variables to be used for state feedback control, will be asymptotically stable.

1.3 Motivation and Purposes

There has been a rapidly growing interest in T–S fuzzy model based robust control for controlling uncertain nonlinear systems. This interest has developed due to the fact that fuzzy logic provides a simple and straight forward way to decompose the task of modeling and control design into a group of local tasks, which makes the whole procedure easier. Moreover, the stability analysis and controller design problems can be reduced to a linear matrix inequality (LMI) problem which can be solved easily and efficiently with the available tools such as LMI toolbox in MATLAB [37] and SeDuMi [56].

To design a T–S fuzzy controller, the primary step involves building the T–S fuzzy model of the uncertain nonlinear system. Therefore constructing a fuzzy model for the uncertain nonlinear system is an important and basic step in this approach. A number of methods has been reported in the literature for fuzzy modeling and most of them deal with the nonlinear system without uncertainties. In robust fuzzy control methodologies used for uncertain nonlinear systems, the controller design is carried out using a class of fuzzy system represented in the Takagi-Sugeno form with uncertainty blocks. Few studies are reported in literature where T–S fuzzy model construction techniques for uncertain systems are discussed. Here the experience or human knowledge to express the uncertainties is utilized. But this technique is very difficult for systems with severe nonlinearity and uncertainties. In some control design literatures [66], the parameters of the nominal fuzzy model are assumed to have certain amount of perturbation and these are not estimated from the uncertainty of the original system. Therefore, fuzzy identification of uncertain nonlinear systems is the preliminary and basic step towards robust fuzzy control. Hence, fuzzy model identification problem is first considered so that a couple of effective robust fuzzy control techniques can be developed for uncertain nonlinear systems.

In the domain of controller design, it is often of primary interest to synthesize a controller to satisfy certain performance function in addition to ensuring stability. Several control schemes were proposed based on this idea of performance function minimization [2,40,43,64,66]. Tracking control is an important problem in practical applications like robotics, missile tracking and attitude tracking of aircrafts. However it is more difficult than the regulatory control problem and only few studies have been carried out in the area of tracking control especially for continuous-time systems [92]. LMI approach has been successfully applied for regulatory control of nonlinear systems with quadratic performance function [8,64]. Our research attempts towards controlling uncertain nonlinear systems to design a fuzzy guaranteed cost controller which satisfies a given quadratic performance function while tracking a specified trajectory.

Several studies have been reported in the literature for stability analysis and controller synthesis of T–S model-based fuzzy control [2, 40, 43, 64, 66], in which the design problem is recast into a set of LMIs based on a fixed or common Lyapunov function. It is required to find a common positive definite matrix to satisfy all the LMIs corresponding to all the local models. Conservatism would remain in this fixed Lyapunov function based approach and in some cases (especially for highly nonlinear complex systems) it might not be possible to find the positive definite matrix. To reduce the conservatism, different types of Lyapunov functions such as piecewise quadratic Lyapunov function, fuzzy Lyapunov function and polynomial Lyapunov function are used in place of common Lyapunov function. The piecewise and fuzzy Lyapunov function based design show better results by allowing the Lyapunov function to vary across different regions while the polynomial Lyapunov function based design is aimed at polynomial fuzzy systems. In the domain of the uncertain nonlinear systems, conservatism can be reduced in the design by varying the Lyapunov function with respect to the uncertainties and this class of function is called as parametric Lyapunov function. Motivated by this concept and encouraged by the results obtained with piecewise and fuzzy Lyapunov function based design, the research work focuses on robust fuzzy controller design using parametric Lyapunov function.

The descriptor representation based systematic design is more difficult than the state-space based design. In recent years, considerable work has been done with fixed Lyapunov function for stability analysis, stabilization, H_∞ stabilization and model following control for fuzzy descriptor systems [86, 87, 89]. The conservatism in these designs can be reduced by using a parametric Lyapunov function in place of fixed Lyapunov function. Hence, a parametric Lyapunov function is considered and the results are extended for synthesizing a controller for uncertain fuzzy descriptor systems.

1.4 Contributions of this Thesis

Main contributions of this thesis can be summarized as follows:

- A fuzzy identification method is proposed for deriving the T–S fuzzy model for uncertain nonlinear systems in a form suitable for robust fuzzy control.
- Robust fuzzy guaranteed cost controller which satisfies a given quadratic performance function is developed for application to the trajectory tracking problem of uncertain nonlinear systems.
- A framework that offers less conservative sufficient conditions for robust stabilization of uncertain nonlinear systems built upon the T-S fuzzy model is developed by using a class of parametric Lyapunov function.
- A controller is developed for robust control of uncertain nonlinear system modeled by fuzzy descriptor systems using parametric Lyapunov function.

1.5 Thesis Organization

This thesis is divided into six chapters. Brief descriptions about the research contributions described in each chapter are outlined in this subsection.

Chapter 2: This chapter describes the interval fuzzy model representation for use in robust fuzzy control problems. A fuzzy logic based model identification method is presented and it is compared with the method proposed by Škrjanc et al. [1]. Suitability of the fuzzy model for application to robust fuzzy control is also discussed and the performance of the proposed technique is demonstrated with different examples.

Chapter 3: This chapter discusses the robust fuzzy guaranteed cost controller design for trajectory tracking control problem in uncertain nonlinear systems. This chapter starts with problem formulation and then the necessary inequality conditions are derived for guaranteed cost control in uncertain nonlinear system. The method of transforming the inequality conditions as LMI conditions are then discussed and finally simulation results are presented.

Chapter 4: This chapter first recalls the robust stabilization results using fixed Lyapunov function. Then robust stability condition for uncertain nonlinear systems by using parametric Lyapunov function is presented. The robust H_∞ controller design methods with parametric Lyapunov function are discussed for stabilization and tracking control problems.

Chapter 5: In this chapter, the fuzzy descriptor system is described and the tracking control problem for uncertain fuzzy descriptor system is presented. Then robust controller design methods are discussed with fixed and parametric Lyapunov functions for controlling uncertain fuzzy descriptor systems.

Chapter 6: Conclusions from the research work and the scope for future research are discussed in this chapter.

Chapter 2

Identification of Uncertain Nonlinear Systems for Robust Fuzzy Control

2.1 Introduction

Fuzzy logic based control is increasingly being applied to uncertain and ill-defined nonlinear systems and fuzzy control strongly competes with other nonlinear control techniques. Several significant research results are reported in the literature which utilize an uncertain fuzzy model to deal with robust fuzzy control problems [62,66,93–96]. Generally an *affine* T-S fuzzy model employs a constant term in the consequent part of each rule. But in most of the aforementioned literature [62,66,93–96], a special type of T-S fuzzy model is used which has linear rule consequence without the constant term. In these robust fuzzy control methodologies, the controller design is carried out using a class of fuzzy system represented in the Takagi-Sugeno form with uncertainty blocks. Apart from robust fuzzy control, these models are also used for fault identification in uncertain nonlinear systems [50].

In the above cited literature, the uncertain fuzzy model uses the experience or human knowledge to express the uncertainties. This may be suitable for small systems or systems with sector nonlinearity like a mass–spring system. But many systems have severe nonlinearity and uncertainties which add difficulty to the identification technique. In some control design literatures [66], the parameters of the nominal fuzzy model are assumed to have certain amount of perturbation and these are not estimated from the uncertainty of the original system. Therefore, for these applications, the interval fuzzy identification of uncertain nonlinear systems has become an important topic of scientific research.

This chapter discusses fuzzy model based identification of uncertain nonlinear systems suitable for robust fuzzy control. Assumption is made that the parameters of the antecedent part are available or the input space is uniformly partitioned for the parameters of the membership functions [96, 97]. Here, the identification of only the consequent part is considered. The fuzzy model is expressed by linear and uncertain terms, which represent the nominal system and parametric uncertainties respectively. The nominal model and the bounds of the uncertain terms of the fuzzy model are identified and the approach is based on the method proposed by Škrjanc et al. [1]. The model identified in this step cannot be directly employed in robust fuzzy control design and it must be expressed in another special form. The uncertain terms are expressed in suitable form as norm bounded uncertain matrices accompanied by constant real matrices. Simulation results show that the fuzzy model representation is suitable for robust fuzzy control of uncertain nonlinear systems.

2.2 Fuzzy Model with Uncertainty

The continuous fuzzy system (CFS) proposed by Takagi and Sugeno [22] represents the dynamics of nonlinear system using fuzzy IF-THEN rules. As in [62, 66, 93–96], rules for the typical fuzzy model of an uncertain nonlinear system employed in robust control design are of the following form:

Plant rule i:

IF $z_1(t)$ is N_{i1} and $z_2(t)$ is N_{i2} and $\cdots$ $z_p(t)$ is N_{ip} *THEN*

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= (\boldsymbol{A}_i + \Delta\boldsymbol{A}_i(t))\boldsymbol{x}(t) + (\boldsymbol{B}_{ui} + \Delta\boldsymbol{B}_{ui}(t))\boldsymbol{u}(t) + (\boldsymbol{B}_{wi} + \Delta\boldsymbol{B}_{wi}(t))\boldsymbol{w}(t) \\
\boldsymbol{y}(t) &= (\boldsymbol{C}_i + \Delta\boldsymbol{C}_i(t))\boldsymbol{x}(t) + (\boldsymbol{D}_{ui} + \Delta\boldsymbol{D}_{ui}(t))\boldsymbol{u}(t) + (\boldsymbol{D}_{wi} + \Delta\boldsymbol{D}_{wi}(t))\boldsymbol{w}(t), i = 1, 2, \ldots, r, \quad (2.1)
\end{aligned}$$

where $z_1(t), \ldots, z_p(t)$ are premise variables, p is the number of premise variables, N_{ij} $(j = 1, \ldots, p)$ is the fuzzy set and r is the number of rules. Here, $\boldsymbol{x}(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ is the state vector, $\boldsymbol{u}(t) = [u_1(t), \ldots, u_{m_u}(t)]^T \in \mathbb{R}^{m_u}$ is the input vector, $\boldsymbol{y}(t) = [y_1(t), \ldots, y_q(t)]^T \in \mathbb{R}^q$ is the output vector, $\boldsymbol{w}(t) = [w_1(t), \ldots, w_{m_w}(t)]^T \in \mathbb{R}^{m_w}$ is the disturbance input vector. $\boldsymbol{A}_i \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}_{ui} \in \mathbb{R}^{n\times m_u}$, $\boldsymbol{B}_{wi} \in \mathbb{R}^{n\times m_w}$, $\boldsymbol{C}_i \in \mathbb{R}^{q\times n}$, $\boldsymbol{D}_{ui} \in \mathbb{R}^{q\times m_u}$, $\boldsymbol{D}_{wi} \in \mathbb{R}^{q\times m_w}$ are constant real matrices describing the nominal system and $\Delta\boldsymbol{A}_i(t)$, $\Delta\boldsymbol{B}_{ui}(t)$, $\Delta\boldsymbol{B}_{wi}(t)$, $\Delta\boldsymbol{C}_i(t)$, $\Delta\boldsymbol{D}_{ui}(t)$ and $\Delta\boldsymbol{D}_{wi}(t)$ are time varying matrices of appropriate dimensions, which represent parametric uncertainties in the plant and modeling errors respectively.

Given a pair of input and output $(\boldsymbol{x}(t), \boldsymbol{u}(t))$, the final output of the fuzzy system is inferred as follows [62]:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \sum_{i=1}^{r} \mu_i(z(t))\{(\boldsymbol{A}_i + \Delta \boldsymbol{A}_i)\boldsymbol{x}(t) + (\boldsymbol{B}_{ui} + \Delta \boldsymbol{B}_{ui})\boldsymbol{u}(t) + (\boldsymbol{B}_{wi} + \Delta \boldsymbol{B}_{wi})\boldsymbol{w}(t)\} \\ \boldsymbol{y}(t) &= \sum_{i=1}^{r} \mu_i(z(t))\{(\boldsymbol{C}_i + \Delta \boldsymbol{C}_i)\boldsymbol{x}(t) + (\boldsymbol{D}_{ui} + \Delta \boldsymbol{D}_{ui})\boldsymbol{u}(t) + (\boldsymbol{D}_{wi} + \Delta \boldsymbol{D}_{wi})\boldsymbol{w}(t)\}, \end{aligned} \tag{2.2}$$

where

$$\mu_i(z(t)) = \frac{\omega_i(z(t))}{\sum_{j=1}^{r} \omega_j(z(t))}, \quad \omega_i(z(t)) = \prod_{j=1}^{p} \mathcal{N}_{ij}(z_j(t)) \tag{2.3}$$

and $N_{ij}(z_j(t))$ is the degree of membership of $z_j(t)$ in the fuzzy set $\mathcal{N}_{ij}$. Some basic properties of $\mu_i(z(t))$ and $\omega_i(z(t))$ are $\omega_i(z(t)) \geq 0$, $\sum_{j=1}^{r} \omega_j(z(t)) > 0$, $\mu_i(z(t)) \geq 0$ and $\sum_{j=1}^{r} \mu_j(z(t)) = 1$.

In robust control design problems, the uncertain matrices are assumed to be norm bounded and are described by [62, 66, 93–96]:

$$\begin{bmatrix} \Delta \boldsymbol{A}_i(t) & \Delta \boldsymbol{B}_{ui}(t) & \Delta \boldsymbol{B}_{wi}(t) \\ \Delta \boldsymbol{C}_i(t) & \Delta \boldsymbol{D}_{ui}(t) & \Delta \boldsymbol{D}_{wi}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{M}_{xi}\boldsymbol{F}_{xi}(t)(\boldsymbol{N}_{x1i} \quad \boldsymbol{N}_{x2i} \quad \boldsymbol{N}_{x3i}) \\ \boldsymbol{M}_{yi}\boldsymbol{F}_{yi}(t)(\boldsymbol{N}_{y1i} \quad \boldsymbol{N}_{y2i} \quad \boldsymbol{N}_{y3i}) \end{bmatrix}, \tag{2.4}$$

where $\boldsymbol{M}_{xi}$, $\boldsymbol{M}_{yi}$, $\boldsymbol{N}_{x1i}$, $\boldsymbol{N}_{x2i}$, $\boldsymbol{N}_{x3i}$, $\boldsymbol{N}_{y1i}$, $\boldsymbol{N}_{y2i}$ and $\boldsymbol{N}_{y3i}$ are known real constant matrices of appropriate dimension and $\boldsymbol{F}_{xi}(t)$, $\boldsymbol{F}_{yi}(t)$ are time varying matrix functions with Lebesgue-measurable elements, satisfying $\boldsymbol{F}_{xi}^T(t)\boldsymbol{F}_{xi}(t) \leq \boldsymbol{I}$, and $\boldsymbol{F}_{yi}^T(t)\boldsymbol{F}_{yi}(t) \leq \boldsymbol{I}$.

2.3 Interval Fuzzy Model Identification

2.3.1 Identification with Homogenous Fuzzy Functions

Let us consider an uncertain nonlinear function $g(\boldsymbol{v}, \boldsymbol{\gamma}(t))$, in which $\boldsymbol{v} = [v_1, \ldots, v_{n_v}]^T \in \mathbb{R}^{n_v}$ is the input vector, $\boldsymbol{\gamma}(t) = [\gamma_1(t), \ldots, \gamma_{n_\gamma}(t)] \in \mathbb{R}^{n_\gamma}$ is a time varying uncertain vector with known lower and upper bounds $\underline{\gamma}_k$, $\overline{\gamma}_k$ of $\gamma_k(t)$ satisfying $\underline{\gamma}_k \leq \gamma_k(t) \leq \overline{\gamma}_k$, $k = 1, \ldots, n_\gamma$.

The uncertain fuzzy model is defined in the following form

$$\psi(\boldsymbol{v}, t) = \sum_{i=1}^{r} \mu_i(z(t))(\boldsymbol{\theta}_i + \Delta \boldsymbol{\theta}_i(t))^T \boldsymbol{v}, \tag{2.5}$$

where $\boldsymbol{\theta}_i = [\theta_{i1}, \theta_{i2}, \ldots, \theta_{in_v}]$ is a constant real vector representing the nominal model of the system and $\Delta \boldsymbol{\theta}_i(t) = [\Delta\theta_{i1}(t), \Delta\theta_{i2}(t), \ldots, \Delta\theta_{in_v}(t)]$ is an interval vector which represents the parametric uncertainties satisfying $|\Delta\theta_{ik}(t)| \leq \delta\theta_{ik}, \quad k = 1, \ldots, n_v$. The upper and lower bounds of the uncertain parameters of the fuzzy model are $\delta\boldsymbol{\theta}_i^T \boldsymbol{\vartheta}$ and $-\delta\boldsymbol{\theta}_i^T \boldsymbol{\vartheta}$ respectively, where $\boldsymbol{\vartheta} = [\mathrm{abs}(v_1), \mathrm{abs}(v_1), \ldots, \mathrm{abs}(v_{n_v})]$. Now the lower and upper bounds of the fuzzy model can be defined as

$$\underline{\psi}(\boldsymbol{v}) = \sum_{i=1}^{r}\mu_i(z(t))(\boldsymbol{\theta}_i^T\boldsymbol{v} - \delta\boldsymbol{\theta}_i^T\boldsymbol{\vartheta}) \tag{2.6}$$

$$\overline{\psi}(\boldsymbol{v}) = \sum_{i=1}^{r}\mu_i(z(t))(\boldsymbol{\theta}_i^T\boldsymbol{v} + \delta\boldsymbol{\theta}_i^T\boldsymbol{\vartheta})). \tag{2.7}$$

Next the following relation from [1] is considered for identification of interval fuzzy model from a finite set of input-output measurements $\boldsymbol{V} = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N]$, where $\boldsymbol{v}_j(j = 1, \ldots, N)$ is the set of inputs collected from the compact set $\mathbf{S}$:

$$\underline{\psi}(\boldsymbol{v}_j) \leq g(\boldsymbol{v}_j, \boldsymbol{\gamma}(t)) \leq \overline{\psi}(\boldsymbol{v}_j), \quad \forall \ j. \tag{2.8}$$

An interval valued function $\underline{\overline{g}}(\boldsymbol{v}_j)$ is defined satisfying $\underline{g}(\boldsymbol{v}_j) \leq g(\boldsymbol{v}_j, \boldsymbol{\gamma}(t)) \leq \overline{g}(\boldsymbol{v}_j)$, where $\underline{g}(\boldsymbol{v}_j)$ and $\overline{g}(\boldsymbol{v}_j)$ represent the lower and upper bounds of $\underline{\overline{g}}(\boldsymbol{v}_i)$. The interval valued function $\underline{\overline{g}}(\boldsymbol{v}_j)$ can be easily formed by replacing the uncertain quantities with interval variables having bounds equal to that of the uncertain term. If the interval valued function $\underline{\overline{g}}(\boldsymbol{v}_j)$ satisfies the following inequalities (2.9) and (2.10), then the lower and upper fuzzy model will also satisfy the relation given in (2.8).

$$\underline{\psi}(\boldsymbol{v}_j) \leq \underline{g}(\boldsymbol{v}_j), \quad \forall \ j \tag{2.9}$$

$$\overline{g}(\boldsymbol{v}_j) \leq \overline{\psi}(\boldsymbol{v}_j), \quad \forall \ j \tag{2.10}$$

If $\underline{e}_j$ and $\overline{e}_j$ are the approximation errors related to lower and upper fuzzy model then

$$\underline{e}_j = \underline{g}(\boldsymbol{v}_j) - \underline{\psi}(\boldsymbol{v}_j), \quad \forall \ j \tag{2.11}$$

$$\overline{e}_j = \overline{\psi}(\boldsymbol{v}_j) - \overline{g}(\boldsymbol{v}_j), \quad \forall \ j. \tag{2.12}$$

If the maximum approximation error related to the fuzzy model and the family of functions is defined as

$$\lambda = \max_{v_j \in V}\left\{\underline{e}_j + \overline{e}_j\right\}, \tag{2.13}$$

then

$$\underline{e}_j + \overline{e}_j \leq \lambda, \quad \forall \ j. \tag{2.14}$$

With (2.11) and (2.12), (2.14) can be written as:

$$\underline{g}(\boldsymbol{v}_j) - \underline{\psi}(\boldsymbol{v}_j) + \overline{\psi}(\boldsymbol{v}_j) - \overline{g}(\boldsymbol{v}_j) \leq \lambda, \quad \forall \ j. \tag{2.15}$$

With the inequalities (2.9), (2.10) and (2.15), the optimization problem for identification of the lower and upper fuzzy functions can be framed as:

$$\begin{aligned} &\min_{\underline{\psi},\overline{\psi}} \quad \lambda \\ &\text{subject to} \\ \underline{g}(\boldsymbol{v}_j) - \underline{\psi}(\boldsymbol{v}_j) + \overline{\psi}(\boldsymbol{v}_j) - \overline{g}(\boldsymbol{v}_j) &\leq \lambda, \quad j = 1, \ldots, N \\ \underline{\psi}(\boldsymbol{v}_j) - \underline{g}(\boldsymbol{v}_j) &\leq 0, \quad j = 1, \ldots, N \\ \overline{g}(\boldsymbol{v}_j) - \overline{\psi}(\boldsymbol{v}_j) &\leq 0, \quad j = 1, \ldots, N \end{aligned} \tag{2.16}$$

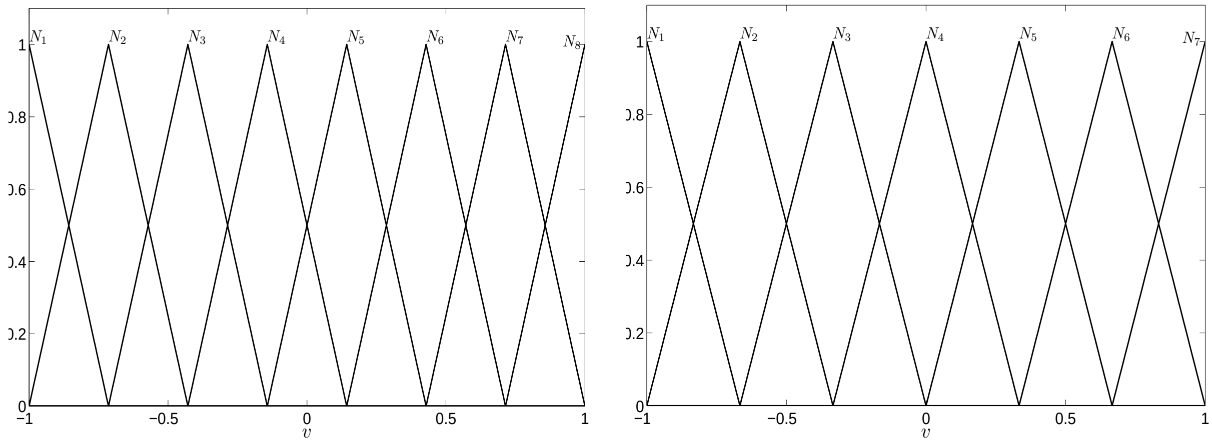
With the lower and upper fuzzy model defined in (2.6) and (2.7), the above optimization problem can be formulated as following convex linear programming problem

$$\begin{aligned} &\min_{\theta_j,\delta\theta_j} \lambda \\ &\text{subject to} \\ \underline{g}(\boldsymbol{v}_j) - \overline{g}(\boldsymbol{v}_j) + 2\sum_{i=1}^{r} \mu_i(z(t))(\delta\boldsymbol{\theta}_i^T \boldsymbol{\vartheta}_j) &\leq \lambda, \quad j = 1, \ldots, N \\ -\underline{g}(\boldsymbol{v}_j) + \sum_{i=1}^{r} \mu_i(z(t))(\boldsymbol{\theta}_i^T \boldsymbol{v}_j - \delta\boldsymbol{\theta}_i^T \boldsymbol{\vartheta}_j) &\leq 0, \quad j = 1, \ldots, N \\ \overline{g}(\boldsymbol{v}_j) - \sum_{i=1}^{r} \mu_i(z(t))(\boldsymbol{\theta}_i^T \boldsymbol{v}_j + \delta\boldsymbol{\theta}_i^T \boldsymbol{\vartheta}_j) &\leq 0, \quad j = 1, \ldots, N \\ &\text{and} \\ -\delta\theta_{ik} &\leq 0, \quad i = 1, \ldots, r, \quad k = 1, \ldots, n_v \end{aligned} \tag{2.17}$$

where λ is a variable that represents the maximum approximation error corresponding to the lower and upper fuzzy models.

Guidelines for selection of input–output data: In the proposed identification method, the model is built using input-output data to replicate the behavior of a given function. A reasonable amount of input-output data needs to be chosen from the input range. The data can be chosen randomly in the input space or it can be obtained at equidistant points. The more the number of data points, the better the representation of the dynamics of the original nonlinear equation will become. To decide the number of data points, start deriving the model with a smaller number of input-output data and again try with a greater number of input-output data until a consistent model is obtained.

Remark 2.1. The model is identified with the input-output data obtained using the nonlinear equation. A similar approach is followed in some literatures for obtaining fuzzy models (without uncertainties); e.g. in Chapter 2 in [10] and also the method in [1] for modeling of uncertain nonlinear systems. The interval fuzzy model approximates the dynamics captured by the input-output data



(a) Set – I (b) Set – II

Fig. 2.1: Membership function.

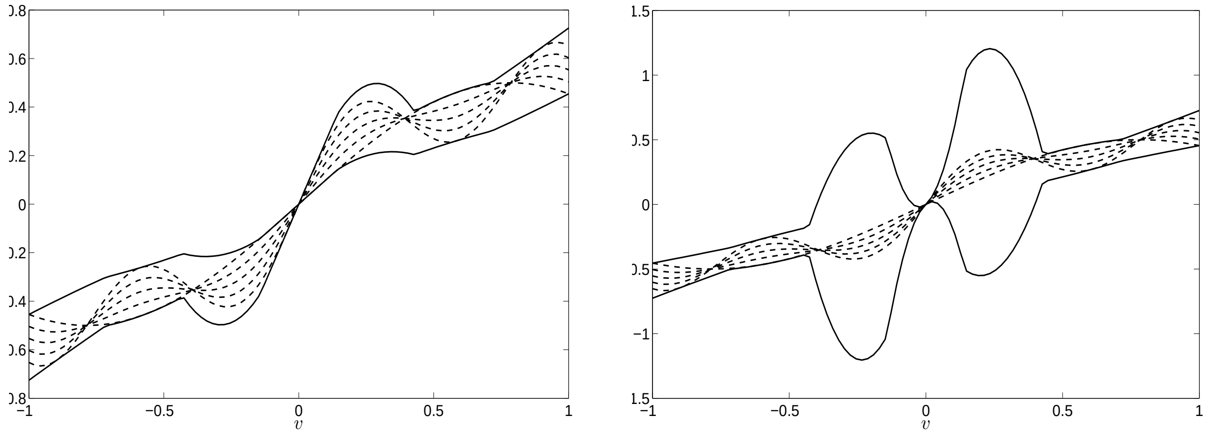
and the approximation may be invalid in some regions where the identification data is not present. This limitation is explained in [1] and the same is applicable to the above fuzzy model identification method.

Remark 2.2. In this work it has been assumed that the parameters of the antecedent part are available or the input space is uniformly partitioned [96,97] for the parameters of the antecedent part. A fuzzy model with narrow error band might be possible if the input space is partitioned in an optimal way. By considering the nominal model of the given system, the antecedent part of the fuzzy model can be identified using the approach in [41,44,47] or by considering the different sectors and applying the method explained in [8,42]. Using the same antecedent parameters for the uncertain fuzzy model, the consequent part can be obtained by the method described in this section.

2.3.2 Example

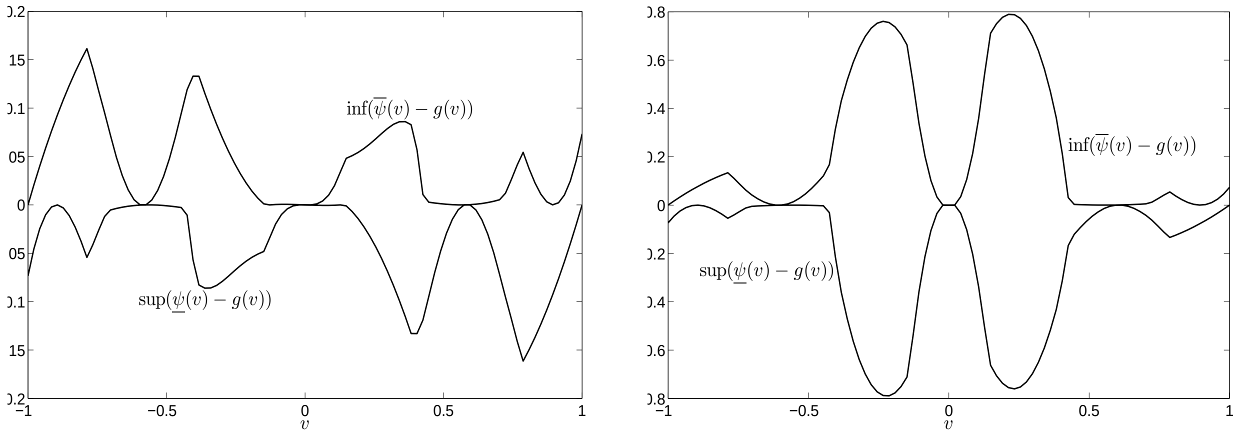
Similar to the uncertain nonlinear function in [1], this example considers the class of $\mathcal{G}$ with $g_{nom}(v) = \cos(v)\sin(v)$ and the uncertainty $\Delta g(v) = \gamma \sin(8v)$, $0 \leq \gamma \leq 0.2$. This function is similar to the one considered in [1] except the presence of a *sine* function in the uncertainty part instead of a *cosine* function and this will make the class of $\mathcal{G}$ to satisfy $g(0) = 0$. The functions from the class are defined in the domain $\mathbf{S} = \{v| -1 \leq v \leq 1\}$ and the set of "measurements" is $\boldsymbol{V} = \{v_j | v_j = 0.021k, k = -47, -46, \ldots, 47\} \subset \mathbf{S}$.

Let us consider the problem of finding the upper and lower fuzzy models in homogenous form for the uncertain nonlinear function $g(z, \gamma(t))$. The dimensionality of the input space is 1 and therefore the premise and consequent variables are same as the measurements, i.e., $z_j = v_j, j = 1, \ldots, N$.



(a) (b)

Fig. 2.2: Data, lower and upper bound of the fuzzy model using membership function set – I (a) by the proposed identification method, (b) by the identification method in [1].



(a) (b)

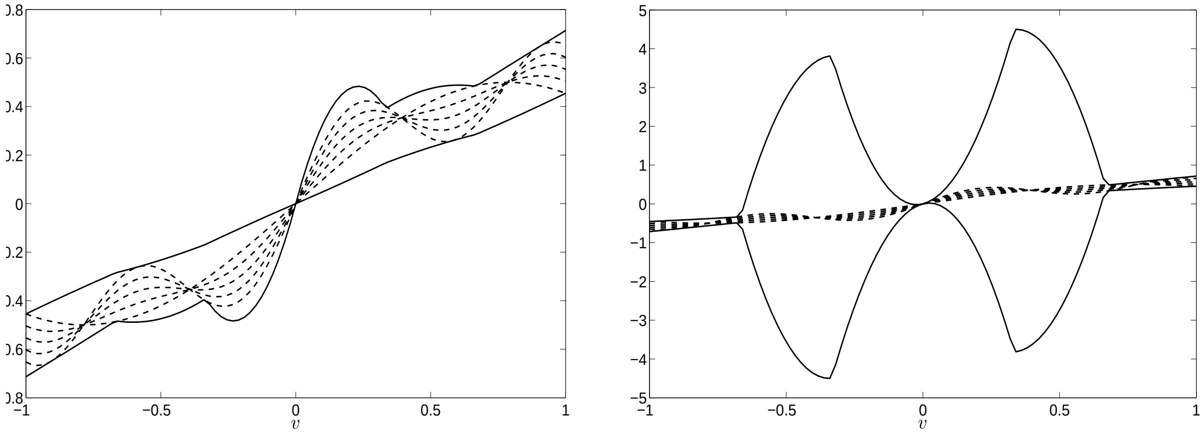
Fig. 2.3: Difference between the bounds of the fuzzy function and the actual envelope of the family of functions (Membership function set – I) (a) by the proposed identification method, (b) by the identification method in [1].

Similar to [1], eight and another seven triangular equidistant membership functions (Set I and II) are considered as shown in Fig. 2.1. Here, the lower and upper fuzzy models take the following form:

$\underline{\boldsymbol{R}}_i$: IF z is N_i THEN $\underline{\psi}_i = \theta_{i1} v - \delta\theta_{i1} \text{abs}(v), \quad i = 1, \ldots, r,$

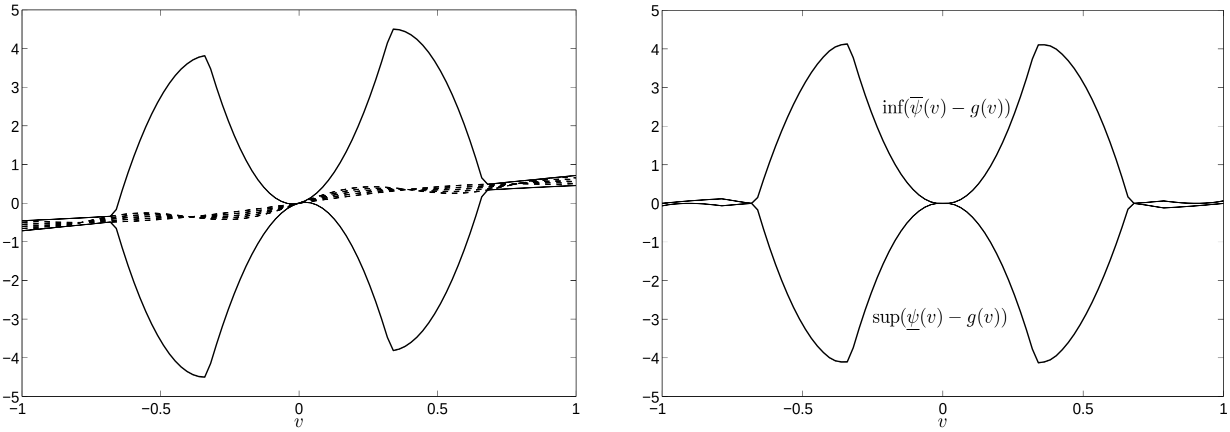
$\overline{\boldsymbol{R}}_i$: IF z is N_i THEN $\overline{\psi}^i = \theta_{i1} v + \delta\theta_{i1} \text{abs}(v), \quad i = 1, \ldots, r.$

The fuzzy model is built for the membership function set – I shown in Fig. 2.1 (a) and the results are shown in Fig. 2.2(a), where the dashed line represents the family of functions and the solid line shows the lower and upper bounds of the fuzzy function. The approximation errors $\sup_{g \in \mathcal{G}}(\underline{\psi}(v) - g(v))$ and $\inf_{g \in \mathcal{G}}(\overline{\psi}(v) - g(v))$ are presented in Fig. 2.3(a). Similarly for the membership function set – II shown in Fig. 2.1 (b), the results and the approximation error obtained using the



(a) (b)

Fig. 2.4: Data, lower and upper bound of the fuzzy model using membership function set – II (a) by the proposed identification method, (b) by the identification method in [1].



(a) (b)

Fig. 2.5: Difference between the bounds of the fuzzy function and the actual envelope of the family of functions (Membership function set – II) (a) by the proposed identification method, (b) by the identification method in [1].

proposed identification method are shown in Figs. 2.4 (a) and 2.5 (a) respectively.

For the purpose of comparison, the method proposed by Škrjanc et al. [1] is considered here. Fuzzy model identification using a l_1 norm is considered in this case and hence, slack variables $\underline{\lambda}_j$, $\overline{\lambda}_j$, $j = 1, \ldots, N$ are introduced. The interval fuzzy model is identified by the following linear programming problem of minimizing the sum of the absolute value of the estimation errors:

$$\sum_{j=1}^{N} \underline{\lambda}_j + \sum_{j=1}^{N} \overline{\lambda}_j \tag{2.18}$$

subject to

$$\inf(g(\boldsymbol{v}_j)) - \sum_{i=1}^{r} \mu_i(z) \underline{\boldsymbol{\theta}}_i^T \boldsymbol{v}_j \quad \leq \quad \underline{\lambda}_j, \quad j = 1, \ldots, N \tag{2.19}$$

$$\inf(g(\boldsymbol{v}_j)) - \sum_{i=1}^{r} \mu_i(z)\underline{\boldsymbol{\theta}}_i^T \boldsymbol{v}_j \geq 0, \quad j = 1, \ldots, N \tag{2.20}$$

$$\underline{\lambda}_j \geq 0, \quad j = 1, \ldots, N \tag{2.21}$$

and

$$-\sup(g(\boldsymbol{v}_j)) + \sum_{i=1}^{r} \mu_i(z)\overline{\boldsymbol{\theta}}_i^T \boldsymbol{v}_j \leq \overline{\lambda}_j, \quad j = 1, \ldots, N \tag{2.22}$$

$$\sup(g(\boldsymbol{v}_j)) - \sum_{i=1}^{r} \mu_i(z)\overline{\boldsymbol{\theta}}_i^T \boldsymbol{v}_j \leq 0, \quad j = 1, \ldots, N \tag{2.23}$$

$$\overline{\lambda_j} \geq 0, \quad j = 1, \ldots, N. \tag{2.24}$$

In the above minimization problem, $\inf(g(\boldsymbol{v}_j))$ and $\sup(g(\boldsymbol{v}_j))$ are considered to minimize the difference between the bounds of the family of functions and the fuzzy model. The identification results for the constructed interval fuzzy model with membership function set – I are shown in Fig. 2.2(b). The solid line represents the upper and lower fuzzy model while the dashed set of lines represents the family of functions $\mathcal{G}$ for some values of γ. The approximation errors $\sup_{g\in\mathcal{G}}(\underline{\psi}(v) - g(v))$ and $\inf_{g\in\mathcal{G}}(\overline{\psi}(v) - g(v))$ are presented in Fig. 2.3(b). Similarly, for the membership function set – II, the results and the approximation error obtained by using the identification method in [1] are shown in Figs. 2.4 (b) and 2.5 (b) respectively. It is observed that the approximation error is very high in the region near the origin. When compared to the results of Škrjanc et al. [1], the upper and lower fuzzy model obtained using the proposed linear programming problem enclose the region closely and it is observed that the model approximates the family of functions in a better way (less conservative result).

The objective function (l_1 - norm) considered in this example is different from the objective function used in [1]. The results with the objective function (l_∞ - norm) in [1] are checked and in this case also the error is very high in the region around the origin.

Comparison of Computational Complexity and Time

The number of linear inequality conditions for solving the problem by the proposed approach is $3N + rn_v$ and for the approach in [1] (with l_∞ - norm), it is $4N + 2$. For the above example with membership function set – I and II, the number of linear inequality conditions in the proposed linear programming problem are $3 \times 95 + 8 = 293$ and $3 \times 95 + 7 = 292$ respectively. With the approach in [1] the number of linear inequality conditions is $4 \times 95 + 2 = 382$ for both set – I and II. The CPU times for solving the proposed linear programming problem for the membership function shown in

Fig. 2.1 are 3.12 sec and 3.04 sec for set - I and set - II respectively. For the approach in [1] with l_∞ - norm, the CPU times are 3.35 sec and 3.32 sec for membership function set - I and set - II respectively.

2.3.3 Lower and Upper Bounds Estimation of Interval Valued Function

The linear programming problem explained in the previous subsection for interval fuzzy model identification can be solved only if the lower bound $\underline{g}(v_j)$ and upper bound $\overline{g}(v_j)$ of the interval valued function $\underline{\overline{g}}(v_j)$ are known. Depending upon the uncertain nonlinear system, the interval valued function can be linear or nonlinear. If it is linear, it can be solved easily by inequality constrained linear programming approach. In the case of nonlinear interval valued functions, the bounds can be estimated from the methods addressed in the literature [98–101].

Some of the basic interval arithmetic rules involved in interval computation on $\underline{\overline{a}}$ and $\underline{\overline{b}}$ are listed below:

$$\text{Addition: } \underline{\overline{a}} + \underline{\overline{b}} = [\underline{a} + \underline{b}, \;\; \overline{a} + \overline{b}] \tag{2.25}$$

$$\text{Subtraction: } \underline{\overline{a}} - \underline{\overline{b}} = [\underline{a} - \overline{b}, \;\; \overline{a} - \underline{b}] \tag{2.26}$$

$$\text{Multiplication: } \underline{\overline{a}} \cdot \underline{\overline{b}} = [\inf(\underline{a}\underline{b}, \;\; \underline{a}\overline{b}, \;\; \overline{a}\underline{b}, \;\; \overline{a}\overline{b}), \;\; \sup(\underline{a}\underline{b}, \;\; \underline{a}\overline{b}, \;\; \overline{a}\underline{b}, \;\; \overline{a}\overline{b})] \tag{2.27}$$

$$\text{Division: } \underline{\overline{a}} / \underline{\overline{b}} = [\underline{a}, \;\; \overline{a}] \cdot [1/\underline{b}, \;\; 1/\overline{b}], 0 \notin \underline{\overline{b}} \tag{2.28}$$

Division by an interval containing zero is not defined under the basic interval arithmetic. For division by an interval including zero, the operation is carried out with multi-intervals by defining $1/[\underline{b}, 0] = [-\infty, 1/\underline{b}]$ and $1/[0, \overline{b}] = [1/\overline{b}, \infty]$.

The bounds (with some overestimation) of the interval valued function can be calculated using the above interval arithmetic rules. Closer bounds for the interval function can be found by using a branch and bound technique [98–101]. The bounds of the interval valued function can be found by using the interval arithmetic rules combined with branch and bound technique which split boxes adaptively until the overestimation becomes insignificant. The main advantage of this technique is that the bounds obtained are global, but this technique suffers from more computation time and hence, the suitability of this approach is restricted to small dimensional problems. It is also possible to find the bounds for the interval function involving nonlinear operations on interval $\underline{\overline{a}}$ like $\sin(\underline{\overline{a}})$, $\cos(\underline{\overline{a}})$, etc. The readers are referred to the literature [98–101] for more details on interval analysis using a branch and bound algorithm.

2.4 Application to Robust Fuzzy Control

The interval fuzzy model obtained using the linear programming method explained in the previous section is not suitable for controller design for robust fuzzy control. The uncertain fuzzy model employed in robust fuzzy control must be expressed in the form given by (2.2) and (2.4).

Let us consider the following nonlinear system described by

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{g}_x(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{w}(t), \boldsymbol{\gamma}(t)) \\ \boldsymbol{y}(t) &= \boldsymbol{g}_y(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{w}(t), \boldsymbol{\gamma}(t)), \end{aligned} \tag{2.29}$$

where $\boldsymbol{x}(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ is the state vector, $\boldsymbol{u}(t) = [u_1(t), \ldots, u_{m_u}(t)]^T \in \mathbb{R}^m$ is the input vector, $\boldsymbol{y}(t) = [y_1(t), \ldots, y_q(t)]^T \in \mathbb{R}^q$ is the output vector, $\boldsymbol{w}(t) = [w_1(t), \ldots, w_{m_w}(t)]^T \in \mathbb{R}^{m_w}$ is the disturbance input vector, $\boldsymbol{\gamma}(t) = [\gamma_1(t), \ldots, \gamma_{n_\gamma}(t)]^T \in \mathbb{R}^{n_\gamma}$ is the uncertain vector with known lower bound $\underline{\gamma}_k$ and upper bound $\overline{\gamma}_k$ satisfying $\underline{\gamma}_k \leq \gamma_k(t) \leq \overline{\gamma_k}$, $(k = 1, \ldots, n_\gamma)$.

With $\boldsymbol{v}(t) = [\boldsymbol{x}^T(t), \boldsymbol{u}^T(t), \boldsymbol{w}^T(t)]^T$, the functions in (2.29) can be expressed in the following form

$$\boldsymbol{g}(\boldsymbol{v}(t), \boldsymbol{\gamma}(t)) = \begin{bmatrix} \boldsymbol{g}_x(\boldsymbol{v}(t), \boldsymbol{\gamma}(t)) \\ \boldsymbol{g}_y(\boldsymbol{v}(t), \boldsymbol{\gamma}(t)) \end{bmatrix}. \tag{2.30}$$

Suppose $\boldsymbol{g}(\boldsymbol{v}(t), \boldsymbol{\gamma}(t))$ is written as $[g_1(\boldsymbol{v}(t), \boldsymbol{\gamma}(t)), \ldots, g_{(n+q)}(\boldsymbol{v}(t), \boldsymbol{\gamma}(t))]^T$ and satisfying the condition

$$g_l(\boldsymbol{0}, \boldsymbol{\gamma}(t)) = 0, \qquad l = 1, \ldots, n+q \tag{2.31}$$

then according to the concept presented in [102], $g_l(\boldsymbol{v}(t), \boldsymbol{\gamma}(t))$, $l = 1, \ldots, n+q$ can be approximated by an interval fuzzy model of the form

$$\psi_l(\boldsymbol{v}, t) = \sum_{i=1}^{r} \mu_i(z(t))(\boldsymbol{\theta}_{li} + \Delta\boldsymbol{\theta}_{li}(t))^T \boldsymbol{v}, \quad l = 1, \ldots, n+q. \tag{2.32}$$

With (2.30) and (2.32), the overall fuzzy model for the nonlinear system (2.29) can be written as follows:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= [\psi_1(\boldsymbol{v}, t), \ldots, \psi_n(\boldsymbol{v}, t)] \\ \boldsymbol{y}(t) &= [\psi_{n+1}(\boldsymbol{v}, t), \ldots, \psi_{n+q}(\boldsymbol{v}, t)]. \end{aligned} \tag{2.33}$$

The terms $\boldsymbol{\theta}_{li}$ and $\Delta\boldsymbol{\theta}_{li}(t)$ corresponding to (2.32) and (2.33) represent the system matrices and uncertain matrices of the fuzzy model (2.2) and are given by

$$\boldsymbol{\theta}_{xi} = \left[\boldsymbol{\theta}_{1i}^T, \ldots, \boldsymbol{\theta}_{ni}^T\right]^T = [\boldsymbol{A}_i \;\; \boldsymbol{B}_{ui} \;\; \boldsymbol{B}_{wi}],$$

$$\begin{aligned}
\boldsymbol{\theta}_{yi} &= \left[\boldsymbol{\theta}_{(n+1)i}^T, \ldots, \boldsymbol{\theta}_{(n+q)i}^T\right]^T = [\boldsymbol{C}_i \ \ \boldsymbol{D}_{ui} \ \ \boldsymbol{D}_{wi}], \\
\Delta\boldsymbol{\theta}_{xi}(t) &= \left[\Delta\boldsymbol{\theta}_{1i}^T(t), \ldots, \Delta\boldsymbol{\theta}_{ni}^T(t)\right]^T = [\Delta\boldsymbol{A}_i(t) \ \ \Delta\boldsymbol{B}_{ui}(t) \ \ \Delta\boldsymbol{B}_{wi}(t)], \\
\Delta\boldsymbol{\theta}_{yi}(t) &= \left[\Delta\boldsymbol{\theta}_{(n+1)i}^T(t), \ldots, \Delta\boldsymbol{\theta}_{(n+q)i}^T(t)\right]^T = [\Delta\boldsymbol{C}_i(t) \ \ \Delta\boldsymbol{D}_{ui}(t) \ \ \Delta\boldsymbol{D}_{wi}(t)].
\end{aligned} \tag{2.34}$$

The parameters $\boldsymbol{\theta}_{xi}$, $\boldsymbol{\theta}_{yi}$ and the bounds $\delta\boldsymbol{\theta}_{xi}$, $\delta\boldsymbol{\theta}_{yi}$ of $\Delta\boldsymbol{\theta}_{xi}(t)$ and $\Delta\boldsymbol{\theta}_{yi}(t)$ can be found from the linear programming problem explained in the previous section. But the uncertain term needs to be expressed in a special form shown in (2.4) for application in robust fuzzy control. Let us assume that the matrices in the right side of the equation (2.4) take the following form

$$\boldsymbol{M}_{xi} = \begin{bmatrix} \boldsymbol{m}_{xi1} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{m}_{xi2} & & \boldsymbol{0} \\ \vdots & & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{m}_{xin} \end{bmatrix}, \quad \boldsymbol{M}_{yi} = \begin{bmatrix} \boldsymbol{m}_{yi1} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{m}_{yi2} & & \boldsymbol{0} \\ \vdots & & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{m}_{yiq} \end{bmatrix}, \tag{2.35}$$

$$\begin{aligned}
&\boldsymbol{F}_{xi}(t) = \mathrm{diag}(f_{xi11}(t), f_{xi12}(t), \ldots, f_{xijk}(t), \ldots),\ j = 1, \ldots, n\ k = 1, \ldots, (n + m_u + m_w), \\
&\boldsymbol{F}_{yi}(t) = \mathrm{diag}(f_{yi11}(t), f_{yi12}(t), \ldots, f_{yijk}(t), \ldots),\ j = 1, \ldots, q,\ k = 1, \ldots, (n + m_u + m_w),
\end{aligned} \tag{2.36}$$

$$\begin{aligned}
\boldsymbol{N}_{xi} &= [\boldsymbol{N}_{x1i} \ \ \boldsymbol{N}_{x2i} \ \ \boldsymbol{N}_{x3i}] = [\boldsymbol{n}_{xi1} \ \ \boldsymbol{n}_{xi2} \ \ \cdots \ \ \boldsymbol{n}_{xin}]^T, \\
\boldsymbol{N}_{yi} &= [\boldsymbol{N}_{y1i} \ \ \boldsymbol{N}_{y2i} \ \ \boldsymbol{N}_{y3i}] = [\boldsymbol{n}_{yi1} \ \ \boldsymbol{n}_{yi2} \ \ \cdots \ \ \boldsymbol{n}_{yin}]^T,
\end{aligned} \tag{2.37}$$

where $\boldsymbol{m}_{xij} = [m_{xij1} \ \ m_{xij2} \ \ \cdots \ \ m_{xij(n+m_u+m_w)}]$ and $\boldsymbol{m}_{yij} = [m_{yij1} \ \ m_{yij2} \ \ \cdots \ \ m_{yij(n+m_u+m_w)}]$. The vector $\boldsymbol{0}$ has a length $n+m_u+m_w$ with all zero entries. The entries of matrices in (2.36) satisfy the condition $|f_{xijk}(t)| \leq 1$, and $|f_{yijk}(t)| \leq 1$. In (2.37), $\boldsymbol{n}_{xij} = \mathrm{diag}(n_{xij1}, n_{xij2}, \ldots, n_{xijn})$ and $\boldsymbol{n}_{yij} = \mathrm{diag}(n_{yij1}, n_{yij2}, \ldots, n_{yijn})$. The above form of expressing the matrices of the uncertain term in the fuzzy model is inspired from [103].

With (2.35), (2.36), (2.37) and defining $l \in \{x, y\}$, the matrices $\Delta\boldsymbol{\theta}_{xi}(t)$ and $\Delta\boldsymbol{\theta}_{yi}(t)$ can be expressed as $\Delta\boldsymbol{\theta}_{li}(t) = \boldsymbol{M}_{li}\boldsymbol{F}_{li}(t)\boldsymbol{N}_{li}$ and

$$\Delta\boldsymbol{\theta}_{li}(t) = \begin{bmatrix} \Delta\theta_{li11}(t) & \Delta\theta_{li12}(t) & \cdots & \Delta\theta_{li1k}(t) & \cdots \\ \Delta\theta_{li21}(t) & \Delta\theta_{li22}(t) & \cdots & \Delta\theta_{li2k}(t) & \cdots \\ \vdots & \vdots & \ddots & & \\ \Delta\theta_{lij1}(t) & \Delta\theta_{lij2}(t) & & \Delta\theta_{lijk}(t) & \\ \vdots & \vdots & & & \ddots \end{bmatrix} \quad \begin{aligned} &k = 1, \ldots, (n + m_u + m_w), \\ &l = x, j = 1, \ldots, n \text{ for } \Delta\theta_{xi}(t), \\ &l = y, j = 1, \ldots, q \text{ for } \Delta\theta_{yi}(t), \end{aligned} \tag{2.38}$$

where $\Delta\theta_{lijk}(t) = m_{lijk} f_{lijk}(t) n_{lijk}$.

Let us assume that m_{lijk} and n_{lijk} take only positive values; then the lower and upper bounds of $m_{lijk}f_{lijk}(t)n_{lijk}$ are given by $-m_{lijk}n_{lijk}$ and $m_{lijk}n_{lijk}$. The band of $\Delta\theta_{lijk}(t)$ can always be found between the lower and upper bounds of $m_{lijk}f_{lijk}(t)n_{lijk}$, if m_{lijk} and n_{lijk} satisfy the following relation,

$$-m_{lijk}n_{lijk} \leq \Delta\theta_{lijk}(t) \leq m_{lijk}n_{lijk}. \tag{2.39}$$

With the above inequality, the relation between the lower and upper bounds of $\Delta\theta_{lijk}(t)$ and $m_{lijk}f_{lijk}(t)n_{lijk}$ is given by

$$\begin{array}{rclcl} \max\left(\Delta\theta_{lijk}(t)\right) & = & \delta\theta_{lijk} & = & m_{lijk}n_{lijk}, \\ \min\left(\Delta\theta_{lijk}(t)\right) & = & -\delta\theta_{lijk} & = & -m_{lijk}n_{lijk}. \end{array} \tag{2.40}$$

Here $\delta\theta_{lijk} \geq 0$. To satisfy the above condition in (2.40), $m_{lijk} = n_{lijk} = \sqrt{\delta\theta_{lijk}}$ is assumed.

Let us consider the uncertain terms in the results of robust fuzzy control [2, 62, 66, 93–96, 104]. The uncertainties terms are eliminated by using in an one of the following inequality

$$\begin{array}{rcl} \Delta\boldsymbol{A}^T(\mu,t)\boldsymbol{P} + \boldsymbol{P}\Delta\boldsymbol{A}(\mu,t) & < & \varepsilon\boldsymbol{N}_{x1}^T(\mu)\boldsymbol{N}_{x1}(\mu) + \dfrac{1}{\varepsilon}\boldsymbol{P}\boldsymbol{M}_x(\mu)\boldsymbol{M}_x^T(\mu)\boldsymbol{P}, \\ \text{or}\ \ \Delta\boldsymbol{A}^T(\mu)\boldsymbol{P} + \boldsymbol{P}\Delta\boldsymbol{A}(\mu) & < & \boldsymbol{N}_{x1}^T(\mu)\boldsymbol{N}_{x1}(\mu) + \boldsymbol{P}\boldsymbol{M}_x(\mu)\boldsymbol{M}_x^T(\mu)\boldsymbol{P} \end{array} \tag{2.41}$$

where $\boldsymbol{P}$ is the Lyapunov variable and ε is some positive scalar. Here $\Delta\boldsymbol{A}(\mu,t) = \sum_{i=1}^{n}\mu_i\Delta\boldsymbol{A}_i(t)$, $\boldsymbol{M}_x(\mu) = \sum_{i=1}^{n}\mu_i\boldsymbol{M}_{xi}$ and $\boldsymbol{N}_{x1}(\mu) = \sum_{i=1}^{n}\mu_i\boldsymbol{N}_{x1i}$. Other uncertain terms ($\Delta\boldsymbol{B}_{ui}(\mu)$, $\Delta\boldsymbol{B}_{wi}(\mu)$, $\Delta\boldsymbol{C}_i(\mu)$, $\Delta\boldsymbol{D}_{ui}(\mu)$, $\Delta\boldsymbol{D}_{wi}(\mu)$) are also transformed in a similar way. The Lyapunov variable $\boldsymbol{P}$ is a control design variable and usually it is not available at the time of modeling. The assumption $h_{lijk} = e_{lijk} = \sqrt{\delta\theta_{lijk}}$ may yield a conservative result while designing the controller. Hence, another variable $\boldsymbol{\epsilon}_i$ is introduced and the robust stability condition is shown in the next subsection.

2.4.1 Robust Stability Condition

Let us consider a nonlinear system which can be described by the T-S fuzzy model given below:

Plant rule i:

IF $z_1(t)$ is N_{i1} and $\cdots$ $z_p(t)$ is N_{ip}, *THEN*

$$\begin{array}{rcl} \dot{\boldsymbol{x}}(t) & = & (\boldsymbol{A}_i + \Delta\boldsymbol{A}_i)\boldsymbol{x}(t) + (\boldsymbol{B}_{ui} + \Delta\boldsymbol{B}_{ui})\boldsymbol{u}(t) + \boldsymbol{B}_{wi}\boldsymbol{w}(t) \\ \boldsymbol{y}(t) & = & \boldsymbol{C}_i\boldsymbol{x}(t), \qquad i = 1, 2, \ldots, r, \end{array} \tag{2.42}$$

where N_{ij} is the fuzzy set, $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state vector, $\boldsymbol{u}(t) \in \mathbb{R}^{m_u}$ is the control input, $\boldsymbol{w}(t) \in \mathbb{R}^{m_w}$ is the unknown but bounded disturbance input and $\boldsymbol{y}(t) \in \mathbb{R}^q$ is the output vector. Here, $\boldsymbol{A}_i \in \mathbb{R}^{n\times n}$,

$\boldsymbol{B}_{ui} \in \mathbb{R}^{n\times m_u}$, $\boldsymbol{B}_{wi} \in \mathbb{R}^{n\times m_w}$ and $\boldsymbol{C}_i \in \mathbb{R}^{q\times n}$, $\Delta\boldsymbol{A}_i(t)$ and $\Delta\boldsymbol{B}_{ui}(t)$ are time-varying matrices with appropriate dimensions and $z_1(t), z_2(t), \ldots, z_p(t)$ are premise variables. The parameters of the uncertain matrices are assumed to be expressed in the form given in (2.35)–(2.37).

Given a pair of $(\boldsymbol{x}(t), \boldsymbol{u}(t))$, the final output of the fuzzy system is inferred as follows:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \sum_{i=1}^{r}\mu_i(z(t))[(\boldsymbol{A}_i+\Delta\boldsymbol{A}_i(t))\boldsymbol{x}(t)+(\boldsymbol{B}_{ui}+\Delta\boldsymbol{B}_{ui}(t))\boldsymbol{u}(t)+\boldsymbol{B}_{wi}\boldsymbol{w}(t)]\\ \boldsymbol{y}(t) &= \sum_{i=1}^{r}\mu_i(z(t))\boldsymbol{C}_i\boldsymbol{x}(t), \end{aligned} \tag{2.43}$$

where

$$\mu_i(z(t)) = \frac{\omega_i(z(t))}{\sum_{i=1}^{r}\omega_i(z(t))}, \quad \omega_i(z(t)) = \prod_{j=1}^{p} N_{ij}(z_j(t)),$$

and $N_{ij}(z_j(t))$ is the degree of the membership of $z_j(t)$ in N_{ij}. Therefore, $\mu_i(z(t)) \geq 0$, for $i = 1, 2, ..., r$ and $\sum_{i=1}^{r}\mu_i(z(t)) > 0$ for all t.

Suppose the following fuzzy control rule is employed to stabilize the system represented by (2.43).

Control Rule i:

IF $z_1(t)$ is N_{i1} and $\ldots z_p(t)$ is N_{ip}, *THEN*

$$\boldsymbol{u}(t) = \boldsymbol{K}_i\boldsymbol{x}(t), \quad i = 1, 2, \ldots, r. \tag{2.44}$$

Then the overall fuzzy control law is represented by

$$\boldsymbol{u}(t) = \sum_{i=1}^{r}\mu_i(z(t))\boldsymbol{K}_i\boldsymbol{x}(t) \tag{2.45}$$

where $\boldsymbol{K}_i$ is the control gain.

The stability condition for the robust fuzzy state feedback control design is presented in the following theorem:

Theorem 2.1. Consider the fuzzy model (2.43) with the T-S state feedback control law (2.45). If there exists a symmetric and positive definite matrix $\boldsymbol{P}$, diagonal matrices $\boldsymbol{\epsilon}_i$, some matrices $\boldsymbol{K}_j$, $(j = 1, 2, \ldots, r)$ such that the following parameterized matrix inequality is satisfied:

$$\begin{aligned}\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big\{&(\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{K}_j)^T\boldsymbol{P}+(*)+\boldsymbol{P}\boldsymbol{M}_{xi}\boldsymbol{\epsilon}_i\boldsymbol{M}_{xi}^T\boldsymbol{P}\\ &+(\boldsymbol{N}_{x1i}+\boldsymbol{N}_{x2i}\boldsymbol{K}_j)^T\boldsymbol{\epsilon}_i^{-1}(\boldsymbol{N}_{x1i}+\boldsymbol{N}_{x2i}\boldsymbol{K}_j)\Big\} \leq \boldsymbol{0}\end{aligned} \tag{2.46}$$

then the uncertain nonlinear system represented by (2.43) is globally stable.

Proof: The proof is given in Appendix. □

The above theorem produces the basic stability condition for an uncertain fuzzy system. Several approaches were discussed in the literature for solving these parametric inequality based problems with less conservative results [105–108].

Based on the concepts presented so far, the following algorithm is proposed for identification and robust fuzzy control design of uncertain nonlinear systems.

Algorithm 2.1: Given the nonlinear equation of the form (2.29) representing the dynamics of the nonlinear system,

Step 1: Select the membership functions, premise and antecedent variables.

Step 2: Construct the vector $\boldsymbol{V} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_N\}$ that represents the input-measurement data. Find the upper and lower bounds of the nonlinear equation (2.29) with the concepts presented in Section 2.3.3.

Step 3: With the input-measurement data $\boldsymbol{V}$ and the bounds of the nonlinear equation obtained in the previous step, solve the linear programming problem (2.17) to find the parameters $\boldsymbol{\theta}_{1i}, \ldots, \boldsymbol{\theta}_{(n+q)i}$ and $\delta\boldsymbol{\theta}_{1i}, \ldots, \delta\boldsymbol{\theta}_{(n+q)i}$ corresponding to the nonlinear equations $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_{n+q}$. If the nonlinear programming problem is not feasible, go to Step 6.

Step 4: Arrange the elements of $\boldsymbol{\theta}_{1i}, \ldots, \boldsymbol{\theta}_{(n+q)i}$ and $\delta\boldsymbol{\theta}_{1i}, \ldots, \delta\boldsymbol{\theta}_{(n+q)i}$ in the form shown in (2.34) to get the system matrices and the bounds of the uncertain matrices.

Step 5: With $h_{lijk} = e_{lijk} = \sqrt{\delta\theta_{lijk}}$, construct the matrices $\boldsymbol{M}_{xi}$, $\boldsymbol{M}_{yi}$, $\boldsymbol{N}_{xi}$ and $\boldsymbol{N}_{yi}$ in the form shown in (2.35) and (2.37). Design the robust fuzzy controller using the concepts presented in Section 2.4.1. If the LMIs for the robust fuzzy control design is feasible, goto Step 7.

Step 6: Check for maximum number of iterations and adjust the membership functions, rules and the premise and antecedent variables and goto Step 3.

Step 7: Stop.

If the above algorithm produces infeasible solution in all the iterations, then the proposed method cannot be applied to derive a fuzzy model for designing a robust fuzzy controller.

Remark 2.3. The vector $\boldsymbol{g}(\boldsymbol{v}(t), \boldsymbol{\gamma}(t))$ contains $n+q$ individual equations and the nonlinearity may not be present in all individual equations. Some equations may be represented as linear combinations of the antecedent variables, e.g., $\dot{x}_2(t) = x_1(t)+x_2(t)$. The sub-models corresponding to these equations can be directly written without solving the linear programming problems. Hence, these simple linear equations can be eliminated and only the nonlinear equations can be considered while identifying the consequent part.

2.5 Illustrative Examples

2.5.1 Inverted Pendulum on a Cart

Let us consider an example of a nonlinear equation with parameter uncertainties given by the equation of motion of an inverted pendulum on a cart [66, 109]:

$$
\begin{aligned}
\dot{x_1}(t) &= x_2(t) \\
\dot{x_2}(t) &= \frac{g_r \sin(x_1(t)) - amlx_2^2(t)\sin(2x_1(t))/2 - a\cos(x_1(t))u(t)}{4l/3 - aml\cos^2(x_1(t))} \\
y(t) &= x_1(t).
\end{aligned} \tag{2.47}
$$

Here $x_1(t)$ and $x_2(t)$ represent the angular displacement about the vertical axis (in rad) and the angular velocity (in rad/sec) respectively, $g_r = 9.8$ m/s^2 is the acceleration due to gravity, $a = 1/(m+M)$, $m \in [m_{\min}\ \ m_{\max}] = [2\ \ 3]$ kg is the mass of the pendulum, $M \in [M_{\min}\ \ M_{\max}] = [8\ \ 10]$ kg is the mass of the cart, $2l = 1$ m is the length of the pendulum and u is the force applied on the cart (in Newton). The actuator dynamics are ignored.

The operating domain is considered as $x_1(t) \in [-\pi/3\ \ \pi/3]$, $x_2(t) \in [-4\ \ 4]$ and the input $u(t) \in [-1000\ \ 1000]$. The membership functions and the rules are as given below.

Plant rule i:

IF x_1 is about N_i THEN

$$
\begin{aligned}
\dot{\boldsymbol{x}}(t) &= (\boldsymbol{A}_i + \Delta\boldsymbol{A}_i)\boldsymbol{x}(t) + (\boldsymbol{B}_{ui} + \Delta\boldsymbol{B}_{ui})u(t) \\
y(t) &= \boldsymbol{C}_i\boldsymbol{x}(t),
\end{aligned}
$$

where N_i is the triangular fuzzy set of x_1 about $0, \pm\pi/12, \pm\pi/6, \pm\pi/4, \pm\pi/3$ for $i = 1, 2, ..., 5$ respectively.

In the equation of motion of the nonlinear system given by (2.47), $\dot{x_1}(t)$ and $y(t)$ are linear equations and the consequent part corresponding to these functions can be written directly. Only the consequent part corresponding to $\dot{x}_2(t)$ needs to be identified. The matrices in the consequent part take the following form:

$$
\begin{aligned}
\boldsymbol{A}_i &= \begin{bmatrix} 0 & 1 \\ a_{i21} & a_{i22} \end{bmatrix}, \quad \boldsymbol{B}_{ui} = \begin{bmatrix} 0 \\ b_{ui21} \end{bmatrix}, \\
\Delta\boldsymbol{A}_i(t) &= \begin{bmatrix} 0 & 0 \\ \Delta a_{i21}(t) & \Delta a_{i22}(t) \end{bmatrix}, \quad \Delta\boldsymbol{B}_{ui}(t) = \begin{bmatrix} 0 \\ \Delta b_{ui21}(t) \end{bmatrix}, \quad \boldsymbol{C}_i = \begin{bmatrix} 1 & 0 \end{bmatrix}.
\end{aligned}
$$

With the uncertainty in m and M, the interval equation for estimating the bounds of $\dot{x}_2(t)$ can

Table. 2.1: Parameters of matrices $\boldsymbol{A}_i$, $\delta\boldsymbol{A}_i$, $\boldsymbol{B}_{ui}$ and $\delta\boldsymbol{B}_{ui}$ – Inverted pendulum on a cart

Rule i	a_{i21}	a_{i22}	b_{ui21}	δa_{i21}	δa_{i22}	δb_{ui21}
$i=1$	16.2703	0	-0.1590	5.6613	0.0062	0.0195
$i=2$	15.5785	0	-0.1516	3.3232	0.0076	0.0179
$i=3$	14.9298	0	-0.1313	1.7417	0.0204	0.0158
$i=4$	13.5178	0	-0.1022	1.2361	0.0090	0.0127
$i=5$	12.1137	0	-0.0692	0.811	0.0040	0.0087

Table. 2.2: Parameters of the feedback gain matrices $\boldsymbol{K}_i$ - Inverted pendulum on a cart

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
K_{i1}	780.0632	800.652	799.3351	861.6303	986.8387
K_{i2}	244.1326	250.8751	247.0909	266.3371	304.1544

be obtained as shown below:

$$\underline{g} = \frac{g_r \sin(x_1) - (\overline{m}/(\overline{m}+\overline{M}))lx_2^2 \sin(2x_1)/2 - (1/(\overline{m}+\overline{M}))\cos(x_1)u}{4l/3 - (\overline{m}/(\overline{m}+\underline{M}))l\cos^2(x_1)}. \tag{2.48}$$

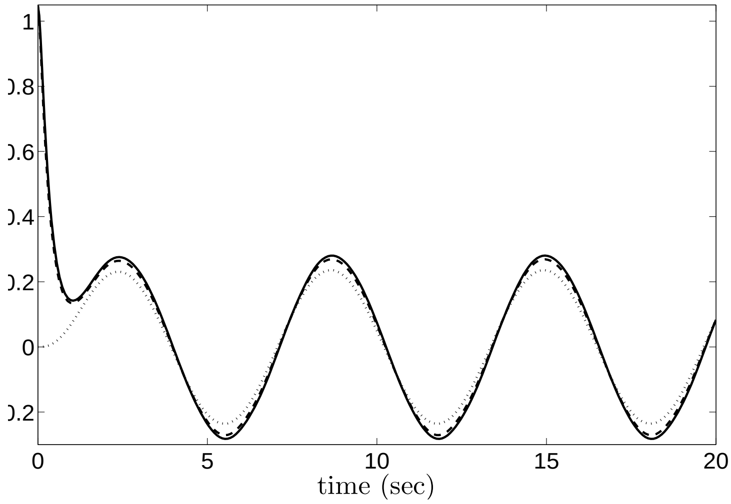
With the guidelines given in Section 2.3, the input data is obtained as $\boldsymbol{V} = \{x_{1i}|x_{1i} = 0.0698l_1, l_1 = -15, -14, \ldots, 15, x_{2j}|x_{2j} = 2l_2, l_2 = -2, -1, \ldots, 2, u_k|u_k = 500l_3, l_3 = -2, -1, \ldots, 2\} \subset \mathbf{S}$ and the output data is estimated from the interval valued function (2.48).

The linear programming problems are solved using SeDuMi [56] with YALMIP [110] interface. The uncertain fuzzy model is obtained by the algorithm presented in Section 2.4 and the parameters of the matrices $\boldsymbol{A}_i$, $\boldsymbol{B}_{ui}$, $\delta\boldsymbol{A}_i$ and $\delta\boldsymbol{B}_{ui}$ are tabulated in Table 2.1. All the computations are performed on a Pentium IV 3.4 GHz processor with 1 GB RAM. The computation time for finding the bounds of the function is 69.29 sec and the linear programming problem for finding the nominal model and the bounds of the fuzzy model is solved in 33.25 sec.

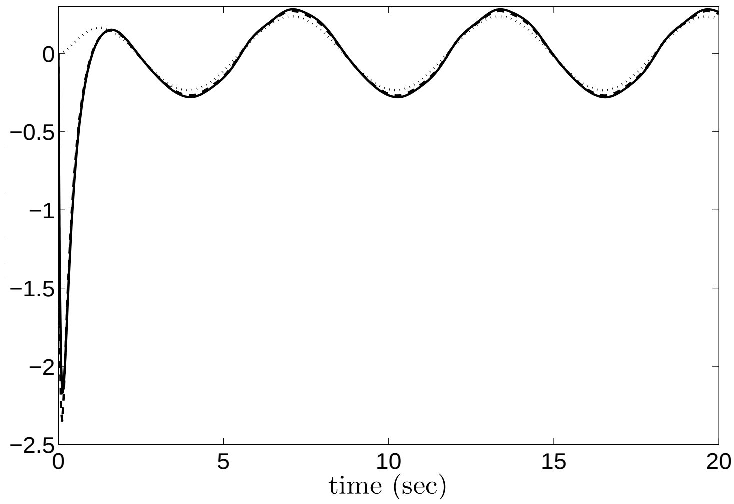
The robust H_∞ tracking control problem with the following reference model and the reference input as given in [66] is considered:

$$\begin{bmatrix} \dot{x}_{r1}(t) \\ \dot{x}_{r2}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} x_{r1}(t) \\ x_{r2}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix}. \tag{2.49}$$

The H_∞ tracking controller is designed with the concept presented in [66] with the disturbance input $\boldsymbol{w} = [0 \quad \Delta g \sin(x_1)/(4l/3 - aml\cos^2(x_1)]^T$, where $\Delta g = g(6370 \times 10^3/6370 \times 10^3 + H), H \in [0, 100]$. The mass of the pendulum and the cart are considered as $m + \Delta m(t) = 2.5 + 0.5\sin(5t)$ and $M + \Delta M(t) = 9 + \sin(4t)$ respectively. The LMI optimization problem is solved using the LMI relaxations shown in Lemma A.1 (Appendix). The parameters of the feedback gain matrices $\boldsymbol{K}_i = [K_{i1} \quad K_{i2}]$ obtained are shown in Table 2.2. The simulation results with the initial condition $x(0) = [\pi/3 \quad 0]^T$ and $x_r(0) = [0 \quad 0]^T$ are shown in Fig. 2.6.



(a) $x_1(t)$ and $x_{r1}(t)$



(b) $x_2(t)$ and $x_{r2}(t)$

Fig. 2.6: Trajectories of state variables $\boldsymbol{x}(t)$ (dashed line: $m = 2$ kg and $M = 8$ kg, solid line: $m = 3$ kg and $M = 10$ kg) and the reference trajectories $\boldsymbol{x}_r(t)$ (dotted line) for an inverted pendulum.

Comparison:

In the example presented in Section 2.3.2, the proposed method was compared with the method in [1] for an uncertain nonlinear function. In this subsection, comparison of the proposed identification method is made with the identification method in [1] for the nonlinear system which is an Inverted pendulum on a cart. The parameters obtained by the proposed method for the homogenous fuzzy model are shown in Table 2.1. But in the case of [1], the linear programming problem provides an infeasible solution for this example (inverted pendulum on a cart).

Next, an affine fuzzy modeling problem is considered for building a fuzzy model of the same nonlinear system (inverted pendulum on a cart) with plant rules in the following form:

Table. 2.3: Parameters of matrices $\boldsymbol{a}_{0i}$, $\delta\boldsymbol{a}_{0i}$, $\boldsymbol{A}_i$, $\boldsymbol{B}_i$, $\delta\boldsymbol{A}_i$ and $\delta\boldsymbol{B}_i$ by the proposed method – Inverted pendulum on a cart (affine fuzzy model)

Rule i	a_{20i}	a_{21}	a_{22}	b_{21}	δa_{20i}	δa_{21}	δa_{22}	δb_{21}
1	0	16.1512	0	-0.159	0.9193	2.2547	0.00795	0.01851
2	0	15.6940	0	-0.1516	0.8973	0.5043	0.00119	0.01780
3	0	14.8059	0	-0.1313	0.7528	0.6499	0.00320	0.01577
4	0	13.5023	0	-0.1022	0.5162	0.7425	0.00173	0.01257
5	0	12.1046	0	-0.0692	0.4747	0.5381	0.01352	0.00874

Table. 2.4: Parameters of matrices of lower and upper fuzzy model by the method in [1] – Inverted pendulum on a cart (affine fuzzy model)

Rule i	$\underline{a}_{20i}$	$\underline{a}_{21}$	$\underline{a}_{22}$	$\underline{b}_{21}$	$\overline{a}_{20i}$	$\overline{a}_{21}$	$\overline{a}_{22}$	$\overline{b}_{21}$
1	-19.279	15.7790	0	-0.1588	19.2785	15.7798	0	-0.1588
2	-18.9947	15.4305	0	-0.1520	18.9953	15.4289	0	-0.1520
3	-18.6561	14.9071	0	-0.1292	18.6558	14.9082	0	-0.1292
4	-16.4212	13.4453	0	-0.1031	16.4215	13.4449	0	-0.1031
5	-14.6017	12.1521	0	-0.0689	14.6018	12.1523	0	-0.0689

Plant rule i:

IF x_1 is about N_i THEN

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= (\boldsymbol{a}_{0i} + \Delta\boldsymbol{a}_{0i}(t)) + (\boldsymbol{A}_i + \Delta\boldsymbol{A}_i(t))\boldsymbol{x}(t) + (\boldsymbol{B}_i + \Delta\boldsymbol{B}_i(t))u(t) \\ y(t) &= \boldsymbol{C}_i\boldsymbol{x}(t)\end{aligned}$$

Here $\boldsymbol{a}_{0i} = [0 \quad a_{20i}]^T$ and $\Delta\boldsymbol{a}_{0i}(t) = [0 \quad \Delta a_{20i}(t)]^T$ are the affine terms of the fuzzy model and the remaining terms are the same as in the homogenous fuzzy model. The parameters of the fuzzy model obtained by the proposed method are shown in Table 2.3.

By the method in [1], the lower fuzzy model (with affine terms) takes the following form:

Plant rule i:

IF x_1 is about N_i THEN

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \underline{\boldsymbol{a}}_{0i} + \underline{\boldsymbol{A}}_i\boldsymbol{x}(t) + \underline{\boldsymbol{B}}_i u(t) \\ y(t) &= \boldsymbol{C}_i\boldsymbol{x}(t)\end{aligned}$$

The upper fuzzy model takes a similar form as the lower fuzzy model. The linear programming problem from [1] is used to identify the parameters of the lower and upper fuzzy model and the results are shown in Table 2.4.

In the robust fuzzy control literature [62, 66, 93–96], the fuzzy controller is designed using the homogenous fuzzy model. In this example, the method in [1] provides an infeasible solution for homogenous fuzzy function identification. Hence a homogenous fuzzy model cannot be derived using

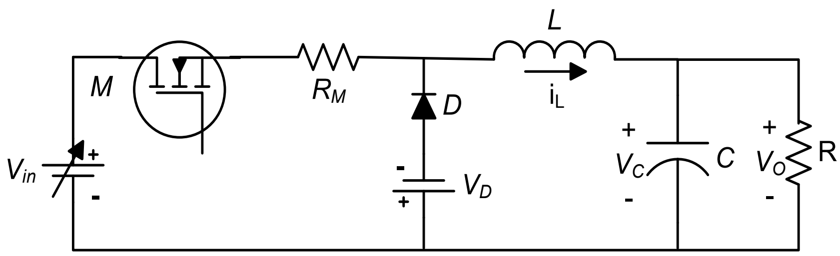


Fig. 2.7: Equivalent circuit of a buck converter

the method in [1]. But the method proposed in this paper provides a feasible solution for fuzzy model identification in homogenous as well as affine forms. Hence the proposed method has better approximation capability than the method in [1].

2.5.2 Regulation of a Pulse-Width Modulated Buck Converter

Next, the example considered is the regulation problem of a pulse-width modulated (PWM) buck converter. The equivalent circuit of a PWM buck converter is shown in Fig. 2.7. The aim of this problem is to regulate the output voltage of the converter by varying its duty ratio. By the averaged modeling method, the dynamic equation for this PWM converter is obtained as [91, 111]:

$$\begin{aligned}
\dot{i}_L(t) &= -\frac{1}{L}v_C(t) - \frac{1}{L}(R_M i_L(t) - V_{in} - V_D)d_u - \frac{V_D}{L} \\
\dot{v}_C(t) &= \frac{1}{C}i_L(t) - \frac{1}{RC}v_C(t)
\end{aligned}$$

where i_L is the inductor current, v_C is the capacitor voltage, $R_M = 0.27\Omega$ is the static drain to source resistance of the power MOSFET, $V_D = 0.82$V is the forward voltage of the power diode and d_u is the duty ratio of the PWM buck converter. Other parameters are $L = 0.09858$mH, $C = 0.2025$mF and $R = 6\Omega$. The input voltage is assumed to have uncertainty as $V_{in} \in [V_{in\,\min} \;\; V_{in\,\max}] = [27 \;\; 33]$V. With $\boldsymbol{x}(t) = [x_1(t) \;\; x_2(t)] = [i_L(t) \;\; v_C(t)]$ and $u(t) = d_u$, the uncertain nonlinear system can be represented as follows:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{RC} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -\frac{1}{L}(R_M x_1(t) - V_{in} - V_D) \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} -\frac{V_D}{L} \\ 0 \end{bmatrix} \tag{2.50}$$

As given in [91], with $x_1(t)$ as the premise variable, the membership functions are defined as $N_1 = (x_1 - \underline{x}_1)/(\overline{x}_1 - \underline{x}_1)$ and $N_2 = 1 - N_1$. $\underline{x}_1$ and $\overline{x}_1$ represent the lower and upper bounds of $x_1(t)$.

Similar to the previous example, the uncertain fuzzy model is obtained by solving the linear programming problem presented in Section 2.3 and the parameters of the fuzzy model are obtained

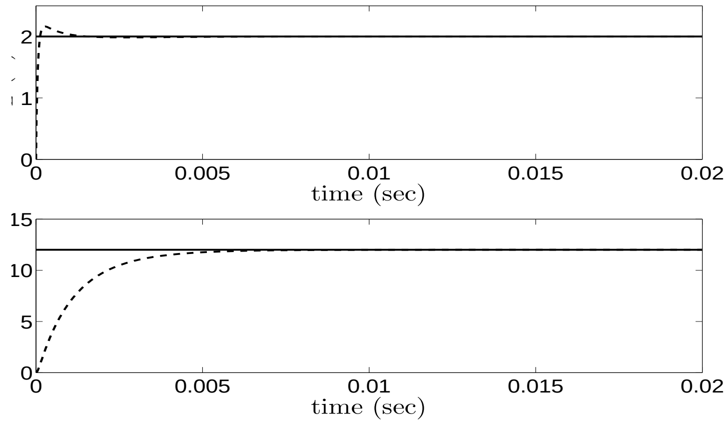


Fig. 2.8: Responses $x_1(t)$ $(i_L(t))$ and $x_2(t)$ $(v_c(t))$ for PWM converter.

as

$$\boldsymbol{A}_1 = 10^3 \times \begin{bmatrix} 0 & -10.14 \\ 4.938 & -0.823 \end{bmatrix}, \quad \boldsymbol{A}_2 = 10^3 \times \begin{bmatrix} 0 & -10.14 \\ 4.938 & -0.823 \end{bmatrix}, \quad \delta\boldsymbol{A}_1 = \boldsymbol{0}, \quad \delta\boldsymbol{A}_2 = \boldsymbol{0},$$

$$\boldsymbol{B}_1 = 10^3 \times \begin{bmatrix} 301.68 \\ 0 \end{bmatrix}, \quad \boldsymbol{B}_2 = 10^3 \times \begin{bmatrix} 312.64 \\ 0 \end{bmatrix}, \quad \delta\boldsymbol{B}_1 = \begin{bmatrix} 211.5 \\ 0 \end{bmatrix}, \quad \delta\boldsymbol{B}_2 = \begin{bmatrix} 209.9 \\ 0 \end{bmatrix}$$

With the above fuzzy model, the set-point tracking control can be designed using the procedure described in [91]. Applying the method described in [91], the state feedback controller for set-point tracking of the PWM converter is obtained as

$$u(t) = -\mu_1 \boldsymbol{K}_1 \begin{bmatrix} x_1 - x_{1d} \\ x_2 - x_{2d} \end{bmatrix} - \mu_2 \boldsymbol{K}_2 \begin{bmatrix} x_1 - x_{1d} \\ x_2 - x_{2d} \end{bmatrix} + \frac{x_{2d} + V_D}{V_{in} + V_D + R_M x_1} \tag{2.51}$$

where $\boldsymbol{K}_1 = [0.0541 \;\; -0.0298]$ and $\boldsymbol{K}_2 = [0.0878 \;\; -0.0339]$ are the controller gains. The parameters of these controller gains are obtained by solving the inequalities given in Theorem A.1 (Appendix) with the terms corresponding to the uncertainty. The responses $x_1(t)$ and $x_2(t)$ are shown in Fig. 2.8.

2.5.3 Translational Oscillator with an Eccentric Rotational Proof Mass Actuator (TORA)

Next, the identification of the translational oscillator with an eccentric rotational proof mass actuator (TORA) [8, 112, 113] shown in Fig. 2.9 is considered. The oscillator consists of a cart of mass M connected by a linear spring. The proof mass actuator affixed to the cart has mass m. The objective of the problem is to damp the base translational oscillations by applying a suitable input. Let $\bar{x}_1(t)$ and $\bar{x}_2(t)$ denote the translational position and velocity of the cart with $\bar{x}_2(t) = \dot{\bar{x}}_1(t)$. Let $\bar{x}_3(t) = \theta(t)$ and $\bar{x}_4(t) = \dot{\bar{x}}_3(t)$ denote the angular position and velocity of the rotational proof mass.

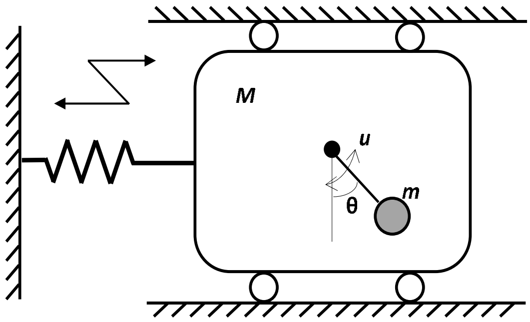


Fig. 2.9: TORA system.

Then the system dynamics can be described by the following equation [8]

$$\dot{\bar{\boldsymbol{x}}}(t) = f(\bar{\boldsymbol{x}}(t)) + g(\bar{\boldsymbol{x}}(t))u(t), \tag{2.52}$$

where u is the torque applied to the eccentric mass and

$$f(\bar{\boldsymbol{x}}(t)) = \begin{bmatrix} \bar{x}_2(t) \\ \dfrac{-\bar{x}_1(t) + \varepsilon\bar{x}_4^2(t)\sin(\bar{x}_3(t))}{1-\varepsilon^2\cos^2(\bar{x}_3(t))} \\ \bar{x}_4(t) \\ \dfrac{\varepsilon\cos(\bar{x}_3(t))(\bar{x}_1(t) - \varepsilon\bar{x}_4^2(t)\sin\bar{x}_3(t))}{1-\varepsilon^2\cos^2(\bar{x}_3(t))} \end{bmatrix}, \quad g(\boldsymbol{x}(t)) = \begin{bmatrix} 0 \\ \dfrac{-\varepsilon\cos(\bar{x}_3(t))}{1-\varepsilon^2\cos^2(\bar{x}_3(t))} \\ 0 \\ \dfrac{1}{1-\varepsilon^2\cos^2(\bar{x}_3(t))} \end{bmatrix},$$

with $\varepsilon = 0.1$ being a constant which depends upon the system parameters.

Let us define the new state variables $z_1(t) = \bar{x}_1(t) + \varepsilon\sin(\bar{x}_3(t))$, $z_2(t) = \bar{x}_2(t) + \varepsilon\bar{x}_4(t)\cos(\bar{x}_3(t))$, $y_1(t) = \bar{x}_3(t)$, $y_2(t) = \bar{x}_4(t)$ and employ the feedback transformation

$$v(t) = \frac{1}{1-\varepsilon^2\cos^2(y_1(t))}\left[\varepsilon\cos(y_1(t))\left(z_1(t) - (1+y_2^2(t))\varepsilon\sin(y_1(t))\right) + u(t)\right] \tag{2.53}$$

to bring the system into the following simpler form:

$$\begin{aligned} \dot{z}_1(t) &= z_2(t), \\ \dot{z}_2(t) &= -z_1(t) + \varepsilon\sin(y_1(t)), \\ \dot{y}_1(t) &= y_2(t), \\ \dot{y}_2(t) &= v(t). \end{aligned} \tag{2.54}$$

The T-S fuzzy model for the TORA system is identified from (2.54) with the fuzzy rules and premise parameters as defined in [8]. Unlike the previous examples, only the approximation error is included in the uncertain blocks of the fuzzy model. With $\boldsymbol{x}(t) = [x_1(t)\ \ x_2(t)\ \ x_3(t)\ \ x_4(t)]^T = [z_1(t)\ \ z_2(t)\ \ y_1(t)\ \ y_2(t)]^T$, the uncertain fuzzy model takes the following form:

Rule 1:

Table. 2.5: Parameters of matrices $\boldsymbol{A}_i$, $\boldsymbol{B}_i$, $\delta\boldsymbol{A}_i$ and $\delta\boldsymbol{B}_i$ – TORA

Rule i	a_{23i}	a_{41i}	a_{43i}	b_{41i}	δa_{23i}	δa_{41i}	δa_{43i}	δa_{44i}	δb_{41i}
1	0.0047	-0.1140	0.0012	1.0095	0.0006	0	0.0055	0	0
2	0.0663	0.0041	0.0061	0.9993	0.0007	0.0038	0.0064	0.0015	0.0007
3	0.1116	0.1132	-0.0127	1.0112	0.0048	0.0105	0.0185	0.0110	0.0027
4	0.0982	0.1114	-0.0627	1.0076	0.0001	0.0004	0.0040	0.0006	0.0006

IF $y_1(t)$ is "about $-\pi$ or π rad" *THEN*

$$\dot{\boldsymbol{x}}(t) = (\boldsymbol{A}_1 + \Delta\boldsymbol{A}_1(t))\boldsymbol{x}(t) + (\boldsymbol{B}_1 + \Delta\boldsymbol{B}_1(t))u(t).$$

Rule 2:

IF $y_1(t)$ is "about $-\pi/2$ or $\pi/2$ rad" *THEN*

$$\dot{\boldsymbol{x}}(t) = (\boldsymbol{A}_2 + \Delta\boldsymbol{A}_2(t))\boldsymbol{x}(t) + (\boldsymbol{B}_2 + \Delta\boldsymbol{B}_2(t))u(t).$$

Rule 3:

IF $y_1(t)$ is "about 0 rad" and $y_2(t)$ is "about 0" *THEN*

$$\dot{\boldsymbol{x}}(t) = (\boldsymbol{A}_3 + \Delta\boldsymbol{A}_3(t))\boldsymbol{x}(t) + (\boldsymbol{B}_3 + \Delta\boldsymbol{B}_3(t))u(t).$$

Rule 4:

IF $y_1(t)$ is "about 0 rad" and $y_2(t)$ is "about $-a$ or a" *THEN*

$$\dot{\boldsymbol{x}}(t) = (\boldsymbol{A}_4 + \Delta\boldsymbol{A}_4(t))\boldsymbol{x}(t) + (\boldsymbol{B}_4 + \Delta\boldsymbol{B}_4(t))u(t).$$

The uncertain fuzzy model is identified by the procedure explained in Section 2.3 and the following system matrices are obtained having the parameters and uncertainty bounds given in Table. 2.5.

$$\boldsymbol{A}_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & a_{23i} & 0 \\ 0 & 0 & 0 & 1 \\ a_{41i} & 0 & a_{43i} & 0 \end{bmatrix}, \quad \boldsymbol{B}_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_{41i} \end{bmatrix}, \quad i = 1, \ldots, 4 \tag{2.55}$$

$$\Delta\boldsymbol{A}_i(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta a_{23i}(t) & 0 \\ 0 & 0 & 0 & 0 \\ \Delta a_{41i}(t) & 0 & \Delta a_{43i}(t) & \Delta a_{44i}(t) \end{bmatrix}, \quad \Delta\boldsymbol{B}_i(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \Delta b_{41i}(t) \end{bmatrix}, \quad i = 1, \ldots, 4 \tag{2.56}$$

A fuzzy parallel distributed compensation controller is designed for the state regulation problem [8] with $[0 \;\; 0 \;\; 0 \;\; 0]^T$ as the desired equilibrium point. With the control input in the form $u(t) =$

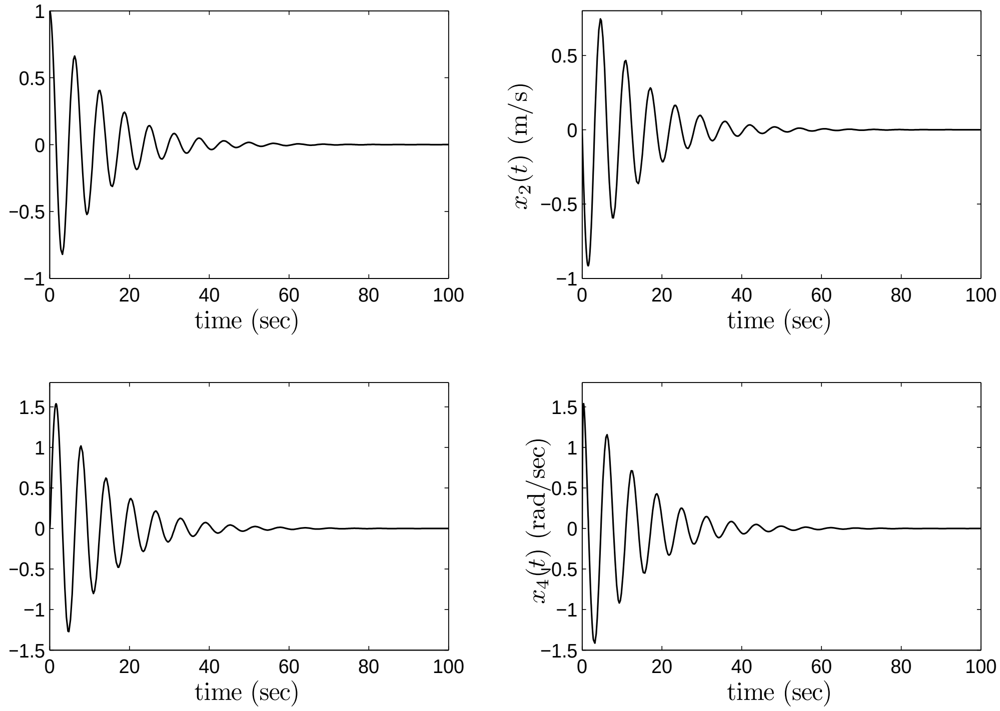


Fig. 2.10: Control results for TORA system.

$\sum_{i=1}^{r} \mu_i \boldsymbol{K}_i \boldsymbol{x}(t)$, the LMIs are solved and the parameters of the feedback gains are obtained as

$$\begin{aligned}
\boldsymbol{K}_1 &= [17.1021 \quad -8.3725 \quad -6.1768 \quad -10.4373] \\
\boldsymbol{K}_2 &= [15.8944 \quad -7.6050 \quad -5.7597 \quad -9.7674] \\
\boldsymbol{K}_3 &= [14.5570 \quad -6.8362 \quad -5.2915 \quad -9.0124] \\
\boldsymbol{K}_4 &= [15.1720 \quad -7.2377 \quad -5.5140 \quad -9.3890]
\end{aligned}$$

Simulation results with the initial condition $\boldsymbol{x}(0) = [1 \ \ 0 \ \ 0 \ \ 0]^T$ are shown in Fig. 2.10.

2.6 Summary

In this chapter, a linear programming based method is proposed for identification of uncertain nonlinear systems using T–S fuzzy model in a form suitable for application in robust fuzzy control. The bounds of the given dynamic equation of the uncertain nonlinear system are estimated using interval analysis technique combined with the branch and bound technique. With the estimated bounds, the nominal model and the uncertainty bounds in the fuzzy model are found using linear programming approach. The uncertain matrices are expressed in a special form using the bounds obtained from the linear programming problem and it has been shown that the obtained model can be used for robust fuzzy control of uncertain nonlinear systems. The number of inequality conditions in the linear programming problem is reduced by the proposed method and the computation time is also less with better approximation than the method proposed by Škrjanc et al. [1]. The limitation of the proposed method lies in determining the bounds of the uncertain equation by a branch and bound technique. The computational complexity of the problem will increase with an increase in dimension of the branching space and the proposed method is not suitable for systems with a large number of uncertain terms. The proposed modeling technique is validated by simulating with examples.

Chapter 3

Fuzzy Guaranteed Cost Controller for Trajectory Tracking in Uncertain Nonlinear Systems

3.1 Introduction

In the past few years, fuzzy logic based control has been applied successfully for controlling nonlinear systems. Takagi and Sugeno [22] suggested a method to describe a system in a fuzzified manner which can represent highly non-linear relations in a simple form. In this model, the local dynamics are represented in each fuzzy implication by a linear model. The overall fuzzy model is obtained by fuzzily "blending" the linear system models. In the domain of fuzzy controller design and analysis, current research is on establishing stringent control norms for guaranteed performance in terms of stability, cost and robustness. For achieving the guaranteed performance, advanced mathematical tools are increasingly being used. Linear matrix inequality (LMI) technique is one such mathematical tool chosen by the research community to design fuzzy controllers in those application areas where a fuzzy model of the process is available in T-S form [22]. T-S model based controllers developed using the LMI approach have been applied to many control problems [8, 106, 114–116] of late for guaranteeing stability and performance.

Finding a solution for an optimization or minimization problem for nonlinear systems involving certain objective function is difficult; but if such a problem is transformed into an LMI problem (convex problem), then it can be solved reliably and effectively [38]. LMI approach has been successfully applied for regulatory control of nonlinear systems satisfying quadratic performance function and the minimization problem was solved utilizing the powerful convex optimization technique [8]. However

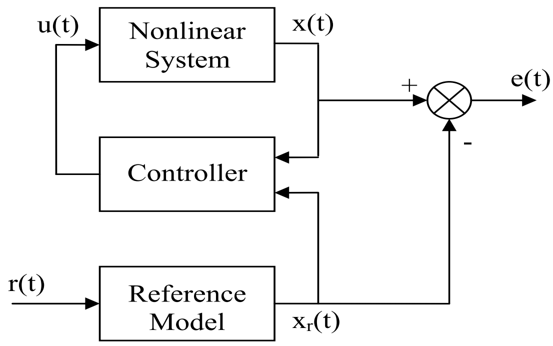


Fig. 3.1: Control scheme

this approach has not been attempted for tracking control problems. Tracking control is an important problem in practical applications like robotics, missile tracking and attitude tracking of aircrafts where the system has to track a certain pre-defined trajectory accurately. However it is more difficult than the regulatory control problem and only few studies are carried out especially for continuous-time systems. A fuzzy controller design for tracking control of nonlinear system with guaranteed H_∞ performance was proposed by Tseng et al. [92]. This method shows a perfect tracking performance, but it is attained at the expense of large control effort. Xu et al. [117] addressed the problem of LMI based tracking control to track a set-point. Here, a state feedback tracking guaranteed cost controller is addressed for a class of norm bounded continuous-time uncertain systems.

In this chapter, a robust fuzzy guaranteed cost controller is designed for a trajectory tracking problem using T-S fuzzy model. Here, a state feedback controller satisfying a quadratic performance function is considered. The Lyapunov function based approach is employed for the design and analysis of the guaranteed cost controller. Our design procedure is motivated by the approach presented by Tong et al. [66]. In our proposed scheme, the controller design results in a minimization problem involving LMIs which can be solved efficiently with the available semi-definite programming technique. The design procedure minimizes the upper bound of the performance function and also ensures that the closed loop system is asymptotically stable.

3.2 System Representation and Problem Formulation

A nonlinear dynamical system described by a T-S fuzzy model and a reference model driven by a reference input are considered in this design. The reference model is considered to be linear and its system states are bounded. The system state of the reference model represents the reference trajectory and it need not be zero or constant. Our design objective is to determine the control input for the nonlinear system to push its system states to track the reference trajectory while minimizing the cost function. The control structure is shown in Fig. 3.1. In this design problem it is assumed that all the system states are available for measurement (state feedback problem).

The dynamic fuzzy model proposed by Takagi and Sugeno [22] described by fuzzy IF-THEN rules is used to represent the local linear input-output relations for the nonlinear system. The overall fuzzy model is achieved by fuzzily blending the linear system models. Given a pair of input-output, the final output of the fuzzy system is inferred by using center of gravity method for defuzzification.

Let us consider a nonlinear system which can be described by the T-S fuzzy model [40] as described below:

Plant rule i:

IF $z_1(t)$ is N_{i1} and $\cdots$ $z_p(t)$ is N_{ip}, *THEN*

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= (\boldsymbol{A}_i + \Delta\boldsymbol{A}_i(t))\boldsymbol{x}(t) + (\boldsymbol{B}_i + \Delta\boldsymbol{B}_i(t))\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}_i\boldsymbol{x}(t), \qquad i = 1, 2, ..., r \end{aligned} \tag{3.1}$$

where N_{ij} is the fuzzy set, $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state vector, $\boldsymbol{u}(t) \in \mathbb{R}^m$ is the control input and $\boldsymbol{y}(t) \in \mathbb{R}^q$ is the output vector. Here, $\boldsymbol{A}_i \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}_i \in \mathbb{R}^{n\times m}$ and $\boldsymbol{C}_i \in \mathbb{R}^{q\times n}$; $z_1(t), z_2(t), ..., z_p(t)$ are premise variables.

Given a pair of $(\boldsymbol{x}(t), \boldsymbol{u}(t))$, the final output of the fuzzy system is inferred as follows [8]:

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} \mu_i(z(t))((\boldsymbol{A}_i + \Delta\boldsymbol{A}_i(t))\boldsymbol{x}(t) + (\boldsymbol{B}_i + \Delta\boldsymbol{B}_i(t))\boldsymbol{u}(t)) \tag{3.2a}$$

$$\boldsymbol{y}(t) = \sum_{i=1}^{r} \mu_i(z(t))\boldsymbol{C}_i\boldsymbol{x}(t), \tag{3.2b}$$

where

$$\mu_i(z(t)) = \frac{\omega_i(z(t))}{\sum_{i=1}^{r}\omega_i(z(t))}, \quad \omega_i(z(t)) = \prod_{j=1}^{p} \mathcal{N}_{ij}(z_j(t))$$

and $\mathcal{N}_{ij}(z_j(t))$ is the degree of the membership of $z_j(t)$ in N_{ij}. Here $\omega_i(z(t)) \geq 0$, for $i = 1, 2, ..., r$ and $\sum_{i=1}^{r}\omega_i(z(t)) > 0$ for all t. Therefore, $\mu_i(z(t)) \geq 0$ for $i = 1, 2, ..., r$ and $\sum_{i=1}^{r}\mu_i(z(t)) = 1$.

The uncertain matrices are assumed to be norm bounded and are described by:

$$\left[\begin{array}{c:c} \Delta\boldsymbol{A}_i(t) & \Delta\boldsymbol{B}_i(t) \end{array}\right] = \boldsymbol{M}_i\boldsymbol{F}_i(t)\left[\begin{array}{c:c} \boldsymbol{N}_{1i} & \boldsymbol{N}_{2i} \end{array}\right], \tag{3.3}$$

where $\boldsymbol{M}_i$, $\boldsymbol{N}_{1i}$ and $\boldsymbol{N}_{2i}$ are known real constant matrices of appropriate dimension and $\boldsymbol{F}_i(t)$ is a time varying matrix function with Lebesgue-measurable elements, satisfying $\boldsymbol{F}_i^T(t)\boldsymbol{F}_i(t) \leq \boldsymbol{I}$.

Let us consider a reference model as follows [92]:

$$\dot{\boldsymbol{x}}_r(t) = \boldsymbol{A}_r\boldsymbol{x}_r(t) + \boldsymbol{r}(t) \tag{3.4}$$

where $\boldsymbol{x}_r(t)$ is the reference state, $\boldsymbol{A}_r$ is a specific asymptotically stable matrix, $\boldsymbol{r}(t)$ is a bounded reference input.

The tracking error is defined as

$$\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_r(t). \tag{3.5}$$

Given positive-definite symmetric matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$, the cost function considered is

$$J = \int_{t_0}^{t_f} \{\boldsymbol{e}^T(t)\boldsymbol{Q}\boldsymbol{e}(t) + \boldsymbol{u}^T(t)\boldsymbol{R}\boldsymbol{u}(t)\}dt \tag{3.6}$$

where t_0 and t_f are the initial and terminal time of control respectively. Associated with the cost function represented in (3.6), the fuzzy guaranteed cost control is defined as follows.

Let us consider the system represented by (3.2). If there exists a fuzzy control law $\boldsymbol{u}(t)$ and a scalar J_o such that the closed-loop system is asymptotically stable and the closed-loop value of the cost function given by (3.6) satisfies $J \leq J_o$, then J_o is said to be a guaranteed cost and the control law $u(t)$ is said to be a guaranteed cost control law for the system.

The objective is to develop a procedure to design a state-feedback guaranteed cost control law for the tracking problem.

3.3 Fuzzy Guaranteed Cost Controller via State-feedback

In this section, a fuzzy guaranteed cost controller design is considered for a nonlinear system without uncertainty. The problem of designing a robust fuzzy guaranteed cost controller for uncertain nonlinear systems will be addressed in the next section. Let us consider a nonlinear system which can be described by the T-S fuzzy model [40] as given below:

Plant rule i:

IF $z_1(t)$ is N_{i1} and $\cdots$ $z_p(t)$ is N_{ip}, *THEN*

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}_i\boldsymbol{x}(t), \qquad i = 1, 2, ..., r \end{aligned} \tag{3.7}$$

The final output of the fuzzy system is inferred as follows [8]:

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} \mu_i(z(t))(\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t)) \tag{3.8a}$$

$$\boldsymbol{y}(t) = \sum_{i=1}^{r} \mu_i(z(t))\boldsymbol{C}_i\boldsymbol{x}(t). \tag{3.8b}$$

Let the following fuzzy control rule be employed to deal with the design of a fuzzy controller for the system represented by (3.8).

Control Rule i:

IF $z_1(t)$ is N_{i1} and $\cdots$ $z_p(t)$ is N_{ip}, *THEN*

$$\boldsymbol{u}(t) = \boldsymbol{K}_{1i}\boldsymbol{x}(t) + \boldsymbol{K}_{2i}\boldsymbol{x}_r(t),\ i = 1, 2, \ldots, r \tag{3.9}$$

The overall fuzzy control law is represented by

$$\boldsymbol{u}(t) = \sum_{i=1}^{r} \mu_i(z(t))(\boldsymbol{K}_{1i}\boldsymbol{x}(t) + \boldsymbol{K}_{2i}\boldsymbol{x}_r(t)) \tag{3.10}$$

where $\boldsymbol{K}_{1i}$ and $\boldsymbol{K}_{2i}$ are controller gains. The design of the fuzzy guaranteed cost controller is to determine the gains $\boldsymbol{K}_{1i}$ and $\boldsymbol{K}_{2i}$ $(i = 1, 2, \ldots, r)$ and a positive scalar J_o such that the resulting closed-loop system is asymptotically stable and the closed-loop value of the cost-function given by (3.6) satisfies $J \leq J_o$. With the control law given by (3.10), the overall closed-loop system can be written as

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(z(t))\mu_j(z(t))[\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i(\boldsymbol{K}_{1j}\boldsymbol{x}(t) + \boldsymbol{K}_{2j}\boldsymbol{x}_r(t))] \tag{3.11}$$

Combining (3.11) and (3.4), the augmented system can be expressed as

$$\dot{\bar{\boldsymbol{x}}}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(z(t))\mu_j(z(t))(\bar{\boldsymbol{A}}_i + \bar{\boldsymbol{B}}_i\bar{\boldsymbol{K}}_j)\bar{\boldsymbol{x}}(t) + \bar{\boldsymbol{r}}(t) \tag{3.12}$$

where

$$\bar{\boldsymbol{x}}(t) = \begin{bmatrix} \boldsymbol{x}(t) \\ \hdashline \boldsymbol{x}_r(t) \end{bmatrix}, \bar{\boldsymbol{r}}(t) = \begin{bmatrix} \boldsymbol{0} \\ \hdashline \boldsymbol{r}(t) \end{bmatrix}$$

$$\bar{\boldsymbol{A}}_i = \left[\begin{array}{c:c} \boldsymbol{A}_i & \boldsymbol{0} \\ \hdashline \boldsymbol{0} & \boldsymbol{A}_r \end{array}\right],\ \bar{\boldsymbol{B}}_i = \begin{bmatrix} \boldsymbol{B}_i \\ \hdashline \boldsymbol{0} \end{bmatrix},\ \bar{\boldsymbol{K}}_j = \left[\begin{array}{c:c} \boldsymbol{K}_{1j} & \boldsymbol{K}_{2j} \end{array}\right]$$

With the augmented system represented by (3.12), the result of the guaranteed cost control law for tracking based on a T-S fuzzy system with state feedback is summarized in the following theorems, followed by the proof. These theorems are based on the concept presented by Tong et al. [66] and [64, 92].

Theorem 3.1. For the fuzzy logic based tracking control of a nonlinear system represented by (3.8), if there exists a symmetric positive definite matrix $\boldsymbol{P}$, matrices $\boldsymbol{K}_{1j}$, $\boldsymbol{K}_{2j}$ ($j = 1, 2, \ldots, r$) and a scalar a such that,

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(z(t))\mu_j(z(t))\left\{\tilde{\boldsymbol{A}}_{ij}^T\boldsymbol{P} + \boldsymbol{P}\tilde{\boldsymbol{A}}_{ij} + \bar{\boldsymbol{Q}} + \bar{\boldsymbol{R}}_j + \frac{\boldsymbol{P}\boldsymbol{P}}{a^2}\right\} < \boldsymbol{0} \tag{3.13}$$

where

$$\tilde{\boldsymbol{A}}_{ij} = \bar{\boldsymbol{A}}_i + \bar{\boldsymbol{B}}_i\bar{\boldsymbol{K}}_j, \quad \bar{\boldsymbol{Q}} = \left[\begin{array}{c:c} \boldsymbol{Q} & -\boldsymbol{Q} \\ \hdashline -\boldsymbol{Q} & \boldsymbol{Q} \end{array}\right], \quad \bar{\boldsymbol{R}}_j = \bar{\boldsymbol{K}}_j^T\boldsymbol{R}\bar{\boldsymbol{K}}_j,$$

then the feedback control law

$$\boldsymbol{u}(t) = \sum_{j=1}^{r}\mu_j(z(t))(\boldsymbol{K}_{1j}\boldsymbol{x}(t) + \boldsymbol{K}_{2j}\boldsymbol{x}_r(t)) \tag{3.14}$$

is a fuzzy guaranteed cost control law with the upper bound for the guaranteed cost

$$J_o = \bar{\boldsymbol{x}}^T(0)\boldsymbol{P}\bar{\boldsymbol{x}}(0) + a^2\int_{t_0}^{t_f}\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t)dt. \tag{3.15}$$

Proof: Let us consider the following Lyapunov function $V(t)$ for the closed loop system given by (3.12). For simplicity, $\mu_i(z(t))$ is denoted as μ_i.

$$\begin{aligned}
V(t) &= \bar{\boldsymbol{x}}^T(t)\boldsymbol{P}\,\bar{\boldsymbol{x}}(t) && (3.16)\\
\dot{V}(t) &= \dot{\bar{\boldsymbol{x}}}^T(t)\boldsymbol{P}\bar{\boldsymbol{x}}(t) + \bar{\boldsymbol{x}}^T(t)\boldsymbol{P}\dot{\bar{\boldsymbol{x}}}(t) && (3.17)\\
&= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\,\mu_j\Big\{\bar{\boldsymbol{x}}^T(t)[\tilde{\boldsymbol{A}}_{ij}^T\boldsymbol{P} + \boldsymbol{P}\tilde{\boldsymbol{A}}_{ij}]\bar{\boldsymbol{x}}(t)\Big\} + \bar{\boldsymbol{r}}^T(t)\boldsymbol{P}\bar{\boldsymbol{x}}(t) + \bar{\boldsymbol{x}}^T(t)\boldsymbol{P}\bar{\boldsymbol{r}}(t)\\
&= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\,\mu_j\Big\{\bar{\boldsymbol{x}}^T(t)[\tilde{\boldsymbol{A}}_{ij}^T\boldsymbol{P} + \boldsymbol{P}\tilde{\boldsymbol{A}}_{ij}]\bar{\boldsymbol{x}}(t)\Big\} + a\bar{\boldsymbol{r}}^T(t)\boldsymbol{P}\bar{\boldsymbol{x}}(t)(1/a) + (1/a)\bar{\boldsymbol{x}}^T(t)\boldsymbol{P}\bar{\boldsymbol{r}}(t)a
\end{aligned}$$

Let $\tilde{\boldsymbol{X}} = \bar{\boldsymbol{r}}(t)a$ and $\tilde{\boldsymbol{Y}} = \boldsymbol{P}\bar{\boldsymbol{x}}(t)(1/a)$. Now using the matrix inequality $\tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{Y}} + \tilde{\boldsymbol{Y}}^T\tilde{\boldsymbol{X}} \leq \tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{X}} + \tilde{\boldsymbol{Y}}^T\tilde{\boldsymbol{Y}}$ given in [66], the following condition is obtained

$$\begin{aligned}
\dot{V}(t) &\leq \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\,\mu_j\Big\{\bar{\boldsymbol{x}}^T(t)[\tilde{\boldsymbol{A}}_{ij}^T\boldsymbol{P} + \boldsymbol{P}\tilde{\boldsymbol{A}}_{ij}]\bar{\boldsymbol{x}}(t)\Big\} + (1/a^2)\bar{\boldsymbol{x}}^T(t)\boldsymbol{P}\boldsymbol{P}\bar{\boldsymbol{x}}(t) + a^2\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t)\\
&\leq \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\,\mu_j\Big\{\bar{\boldsymbol{x}}^T(t)[\tilde{\boldsymbol{A}}_{ij}^T\boldsymbol{P} + \boldsymbol{P}\tilde{\boldsymbol{A}}_{ij} + (1/a^2)\boldsymbol{P}\boldsymbol{P}]\bar{\boldsymbol{x}}(t)\Big\} + a^2\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t) && (3.18)
\end{aligned}$$

Using the matrix inequality given in (3.13), the above inequality can be rewritten as

$$\begin{aligned}
\dot{V}(t) &\leq \sum_{j=1}^{r}\mu_j\{\bar{\boldsymbol{x}}^T(t)[-\bar{\boldsymbol{Q}} - \bar{\boldsymbol{R}}_j]\bar{\boldsymbol{x}}(t)\} + a^2\bar{\boldsymbol{r}}^T(t)\boldsymbol{r}(t)\\
&\leq -\begin{bmatrix}\boldsymbol{x}^T(t) & \boldsymbol{x}_r^T(t)\end{bmatrix}\begin{bmatrix}\boldsymbol{Q} & -\boldsymbol{Q}\\ -\boldsymbol{Q} & \boldsymbol{Q}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{x}_r(t)\end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
& -\sum_{j=1}^{r}\mu_j \begin{bmatrix}\boldsymbol{x}^T(t) & \boldsymbol{x}_r^T(t)\end{bmatrix} \begin{bmatrix}\boldsymbol{K}_{1j}^T\boldsymbol{R}\boldsymbol{K}_{1j} & \boldsymbol{K}_{1j}^T\boldsymbol{R}\boldsymbol{K}_{2j} \\ \boldsymbol{K}_{2j}^T\boldsymbol{R}\boldsymbol{K}_{1j} & \boldsymbol{K}_{2j}^T\boldsymbol{R}\boldsymbol{K}_{2j}\end{bmatrix} \begin{bmatrix}\boldsymbol{x}(t) \\ \boldsymbol{x}_r(t)\end{bmatrix} \\
& + a^2\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t) \\
\leq\ & -\Big[(\boldsymbol{x}^T(t)-\boldsymbol{x}_r^T(t))\boldsymbol{Q}(\boldsymbol{x}(t)-\boldsymbol{x}_r(t))\Big] \\
& -\sum_{j=1}^{r}\mu_j\Big\{(\boldsymbol{x}^T(t)\boldsymbol{K}_{1j}^T+\boldsymbol{x}_r^T(t)\boldsymbol{K}_{2j}^T)\boldsymbol{R}(\boldsymbol{K}_{1j}\boldsymbol{x}(t)+\boldsymbol{K}_{2j}\boldsymbol{x}_r(t))\Big\}+a^2\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t) \\
\dot{V}(t) \ \leq\ & -\Big[\boldsymbol{e}^T(t)\boldsymbol{Q}\boldsymbol{e}(t)+\boldsymbol{u}^T(t)\boldsymbol{R}\boldsymbol{u}(t)\Big]+a^2\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t)
\end{aligned} \tag{3.19}
$$

Integrating (3.19) from $t=0$ to $t=t_f$ gives,

$$
V(t_f)-V(0) \leq -\int_0^{t_f}(\boldsymbol{e}^T(t)\boldsymbol{Q}\boldsymbol{e}(t)+\boldsymbol{u}^T(t)\boldsymbol{R}\boldsymbol{u}(t))dt + a^2\int_0^{t_f}\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t)dt \tag{3.20}
$$

$$
\text{or,}\quad \int_0^{t_f}(\boldsymbol{e}^T(t)\boldsymbol{Q}\boldsymbol{e}(t)+\boldsymbol{u}^T(t)\boldsymbol{R}\boldsymbol{u}(t))dt \leq V(0)-V(t_f)+a^2\int_0^{t_f}\bar{\boldsymbol{r}}^T(t)\boldsymbol{r}(t)dt
$$

Since $V(t_f)\geq 0$, the above inequality can be written as

$$
\begin{aligned}
\int_0^{t_f}(\boldsymbol{e}^T(t)\boldsymbol{Q}\boldsymbol{e}(t)+\boldsymbol{u}^T(t)\boldsymbol{R}\boldsymbol{u}(t))dt \ &\leq\ V(0)+a^2\int_0^{t_f}\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t)dt \\
\text{or,}\ \int_0^{t_f}(\boldsymbol{e}^T(t)\boldsymbol{Q}\boldsymbol{e}(t)+\boldsymbol{u}^T(t)\boldsymbol{R}\boldsymbol{u}(t))dt \ &\leq\ \bar{\boldsymbol{x}}^T(0)\boldsymbol{P}\bar{\boldsymbol{x}}(0)+a^2\int_0^{t_f}\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t)dt
\end{aligned} \tag{3.21}
$$

Hence it follows from (3.21), (3.6) and (3.15) that $J \leq J_0$. This completes the proof. □

3.3.1 Stability

The above proposed theorem states the minimization problem for guaranteed cost control of the nonlinear fuzzy system. If there exists a common symmetric positive definite matrix $\boldsymbol{P}$ satisfying the matrix inequality given in (3.13), then the closed loop fuzzy system given in (3.12) is quadratically stable.

Proof: The closed loop system represented by (3.12) is associated with an external input $\boldsymbol{r}(t)$. Hence the "input to state stability" (ISS) property [118] is used to show the stability of the system.

From (3.19),

$$
\dot{V}(t) \leq -\Big[\boldsymbol{e}^T(t)\boldsymbol{Q}\boldsymbol{e}(t)+\boldsymbol{u}^T(t)\boldsymbol{R}\boldsymbol{u}(t)\Big]+a^2\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t) \tag{3.22}
$$

The above inequality is a dissipation inequality with $V(t)$ as storage function. The right hand side of the inequality is the supply function. In this inequality, $\boldsymbol{Q}$ and $\boldsymbol{R}$ are positive definite matrices. Hence $\dot{V} \leq a^2\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t)$ or $V(t)$ is less than the integral of $a^2\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t)$. This inequality satisfies the "input to state stability condition" (ISSC) [118] and hence the system is asymptotically stable. This completes the proof. □

The theorem described above provides the sufficient conditions for stability of the fuzzy system with minimization of the performance function. But it does not show the method for finding the control law which minimizes the upper bound and satisfies the inequalities with common symmetric positive definite matrix $\boldsymbol{P}$. The method of transforming the matrix inequality given in (3.13) into standard LMI will solve this problem and it is shown in the following subsection.

3.3.2 Optimal Fuzzy Guaranteed Cost Control

Theorem 3.2. Let us consider the system (3.8) associated with the cost function (3.6). Suppose the following minimization problem

$$\underset{\boldsymbol{Y},\bar{\boldsymbol{X}}_j}{min} \quad \{\alpha + a^2 \int_0^{t_f} \bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t)dt\} \tag{3.23}$$

subject to the following inequalities

$$\begin{aligned} \boldsymbol{\phi}_{ii} &< 0, \quad i = 1, 2, \ldots, r && (3.24)\\ \frac{1}{r-1}\phi_{ii} + \frac{1}{2}(\boldsymbol{\phi}_{ij} + \boldsymbol{\phi}_{ji}) &< 0, \quad 1 \le i \ne j \le r && (3.25)\\ \begin{bmatrix} -\alpha & \bar{\boldsymbol{x}}^T(0) \\ \bar{\boldsymbol{x}}(0) & -\boldsymbol{Y} \end{bmatrix} &< 0 && (3.26) \end{aligned}$$

where

$$\boldsymbol{\phi}_{ij} = \begin{bmatrix} \boldsymbol{H}_{ij} & * & * \\ \boldsymbol{W} & -\boldsymbol{Q}^{-1} & * \\ \bar{\boldsymbol{X}}_j & \boldsymbol{0} & -\boldsymbol{R}^{-1} \end{bmatrix}$$

and $\boldsymbol{H}_{ij} = \boldsymbol{Y}\bar{\boldsymbol{A}}_i^T + \bar{\boldsymbol{X}}_j^T\bar{\boldsymbol{B}}_i^T + (*) + \frac{\boldsymbol{I}}{a^2}$ has a solution set a, α, $\boldsymbol{Y}$ and $\bar{\boldsymbol{X}}_j$, where $\bar{\boldsymbol{r}}(t)$ is the reference trajectory. Then the control law (3.10) is a fuzzy guaranteed cost control law with minimal upper bound for the performance function.

Proof: By Theorem 3.1, the control law $u(t)$ satisfying (3.13) provides the fuzzy guaranteed cost control law for the system (3.8). Pre-multiplying and post-multiplying (3.13) by $\boldsymbol{Y}$, where $\boldsymbol{Y} = \boldsymbol{P}^{-1}$, yields

$$\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j \left\{ \boldsymbol{Y}(\bar{\boldsymbol{A}}_i + \bar{\boldsymbol{B}}_i\bar{\boldsymbol{K}}_j)^T + (\bar{\boldsymbol{A}}_i + \bar{\boldsymbol{B}}_i\bar{\boldsymbol{K}}_j)\boldsymbol{Y} + \boldsymbol{Y}\bar{\boldsymbol{Q}}\boldsymbol{Y} + \boldsymbol{Y}\bar{\boldsymbol{R}}_j\boldsymbol{Y} + \frac{\boldsymbol{I}}{a^2} \right\} < \boldsymbol{0} \tag{3.27}$$

Substituting $\bar{\boldsymbol{X}}_j = \bar{\boldsymbol{K}}_j\boldsymbol{Y}$, $\boldsymbol{W} = \left[\begin{array}{c:c} \boldsymbol{I} & -\boldsymbol{I} \end{array}\right]\boldsymbol{Y}$, it follows

$$\begin{aligned} \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\left\{\boldsymbol{Y}(\bar{\boldsymbol{A}}_i+\bar{\boldsymbol{B}}_i\bar{\boldsymbol{K}}_j)^T+(\bar{\boldsymbol{A}}_i+\bar{\boldsymbol{B}}_i\bar{\boldsymbol{K}}_j)\boldsymbol{Y}+\boldsymbol{W}^T\boldsymbol{Q}\boldsymbol{W}+\boldsymbol{Y}\bar{\boldsymbol{K}}_j^T\boldsymbol{R}\bar{\boldsymbol{K}}_j\boldsymbol{Y}+\frac{\boldsymbol{I}}{a^2}\right\} &< \boldsymbol{0} \\ \text{or,}\quad \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\left\{\boldsymbol{Y}\bar{\boldsymbol{A}}_i^T+\bar{\boldsymbol{X}}_j^T\bar{\boldsymbol{B}}_i^T+\bar{\boldsymbol{A}}_i\boldsymbol{Y}+\bar{\boldsymbol{B}}_i\bar{\boldsymbol{X}}_j+\boldsymbol{W}^T\boldsymbol{Q}\boldsymbol{W}+\bar{\boldsymbol{X}}_j^T\boldsymbol{R}\bar{\boldsymbol{X}}_j+\frac{\boldsymbol{I}}{a^2}\right\} &< \boldsymbol{0} \end{aligned} \tag{3.28}$$

By Schur Complement, (3.28) is equivalent to the following LMIs,

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\begin{bmatrix} \boldsymbol{H}_{ij} & * & * \\ \boldsymbol{W} & -\boldsymbol{Q}^{-1} & * \\ \bar{\boldsymbol{X}}_j & \boldsymbol{0} & -\boldsymbol{R}^{-1} \end{bmatrix} < \boldsymbol{0} \tag{3.29}$$

where, $*$ represents the transposed elements in symmetric positions. Applying Lemma A.1 (Appendix) to the above inequality, the conditions (3.24) and (3.25) are obtained. Hence if the conditions (3.24) and (3.25) are satisfied, then $u(t)$ is the guaranteed cost control law for the fuzzy system.

Also, it follows from Schur complement that (3.26) is equivalent to $\bar{\boldsymbol{x}}^T(0)\boldsymbol{Y}^{-1}\bar{\boldsymbol{x}}(0) < \alpha$. Hence

$$\bar{\boldsymbol{x}}^T(0)\boldsymbol{P}\bar{\boldsymbol{x}}(0) + a^2\int_0^{t_f}\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t)dt < \alpha + a^2\int_0^{t_f}\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t)dt$$

From (3.15), it follows that

$$J_0 < \alpha + a^2\int_0^{t_f}\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t)dt \tag{3.30}$$

Thus minimization of $\alpha + a^2\int_0^{t_f}\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t)dt$ implies minimization of the upper bound of the guaranteed cost J_0 for the system (3.8). This completes the proof. □

The matrix inequalities given by (3.13) are expressed as standard LMIs in Theorem 3.2 and these can be solved easily and efficiently for a, $\boldsymbol{P}$, $\boldsymbol{K}_{1j}$and $\boldsymbol{K}_{2j}$. In the upper bound of the total cost J_0, the term α is the transient cost component. The steady state cost component is given by the term $a^2\int_0^{t_f}\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t)dt$.

Remark 3.1. The above LMI design condition depends on the initial state of the system. If the initial state is changed or unknown, the feedback gains $\boldsymbol{K}_{1j}$ and $\boldsymbol{K}_{2j}$ need to be changed. This is a disadvantage in this method. The way of overcoming the problem of initial state dependence is discussed in [8] and the above theorem can be used by adding the conditions provided in [8] to overcome this disadvantage.

Remark 3.2. In Theorem 3.2, the LMI conditions were presented using the relaxations in Lemma A.1 (Appendix) from [105]. Solving these LMI based problems to get less conservative results is an interesting problem and several approaches were proposed in the literature [106, 107, 119]. Some of these methods provide relaxations with additional artificial decision variables. The number of artificial decision variables increases with the increase in the number of rules. This results in more computation time for solving the LMIs. These factors must be considered before using these LMI relaxations for problems with more number of rules.

3.4 Robust Fuzzy Guaranteed Cost Controller via State-feedback

In the previous section, the fuzzy guaranteed cost controller design for nonlinear systems without any uncertainty was discussed. Now, the design of a robust fuzzy guaranteed cost controller via state-feedback for an uncertain nonlinear system is considered. Let us consider the uncertain fuzzy system in the form (3.2), with the control law given by (3.10). The overall closed-loop system can be written as

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big\{(\boldsymbol{A}_i + \Delta\boldsymbol{A}_i(t))\boldsymbol{x}(t) + (\boldsymbol{B}_i + \Delta\boldsymbol{B}_i(t))(\boldsymbol{K}_{1j}\boldsymbol{x}(t) + \boldsymbol{K}_{2j}\boldsymbol{x}_r(t))\Big\} \tag{3.31}$$

Combining (3.31) and (3.4), the augmented system can be expressed as

$$\dot{\bar{\boldsymbol{x}}}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big\{(\bar{\boldsymbol{A}}_i + \Delta\bar{\boldsymbol{A}}_i(t)) + (\bar{\boldsymbol{B}}_i + \Delta\bar{\boldsymbol{B}}_i(t))\bar{\boldsymbol{K}}_j\bar{\boldsymbol{x}}(t) + \bar{\boldsymbol{r}}(t)\Big\} \tag{3.32}$$

where

$$\bar{\boldsymbol{x}}(t) = \left[\begin{array}{c}\boldsymbol{x}(t)\\ \hdashline \boldsymbol{x}_r(t)\end{array}\right],\ \bar{\boldsymbol{r}}(t) = \left[\begin{array}{c}\boldsymbol{0}\\ \hdashline \boldsymbol{r}(t)\end{array}\right],\ \bar{\boldsymbol{A}}_i = \left[\begin{array}{c:c}\boldsymbol{A}_i & \boldsymbol{0}\\ \hdashline \boldsymbol{0} & \boldsymbol{A}_r\end{array}\right],\ \Delta\bar{\boldsymbol{A}}_i(t) = \left[\begin{array}{c:c}\Delta\boldsymbol{A}_i(t) & \boldsymbol{0}\\ \hdashline \boldsymbol{0} & \boldsymbol{0}\end{array}\right],$$

$$\bar{\boldsymbol{B}}_i = \left[\begin{array}{c}\boldsymbol{B}_i\\ \hdashline \boldsymbol{0}\end{array}\right],\ \Delta\bar{\boldsymbol{B}}_i(t) = \left[\begin{array}{c}\Delta\boldsymbol{B}_i(t)\\ \hdashline \boldsymbol{0}\end{array}\right],\ \bar{\boldsymbol{K}}_j = \left[\begin{array}{c:c}\boldsymbol{K}_{1j} & \boldsymbol{K}_{2j}\end{array}\right]$$

$$\left[\begin{array}{cc}\Delta\bar{\boldsymbol{A}}_i(t) & \Delta\bar{\boldsymbol{B}}_i(t)\end{array}\right] = \bar{\boldsymbol{M}}_i\boldsymbol{F}_i(t)\left[\begin{array}{cc}\bar{\boldsymbol{N}}_{1i} & \bar{\boldsymbol{N}}_{2i}\end{array}\right]$$

$$\bar{\boldsymbol{M}}_i = \left[\begin{array}{c}\boldsymbol{M}_i\\ \boldsymbol{0}\end{array}\right],\ \ \bar{\boldsymbol{N}}_{1i} = \left[\begin{array}{cc}\boldsymbol{N}_{1i} & \boldsymbol{0}\end{array}\right],\ \ \bar{\boldsymbol{N}}_{2i} = \boldsymbol{N}_{2i}$$

With the augmented system represented by (3.32), the result of the guaranteed cost control law for a T-S fuzzy system with state feedback applied for trajectory tracking is summarized in the following theorems, followed by the proof.

Theorem 3.3. Let us consider the nonlinear system and the reference model represented by (3.2) and (3.4). With the augmented system represented by (3.32), if there exists a symmetric positive definite matrix $\boldsymbol{P}$, matrices $\boldsymbol{K}_{1j}$, $\boldsymbol{K}_{2j}$ $(j = 1, \ldots, r)$ of appropriate dimension and a scalar 'a' such that,

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(z(t))\mu_j(z(t))\Big\{(\bar{\boldsymbol{A}}_i^T + \bar{\boldsymbol{B}}_i^T\bar{\boldsymbol{K}}_j)^T\boldsymbol{P} + (*) + \frac{\boldsymbol{P}\boldsymbol{P}}{a^2} + \boldsymbol{P}\bar{\boldsymbol{M}}_i\boldsymbol{\epsilon}_i\bar{\boldsymbol{M}}_i^T\boldsymbol{P} \\ +(\bar{\boldsymbol{N}}_{1i} + \bar{\boldsymbol{N}}_{2i}\bar{\boldsymbol{K}}_j)^T\boldsymbol{\epsilon}_i^{-1}(\bar{\boldsymbol{N}}_{1i} + \bar{\boldsymbol{N}}_{2j}\bar{\boldsymbol{K}}_j) + \bar{\boldsymbol{Q}} + \bar{\boldsymbol{R}}_j\Big\} \;<\; \boldsymbol{0} \quad (3.33)$$

where

$$\bar{\boldsymbol{Q}} = \left[\begin{array}{c|c} \boldsymbol{Q} & -\boldsymbol{Q} \\ \hline -\boldsymbol{Q} & \boldsymbol{Q} \end{array}\right], \quad \bar{\boldsymbol{R}}_j = \bar{\boldsymbol{K}}_j^T\boldsymbol{R}\bar{\boldsymbol{K}}_j$$

then the feedback control law

$$\boldsymbol{u}(t) = \sum_{j=1}^{r}\mu_j(z(t))(\boldsymbol{K}_{1j}\boldsymbol{x}(t) + \boldsymbol{K}_{2j}\boldsymbol{x}_r(t)) \quad (3.34)$$

is a fuzzy guaranteed cost control law with the upper bound for the guaranteed cost

$$J_o = \bar{\boldsymbol{x}}^T(0)\boldsymbol{P}\bar{\boldsymbol{x}}(0) + a^2\int_{t_0}^{t_f}\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t)dt. \quad (3.35)$$

Proof: Let us consider the following Lyapunov function $V(t)$ for the closed loop system given by (3.32).

$$V(t) = \bar{\boldsymbol{x}}^T(t)\boldsymbol{P}\,\bar{\boldsymbol{x}}(t) \quad (3.36)$$

$$\begin{aligned}\dot{V}(t) &= \dot{\bar{\boldsymbol{x}}}^T(t)\boldsymbol{P}\bar{\boldsymbol{x}}(t) + \bar{\boldsymbol{x}}^T(t)\boldsymbol{P}\dot{\bar{\boldsymbol{x}}}(t) \quad (3.37)\\ &= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big\{\bar{\boldsymbol{x}}^T(t)\Big((\bar{\boldsymbol{A}}_i + \Delta\bar{\boldsymbol{A}}_i(t) + (\bar{\boldsymbol{B}}_i + \Delta\bar{\boldsymbol{B}}_i(t))\bar{\boldsymbol{K}}_j)^T\boldsymbol{P} + (*)\Big)\bar{\boldsymbol{x}}(t) \\ &\qquad + \bar{\boldsymbol{x}}^T(t)\boldsymbol{P}\bar{\boldsymbol{r}}(t) + \bar{\boldsymbol{r}}^T(t)\boldsymbol{P}\bar{\boldsymbol{x}}(t)\Big\} \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big\{\bar{\boldsymbol{x}}^T(t)\Big((\bar{\boldsymbol{A}}_i + \Delta\bar{\boldsymbol{A}}_i(t) + (\bar{\boldsymbol{B}}_i + \Delta\bar{\boldsymbol{B}}_i(t))\bar{\boldsymbol{K}}_j)^T\boldsymbol{P} + (*) + (1/a^2)\boldsymbol{P}\boldsymbol{P}\Big)\bar{\boldsymbol{x}}(t) \\ &\qquad + a^2\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t) \quad (3.38)\end{aligned}$$

With (3.32) and using the matrix inequality $\tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{Y}} + \tilde{\boldsymbol{Y}}^T\tilde{\boldsymbol{X}} \leq \tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{X}} + \tilde{\boldsymbol{Y}}^T\tilde{\boldsymbol{Y}}$ given in [66], the uncertain terms in the above equation can be represented as,

$$\begin{aligned}&\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\bar{\boldsymbol{x}}^T(t)\Big((\Delta\bar{\boldsymbol{A}}_i(t) + \Delta\bar{\boldsymbol{B}}_i(t)\bar{\boldsymbol{K}}_j)^T\boldsymbol{P} + (*)\Big)\bar{\boldsymbol{x}}(t) \\ &\quad = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\bar{\boldsymbol{x}}^T(t)\Big((\bar{\boldsymbol{M}}_i\boldsymbol{F}_i(t)(\bar{\boldsymbol{N}}_{1i} + \bar{\boldsymbol{N}}_{2i}\bar{\boldsymbol{K}}_j))^T\boldsymbol{P} + (*)\Big)\bar{\boldsymbol{x}}(t) \\ &\quad < \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\bar{\boldsymbol{x}}^T(t)\Big(\boldsymbol{P}\bar{\boldsymbol{M}}_i\boldsymbol{\epsilon}_i\bar{\boldsymbol{M}}_i^T\boldsymbol{P} + (\bar{\boldsymbol{N}}_{1i} + \bar{\boldsymbol{N}}_{2i}\bar{\boldsymbol{K}}_j)^T\boldsymbol{\epsilon}_i^{-1}(\bar{\boldsymbol{N}}_{1i} + \bar{\boldsymbol{N}}_{2j}\bar{\boldsymbol{K}}_j)\Big)\bar{\boldsymbol{x}}(t) \quad (3.39)\end{aligned}$$

With the above inequality (3.39), the relation in (3.38) can be written as

$$\begin{aligned}\dot{V}(t) \quad < \quad & \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j\Big\{\bar{\boldsymbol{x}}^T(t)\Big((\bar{\boldsymbol{A}}_i + \bar{\boldsymbol{B}}_i\bar{\boldsymbol{K}}_j)^T\boldsymbol{P} + (*) + (1/a^2)\boldsymbol{P}\boldsymbol{P} + \boldsymbol{P}\bar{\boldsymbol{M}}_i\boldsymbol{\epsilon}_i\bar{\boldsymbol{M}}_i^T\boldsymbol{P} \\ & +(\bar{\boldsymbol{N}}_{1i} + \bar{\boldsymbol{N}}_{2i}\bar{\boldsymbol{K}}_j)^T\boldsymbol{\epsilon}_i^{-1}(\bar{\boldsymbol{N}}_{1i} + \bar{\boldsymbol{N}}_{2j}\bar{\boldsymbol{K}}_j)\Big)\bar{\boldsymbol{x}}(t) + a^2\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t) \qquad (3.40)\end{aligned}$$

Using Schur complement and applying similar steps as in the proof of Theorem 3.1 to the matrix inequality given in (3.33), the following condition is obtained

$$\begin{aligned}\dot{V}(t) \quad & \leq \quad \sum_{i=1}^{r}\mu_i\{\bar{\boldsymbol{x}}^T(t)[-\bar{\boldsymbol{Q}} - \bar{\boldsymbol{R}}_i]\bar{\boldsymbol{x}}(t)\} + a\boldsymbol{r}^T(t)\boldsymbol{r}(t) \\ & \leq \quad -\Big[\boldsymbol{e}^T(t)\boldsymbol{Q}\boldsymbol{e}(t) + \boldsymbol{u}^T(t)\boldsymbol{R}\boldsymbol{u}(t)\Big] + a\boldsymbol{r}^T(t)\boldsymbol{r}(t) \qquad (3.41)\end{aligned}$$

Integrating (3.41) from $t = 0$ to $t = t_f$, yields

$$V(t_f) - V(0) \leq -\int_0^{t_f} (\boldsymbol{e}^T(t)\boldsymbol{Q}\boldsymbol{e}(t) + \boldsymbol{u}^T(t)\boldsymbol{R}\boldsymbol{u}(t))dt + a\int_0^{t_f} \boldsymbol{r}^T(t)\boldsymbol{r}(t)dt \qquad (3.42)$$

Since $V(t_f) \geq 0$, it follows

$$\int_0^{t_f} (\boldsymbol{e}^T(t)\boldsymbol{Q}\boldsymbol{e}(t) + \boldsymbol{u}^T(t)\boldsymbol{R}\boldsymbol{u}(t))dt \quad \leq \quad \bar{\boldsymbol{x}}^T(0)\boldsymbol{P}\bar{\boldsymbol{x}}(0) + a\int_0^{t_f} \boldsymbol{r}^T(t)\boldsymbol{r}(t)dt \qquad (3.43)$$

Hence it follows from (3.43), (3.6) and (3.35) that $J \leq J_0$. This completes the proof. □

3.4.1 Robust Optimal Fuzzy Guaranteed Cost Control

Theorem 3.4. Let us consider the system (3.2) associated with the cost function (3.6). Suppose the following minimization problem

$$\min_{\boldsymbol{Y},\bar{\boldsymbol{X}}_j} \quad \{\alpha + a^2\int_0^{t_f} \bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t)dt\} \qquad (3.44)$$

subject to the following inequalities

$$\begin{aligned}\hat{\boldsymbol{\phi}}_{ii} \quad & < \quad 0, \quad i = 1, 2, \ldots, r \qquad (3.45) \\ \frac{1}{r-1}\hat{\boldsymbol{\phi}}_{ii} + \frac{1}{2}(\hat{\boldsymbol{\phi}}_{ij} + \hat{\boldsymbol{\phi}}_{ji}) \quad & < \quad 0, \quad 1 \leq i \neq j \leq r \qquad (3.46) \\ \begin{bmatrix} -\alpha & \bar{\boldsymbol{x}}^T(0) \\ \bar{\boldsymbol{x}}(0) & -\boldsymbol{Y} \end{bmatrix} \quad & < \quad 0 \qquad (3.47)\end{aligned}$$

where

$$\hat{\boldsymbol{\phi}}_{ij} = \begin{bmatrix} \hat{\boldsymbol{H}}_{ij} & * & * & * \\ \bar{\boldsymbol{N}}_{1i}\boldsymbol{Y} + \bar{\boldsymbol{N}}_{2i}\bar{\boldsymbol{X}}_j & -\boldsymbol{\epsilon}_i & * & * \\ \boldsymbol{W} & \boldsymbol{0} & -\boldsymbol{Q}^{-1} & * \\ \bar{\boldsymbol{X}}_j & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{R}^{-1} \end{bmatrix}$$

and $\hat{\boldsymbol{H}}_{ij} = \boldsymbol{Y}\bar{\boldsymbol{A}}_i^T + \bar{\boldsymbol{X}}_j^T\bar{\boldsymbol{B}}_i^T + (*) + \frac{\boldsymbol{I}}{a^2} + \bar{\boldsymbol{M}}_i\boldsymbol{\epsilon}_i\bar{\boldsymbol{M}}_i^T$ has a solution set a, α, $\boldsymbol{Y}$ and $\bar{\boldsymbol{X}}_j$, where $\bar{\boldsymbol{r}}(t)$ is the reference trajectory. Then the control law (3.10) is a fuzzy guaranteed cost control law with minimal upper bound for the performance function.

Proof: By Theorem 3.3, the control law $u(t)$ satisfying (3.33) yields the fuzzy guaranteed cost control law for the system (3.8). Pre-multiplying and post-multiplying (3.33) by $\boldsymbol{Y}$, where $\boldsymbol{Y} = \boldsymbol{P}^{-1}$, produces the following inequality,

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big\{\boldsymbol{Y}(\bar{\boldsymbol{A}}_i + \bar{\boldsymbol{B}}_i\bar{\boldsymbol{K}}_j)^T + (*) + \frac{\boldsymbol{I}}{a^2} + \bar{\boldsymbol{M}}_i\boldsymbol{\epsilon}_i\bar{\boldsymbol{M}}_i^T \\ +\boldsymbol{Y}(\bar{\boldsymbol{N}}_{1i} + \bar{\boldsymbol{N}}_{2i}\bar{\boldsymbol{K}}_j)^T\boldsymbol{\epsilon}_i^{-1}(\bar{\boldsymbol{N}}_{1i} + \bar{\boldsymbol{N}}_{2i}\bar{\boldsymbol{K}}_j)\boldsymbol{Y} + \boldsymbol{Y}\bar{\boldsymbol{Q}}\boldsymbol{Y} + \boldsymbol{Y}\bar{\boldsymbol{R}}_j\boldsymbol{Y}\Big\} < \boldsymbol{0} \tag{3.48}$$

Substituting $\bar{\boldsymbol{X}}_j = \bar{\boldsymbol{K}}_j\boldsymbol{Y}$, $\boldsymbol{W} = \left[\begin{array}{c|c}\boldsymbol{I} & -\boldsymbol{I}\end{array}\right]\boldsymbol{Y}$, yields,

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big\{\boldsymbol{Y}(\bar{\boldsymbol{A}}_i + \bar{\boldsymbol{B}}_i\bar{\boldsymbol{K}}_j)^T + (*) + \bar{\boldsymbol{M}}_i\boldsymbol{\epsilon}_i\bar{\boldsymbol{M}}_i^T + \frac{\boldsymbol{I}}{a^2} \\ +\boldsymbol{Y}(\bar{\boldsymbol{N}}_{1i} + \bar{\boldsymbol{N}}_{2i}\bar{\boldsymbol{K}}_j)^T\boldsymbol{\epsilon}_i^{-1}(\bar{\boldsymbol{N}}_{1i} + \bar{\boldsymbol{N}}_{2i}\bar{\boldsymbol{K}}_j)\boldsymbol{Y} + \boldsymbol{W}^T\boldsymbol{Q}\boldsymbol{W} + \boldsymbol{Y}\bar{\boldsymbol{K}}_j^T\boldsymbol{R}\bar{\boldsymbol{K}}_j\boldsymbol{Y}\Big\} < \boldsymbol{0}$$

$$\text{or,}\quad \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big\{\boldsymbol{Y}\bar{\boldsymbol{A}}_i^T + \bar{\boldsymbol{X}}_j^T\bar{\boldsymbol{B}}_i^T + (*) + \bar{\boldsymbol{M}}_i\boldsymbol{\epsilon}_i\bar{\boldsymbol{M}}_i^T + \frac{\boldsymbol{I}}{a^2} \\ +(\bar{\boldsymbol{N}}_{1i}\boldsymbol{Y} + \bar{\boldsymbol{N}}_{2i}\bar{\boldsymbol{X}}_j)^T\boldsymbol{\epsilon}_i^{-1}(\bar{\boldsymbol{N}}_{1i}\boldsymbol{Y} + \bar{\boldsymbol{N}}_{2i}\bar{\boldsymbol{X}}_j) + \boldsymbol{W}^T\boldsymbol{Q}\boldsymbol{W} + \bar{\boldsymbol{X}}_j^T\boldsymbol{R}\bar{\boldsymbol{X}}_j\Big\} < \boldsymbol{0} \tag{3.49}$$

By Schur Complement, (3.49) is equivalent to the following LMIs,

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\begin{bmatrix}\hat{\boldsymbol{H}}_{ij} & * & * & * \\ \bar{\boldsymbol{N}}_{1i}\boldsymbol{Y} + \bar{\boldsymbol{N}}_{2i}\bar{\boldsymbol{X}}_j & -\boldsymbol{\epsilon}_i & * & * \\ \boldsymbol{W} & \boldsymbol{0} & -\boldsymbol{Q}^{-1} & * \\ \bar{\boldsymbol{X}}_j & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{R}^{-1}\end{bmatrix} < \boldsymbol{0} \tag{3.50}$$

where, $*$ represents the transposed elements in symmetric positions. Applying Lemma A.1 to the above inequality, the conditions (3.45) and (3.46) are obtained. Hence if the conditions (3.45) and (3.46) are satisfied, then $u(t)$ is the guaranteed cost control law for the fuzzy system.

Also, it follows from Schur complement that (3.47) is equivalent to $\bar{\boldsymbol{x}}^T(0)\boldsymbol{Y}^{-1}\bar{\boldsymbol{x}}(0) < \alpha$. Hence

$$\bar{\boldsymbol{x}}^T(0)\boldsymbol{P}\bar{\boldsymbol{x}}(0) + a^2\int_0^{t_f}\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t)dt < \alpha + a^2\int_0^{t_f}\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t)dt$$

From (3.35), it follows that

$$J_0 < \alpha + a^2\int_0^{t_f}\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t)dt \tag{3.51}$$

Thus minimization of $\alpha + a^2\int_0^{t_f}\bar{\boldsymbol{r}}^T(t)\bar{\boldsymbol{r}}(t)dt$ implies minimization of the upper bound of the guaranteed cost J_0 for the system (3.2). This completes the proof. □

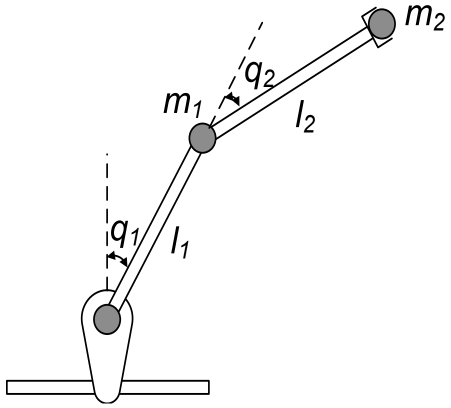


Fig. 3.2: Configuration of a two-link robotic manipulator

The matrix inequalities given by (3.33) are expressed as standard LMIs in Theorem 3.4 and these can be solved easily and efficiently for a, $\boldsymbol{P}$, $\boldsymbol{K}_{1j}$and $\boldsymbol{K}_{2j}$.

3.5 Simulation Results

3.5.1 Optimal Fuzzy Guaranteed Cost Control

The performance of the proposed fuzzy logic based controller is tested on the tracking control problem for a two-link robotic manipulator. The configuration of a two-link robotic manipulator is shown in Fig. 3.2. The dynamics of a two-link robotic manipulator can be expressed as [92],

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{\tau} \tag{3.52}$$

where

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{bmatrix} (m_1 + m_2)l_1^2 & m_2 l_1 l_2 (s_1 s_2 + c_1 c_2) \\ m_2 l_1 l_2 (s_1 s_2 + c_1 c_2) & m_2 l_2^2 \end{bmatrix}$$

$$\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = m_2 l_1 l_2 (c_1 s_2 - s_1 c_2) \begin{bmatrix} 0 & -\dot{q}_2 \\ -\dot{q}_1 & 0 \end{bmatrix}, \quad \boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} -(m_1 + m_2) l_1 g s_1 \\ m_2 l_2 g s_2 \end{bmatrix}$$

Here, $\boldsymbol{q}(t) = [q_1(t), q_2(t)]$, where $q_1(t)$ and $q_2(t)$ are angular positions of joints 1 and 2, $\boldsymbol{M}(\boldsymbol{q})$ is the inertia matrix, $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is the centripetal Coriolis matrix, $\boldsymbol{G}(\boldsymbol{q})$ is the gravity vector and $\boldsymbol{\tau} = [\tau_1, \tau_2]^T$ is the applied torque. The two links of the manipulator have masses m_1, m_2 in kilograms and lengths l_1, l_2 in meters. The acceleration due to gravity g is 9.81 m/s^2. Here $s_1 = \sin(q_1)$, $s_2 = \sin(q_2)$, $c_1 = \cos(q_1)$, and $c_2 = \cos(q_2)$. Friction terms are ignored.

The two-link robotic manipulator has four inner states $x_1(t) = q_1(t)$, $x_2(t) = \dot{q}_1(t)$, $x_3(t) = q_2(t)$

and $x_4(t) = \dot{q}_2(t)$, two output states $y_1(t) = q_1(t)$ and $y_2(t) = q_2(t)$, and two inputs $u_1(t) = \tau_1$ and $u_2(t) = \tau_2$. The nine rule T-S fuzzy model as reported in [92] is used to represent the dynamics of this nonlinear system. In this model, link masses $m_1 = 1\,kg$, $m_2 = 1\,kg$ and link lengths $l_1 = 1\,m$, $l_2 = 1\,m$ are considered. The angular positions $q_1(t)$ and $q_2(t)$ are constrained within $[-\pi/2, \pi/2]$. Triangular type of membership function as shown in Fig. 3.3 is assumed for $x_1(t)$ and $x_3(t)$ in rule 1 to rule 9.

The nine rule fuzzy model is given below:

Rule 1: *IF* $x_1(t)$ is about $-\frac{\pi}{2}$ and $x_3(t)$ is about $-\frac{\pi}{2}$, *THEN*

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_1\boldsymbol{x}(t) + \boldsymbol{B}_1\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}_1\boldsymbol{x}(t) \end{aligned}$$

Rule 2: *IF* $x_1(t)$ is about $-\frac{\pi}{2}$ and $x_3(t)$ is about *zero*, *THEN*

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_2\boldsymbol{x}(t) + \boldsymbol{B}_2\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}_2\boldsymbol{x}(t) \end{aligned}$$

Rule 3: *IF* $x_1(t)$ is about $-\frac{\pi}{2}$ and $x_3(t)$ is about $\frac{\pi}{2}$, *THEN*

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_3\boldsymbol{x}(t) + \boldsymbol{B}_3\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}_3\boldsymbol{x}(t) \end{aligned}$$

Rule 4: *IF* $x_1(t)$ is about *zero* and $x_3(t)$ is about $-\frac{\pi}{2}$, *THEN*

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_4\boldsymbol{x}(t) + \boldsymbol{B}_4\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}_4\boldsymbol{x}(t) \end{aligned}$$

Rule 5: *IF* $x_1(t)$ is about *zero* and $x_3(t)$ is about *zero*, *THEN*

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_5\boldsymbol{x}(t) + \boldsymbol{B}_5\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}_5\boldsymbol{x}(t) \end{aligned}$$

Rule 6: *IF* $x_1(t)$ is about *zero* and $x_3(t)$ is about $\frac{\pi}{2}$, *THEN*

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_6\boldsymbol{x}(t) + \boldsymbol{B}_6\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}_6\boldsymbol{x}(t) \end{aligned}$$

Rule 7: *IF* $x_1(t)$ is about $\frac{\pi}{2}$ and $x_3(t)$ is about $-\frac{\pi}{2}$, *THEN*

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_7\boldsymbol{x}(t) + \boldsymbol{B}_7\boldsymbol{u}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}_7\boldsymbol{x}(t)\end{aligned}$$

Rule 8: *IF* $x_1(t)$ is about $\frac{\pi}{2}$ and $x_3(t)$ is about *zero*, *THEN*

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_8\boldsymbol{x}(t) + \boldsymbol{B}_8\boldsymbol{u}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}_8\boldsymbol{x}(t)\end{aligned}$$

Rule 9: *IF* $x_1(t)$ is about $\frac{\pi}{2}$ and $x_3(t)$ is about $\frac{\pi}{2}$, *THEN*

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_9\boldsymbol{x}(t) + \boldsymbol{B}_9\boldsymbol{u}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}_9\boldsymbol{x}(t)\end{aligned}$$

where

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 5.927 & -0.001 & -0.315 & -8.4\times10^{-6} \\ 0 & 0 & 0 & 1 \\ -6.859 & 0.002 & 3.155 & 6.2\times10^{-6} \end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3.0428 & -0.001 & 0.1791 & -0.0002 \\ 0 & 0 & 0 & 1 \\ 3.5436 & 0.0313 & 2.5611 & 1.14\times10^{-5} \end{bmatrix},$$

$$\boldsymbol{A}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.2728 & 0.003 & 0.4339 & -0.0001 \\ 0 & 0 & 0 & 1 \\ 9.1041 & 0.0158 & -1.0574 & -3.2\times10^{-5} \end{bmatrix}, \quad \boldsymbol{A}_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.4535 & 0.0017 & 1.243 & 0.0002 \\ 0 & 0 & 0 & 1 \\ -3.187 & -0.031 & 5.191 & -1.8\times10^{-5} \end{bmatrix},$$

$$\boldsymbol{A}_5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 11.1336 & 0 & -1.8145 & 0 \\ 0 & 0 & 0 & 1 \\ -9.0918 & 0 & 9.1638 & 0 \end{bmatrix}, \quad \boldsymbol{A}_6 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.1702 & -0.001 & 1.6870 & -0.0002 \\ 0 & 0 & 0 & 1 \\ -2.3559 & 0.0314 & 4.5298 & 1.1\times10^{-5} \end{bmatrix},$$

$$\boldsymbol{A}_7 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.1206 & -0.0041 & 0.6205 & 0.0001 \\ 0 & 0 & 0 & 1 \\ 8.8794 & -0.0193 & -1.0119 & 4.4\times10^{-5} \end{bmatrix}, \quad \boldsymbol{A}_8 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3.6421 & 0.0018 & 0.0721 & 0.0002 \\ 0 & 0 & 0 & 1 \\ 2.4290 & -0.031 & 2.9832 & -1.9\times10^{-5} \end{bmatrix},$$

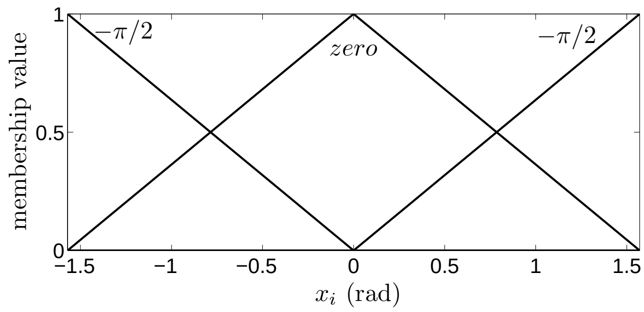


Fig. 3.3: Membership for the input variables x_1 and x_3

$$\boldsymbol{A}_9 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.293 & -0.001 & -0.218 & -1.2\times 10^{-5} \\ 0 & 0 & 0 & 1 \\ -7.464 & 0.0024 & 3.2693 & 9.2\times 10^{-6} \end{bmatrix},$$

$$\boldsymbol{B}_1 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ -1 & 2 \end{bmatrix}, \quad \boldsymbol{B}_2 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{B}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}, \quad \boldsymbol{B}_4 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{B}_5 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ -1 & 2 \end{bmatrix},$$

$$\boldsymbol{B}_6 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{B}_7 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}, \quad \boldsymbol{B}_8 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{B}_9 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ -1 & 2 \end{bmatrix},$$

$$\boldsymbol{C}_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ for } i = 1, ..., 9.$$

The reference model and reference input in [92] are given as:

$$\dot{\boldsymbol{x}}_r(t) = \boldsymbol{A}_r \boldsymbol{x}_r(t) + \boldsymbol{r}(t) \tag{3.53}$$

where

$$\boldsymbol{A}_r = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -6 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & -5 \end{bmatrix} \tag{3.54}$$

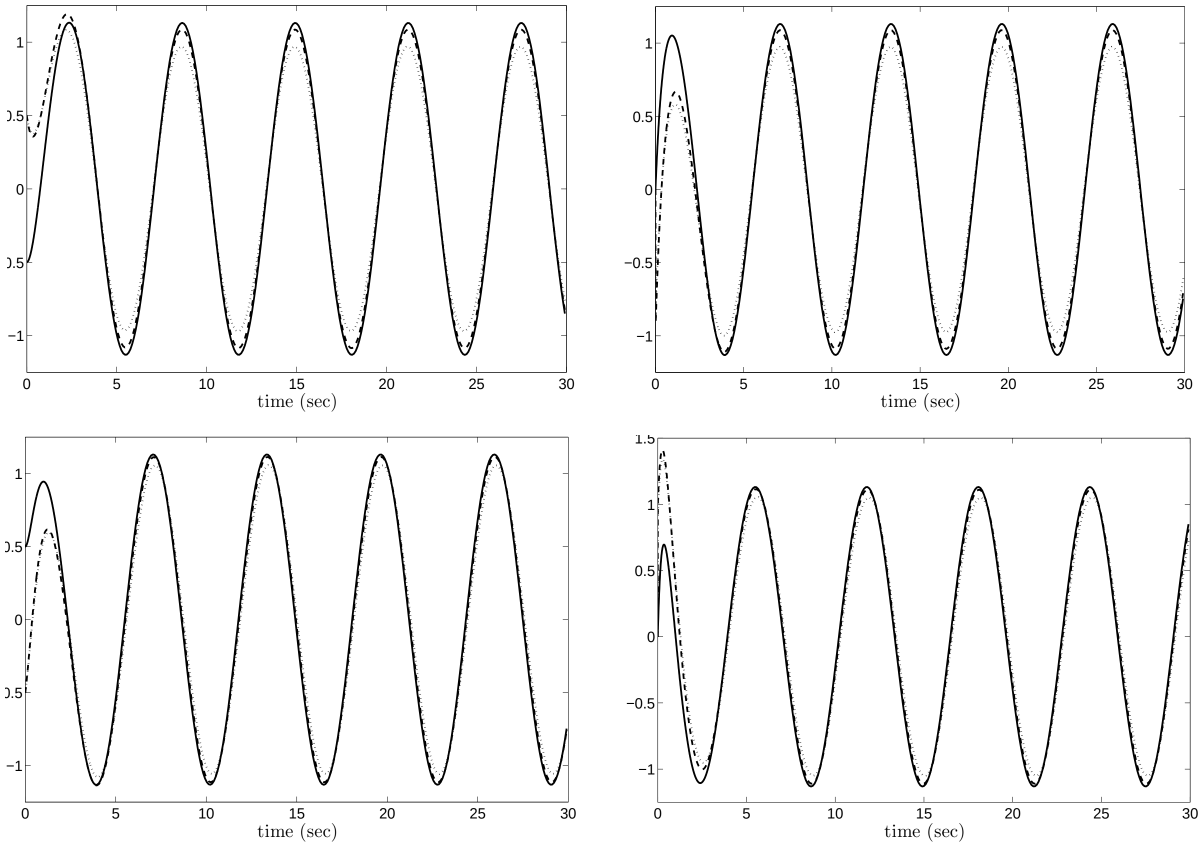


Fig. 3.4: Trajectories of state variables $\boldsymbol{x}(t)$ (dashed line and dotted line for $\boldsymbol{Q} = 25 \times 10^3\boldsymbol{I}$, $\boldsymbol{R} =$ diag$[0.5 \;\; 0.5]$, and $\boldsymbol{Q} = 10 \times 10^3\boldsymbol{I}$, $\boldsymbol{R} =$ diag$[2 \;\; 2]$ respectively) and the reference trajectories $\boldsymbol{x}_r(t)$ (solid line) with $\boldsymbol{A}_r$, $\boldsymbol{r}(t)$ as defined in (3.54) and (3.55).

and

$$\boldsymbol{r}(t) = [0, \;\; 8sin(t), \;\; 0, \;\; 8cos(t)]^T. \tag{3.55}$$

The matrix inequalities in Theorem 3.2 are solved using the LMI solver *'mincx'* available in MATLAB Robust Control Toolbox while minimizing the performance function J for $\boldsymbol{Q} = 25 \times 10^3\boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix of appropriate dimension and $\boldsymbol{R} =$ diag$[0.5 \;\; 0.5]$ (Desired control results can be obtained by proper choice of $\boldsymbol{Q}$ and $\boldsymbol{R}$). The initial conditions are assumed as $x_1(0) = 0.5$, $x_2(0) = 0$, $x_3(0) = -0.5$, $x_4(0) = 0$ and $x_{r1}(0) = -0.5$, $x_{r2}(0) = 0$, $x_{r3}(0) = 0.5$, $x_{r4}(0) = 0$. The value of a, $\boldsymbol{P}$ and the feedback gain matrices $\boldsymbol{K}_{1j}$ and $\boldsymbol{K}_{2j}$ obtained are given below.

$$\begin{aligned} a &= 1838.4 \\ \boldsymbol{P} &= \begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ \boldsymbol{P}_{12}^T & \boldsymbol{P}_{22} \end{bmatrix} \end{aligned}$$

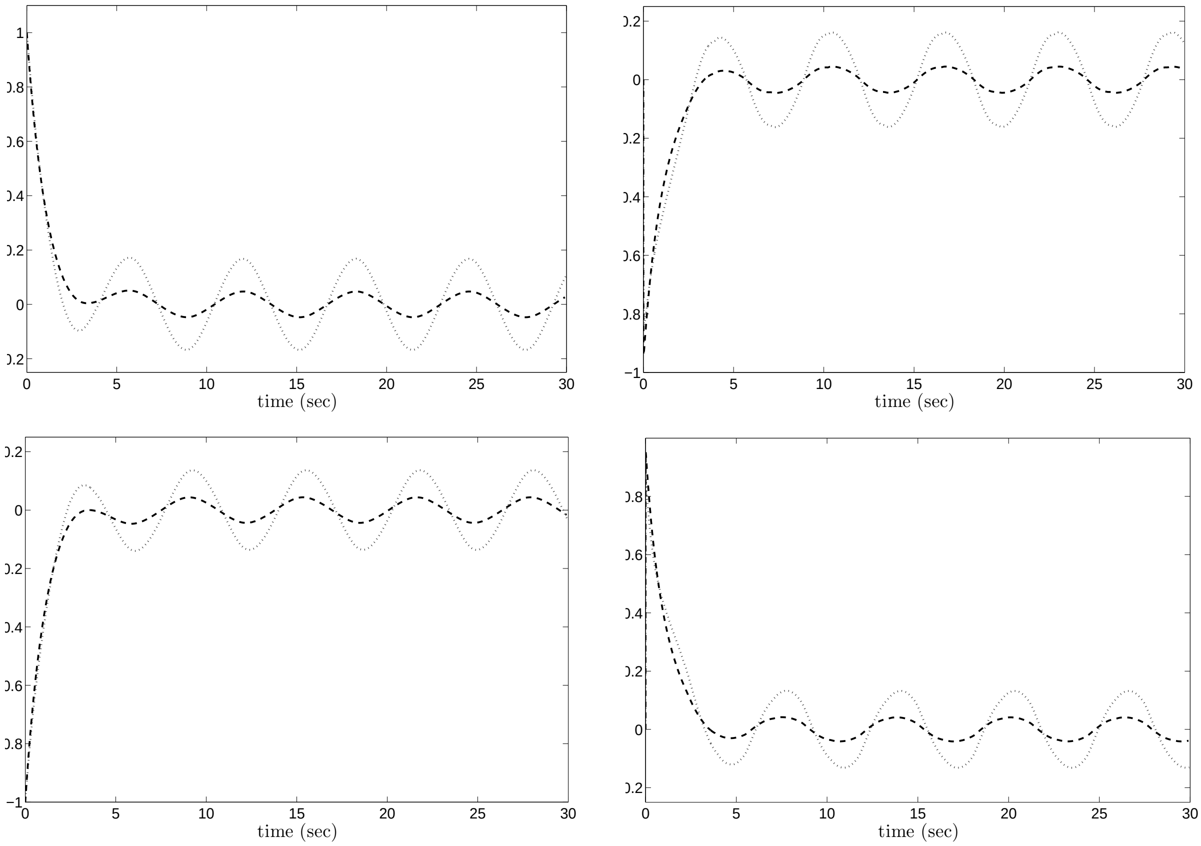


Fig. 3.5: Tracking error between actual and reference trajectories for different state variables (Dashed and dotted line for $\boldsymbol{Q} = 25 \times 10^3 \boldsymbol{I}$, $\boldsymbol{R} = \mathrm{diag}[0.5 \;\; 0.5]$ and $\boldsymbol{Q} = 10 \times 10^3 \boldsymbol{I}$, $\boldsymbol{R} = \mathrm{diag}[2 \;\; 2]$ respectively) with $\boldsymbol{A}_r$, $\boldsymbol{r}(t)$ as defined in (3.54) and (3.55)

where

$$\boldsymbol{P}_{11} = \begin{bmatrix} 110773 & * & * & * \\ 4195.4 & 854.46 & * & * \\ 85109 & 3847.5 & 110482 & * \\ 3512.9 & 496.54 & 3748.6 & 723.8 \end{bmatrix}, \quad \boldsymbol{P}_{22} = \begin{bmatrix} 110195 & * & * & * \\ 4036.7 & 825.7 & * & * \\ 84735 & 3719 & 110242 & * \\ 3448.8 & 490.1 & 3685.5 & 712.3 \end{bmatrix}$$

$$\boldsymbol{P}_{12} = \begin{bmatrix} -110414 & -4031.8 & -84981 & -3456.2 \\ -4141.3 & -824.6 & -3833.2 & -493.49 \\ -84834 & -3716.7 & -110336 & -3682.9 \\ -3491.5 & -488.38 & -3726.9 & -710.59 \end{bmatrix}$$

$$\boldsymbol{K}_{11} = \begin{bmatrix} -2098 & -713.1 & -1236 & 153.7 \\ -4640 & -456.7 & -5429 & -1236 \end{bmatrix} \qquad \boldsymbol{K}_{21} = \begin{bmatrix} 2040 & 676.3 & 1242 & -141.3 \\ 4632 & 461.1 & 5391 & 1209 \end{bmatrix}$$

$$\boldsymbol{K}_{12} = \begin{bmatrix} -4614 & -938.6 & -4233 & -547.4 \\ -5185 & -728.4 & -5545 & -1074 \end{bmatrix} \qquad \boldsymbol{K}_{22} = \begin{bmatrix} 4554 & 906 & 4217 & 544 \\ 5155 & 716.8 & 5512 & 1055 \end{bmatrix}$$

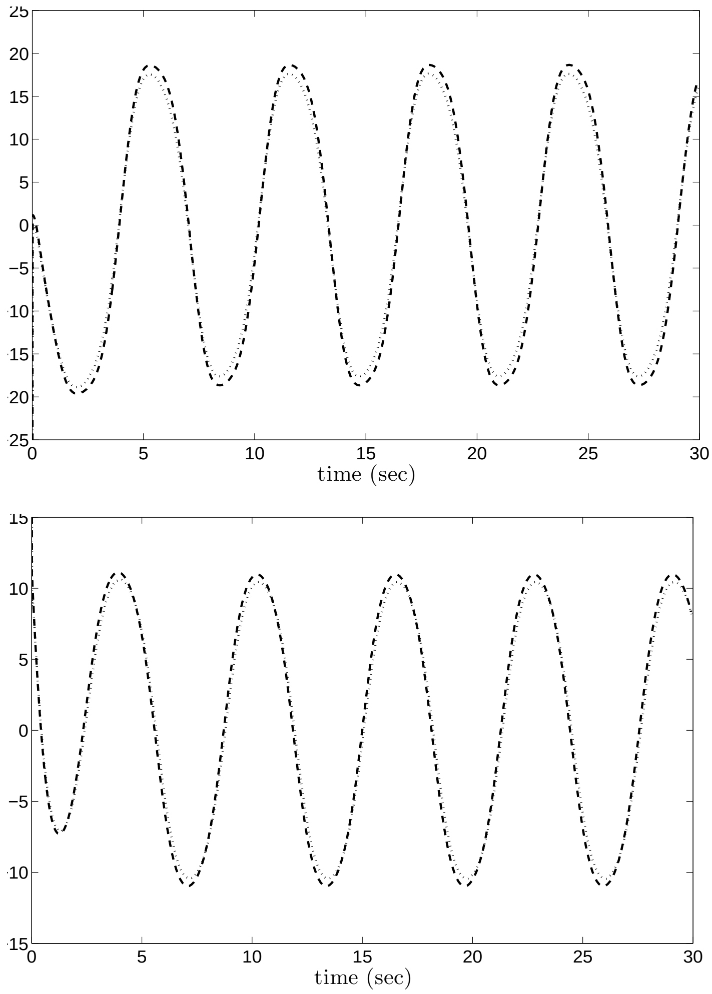


Fig. 3.6: Control input (Dashed and dotted line for $\boldsymbol{Q} = 25 \times 10^3\boldsymbol{I}$, $\boldsymbol{R} = \mathrm{diag}[0.5 \quad 0.5]$, and $\boldsymbol{Q} = 10 \times 10^3\boldsymbol{I}$, $\boldsymbol{R} = \mathrm{diag}[2 \quad 2]$ respectively) with $\boldsymbol{A}_r$, $\boldsymbol{r}(t)$ as defined in (3.54) and (3.55)

$$\boldsymbol{K}_{13} = \begin{bmatrix} -9965 & -1788 & -9719 & -1519 \\ -7363 & -1154 & -7590 & -1359 \end{bmatrix} \qquad \boldsymbol{K}_{23} = \begin{bmatrix} 9862 & 1736 & 9674 & 1500 \\ 7308 & 1128 & 7548 & 1337 \end{bmatrix}$$

$$\boldsymbol{K}_{14} = \begin{bmatrix} -4474 & -901.8 & -4126 & -542.8 \\ -4828 & -702 & -5110 & -968.3 \end{bmatrix} \qquad \boldsymbol{K}_{24} = \begin{bmatrix} 4417 & 870.8 & 4110 & 539.1 \\ 4799 & 689.3 & 5079 & 951.0 \end{bmatrix}$$

$$\boldsymbol{K}_{15} = \begin{bmatrix} -2389 & -684.8 & -1712 & -3.43 \\ -3948 & -436.1 & -4507 & -985 \end{bmatrix} \qquad \boldsymbol{K}_{25} = \begin{bmatrix} 2335 & 652.7 & 1713 & 11.34 \\ 3937 & 436.3 & 4477 & 964.3 \end{bmatrix}$$

$$\boldsymbol{K}_{16} = \begin{bmatrix} -4398 & -892.2 & -4041 & -525.5 \\ -4755 & -681.9 & -5053 & -966.4 \end{bmatrix} \qquad \boldsymbol{K}_{26} = \begin{bmatrix} 4342 & 861.2 & 4027 & 522.1 \\ 4726 & 670.1 & 5023 & 949 \end{bmatrix}$$

$$\boldsymbol{K}_{17} = \begin{bmatrix} -8334 & -1513 & -8084 & -1245 \\ -6016 & -926.6 & -6241 & -1133 \end{bmatrix} \qquad \boldsymbol{K}_{27} = \begin{bmatrix} 8246 & 1468 & 8048 & 1230 \\ 5974 & 906.8 & 6206 & 1114 \end{bmatrix}$$

$$\boldsymbol{K}_{18} = \begin{bmatrix} -4488 & -911.2 & -4121 & -534.9 \\ -4733 & -666.9 & -5057 & -978.3 \end{bmatrix} \qquad \boldsymbol{K}_{28} = \begin{bmatrix} 4430 & 879.5 & 4106 & 531.5 \\ 4705 & 656.1 & 5027 & 960.4 \end{bmatrix}$$

$$\boldsymbol{K}_{19} = \begin{bmatrix} -2700 & -698.3 & -2118 & -110.6 \\ -4282 & -495.6 & -4831 & -1036 \end{bmatrix} \qquad \boldsymbol{K}_{29} = \begin{bmatrix} 2648 & 667.8 & 2116 & 115.8 \\ 4266 & 494 & 4800 & 1015 \end{bmatrix}$$

The value of $\boldsymbol{K}_{1j}$ is closer to the value of $-\boldsymbol{K}_{2j}$, which is in agrement with the control structure $\boldsymbol{u}(t) = \sum_{j=1}^{r} \mu_j [\boldsymbol{K}_j(x(t) - x_r(t))]$ considered in [92]. The value of α is 5.1×10^4. The upper bound for the performance function J_0 is obtained as 3.108×10^8. The closed loop system is simulated with the initial conditions of the system state and reference state as $x_1(0) = 0.5$, $x_2(0) = 0$, $x_3(0) = -0.5$, $x_4(0) = 0$ and $x_{r1}(0) = -0.5$, $x_{r2}(0) = 0$, $x_{r3}(0) = 0.5$, $x_{r4}(0) = 0$.

Next the system with different values for $\boldsymbol{Q}$ and $\boldsymbol{R}$ matrices are considered. Let $\boldsymbol{Q} = 10 \times 10^3 \boldsymbol{I}$ and $\boldsymbol{R} = diag[2 \;\; 2]$. When compared to the previous case with $\boldsymbol{Q} = 25 \times 10^3 \boldsymbol{I}$ and $\boldsymbol{R} = diag[0.5 \;\; 0.5]$, now lower weightage is given for the error and higher weightage is given for the input. The controller is designed for this performance measure where the value of α is 2.1×10^4, the upper bound for J_0 is found to be 1.2716×10^8 and some of the obtained controller parameters are given below.

$$\boldsymbol{K}_{11} = \begin{bmatrix} -656 & -242.2 & -368.3 & 45.82 \\ -950.7 & -89.21 & -1183 & -325.5 \end{bmatrix} \qquad \boldsymbol{K}_{21} = \begin{bmatrix} 578.9 & 197.7 & 369.4 & -34.94 \\ 920.9 & 82.7 & 1141 & 297.3 \end{bmatrix}$$

$$\boldsymbol{K}_{15} = \begin{bmatrix} -690.5 & -224.2 & -462.8 & 2.455 \\ -844.5 & -98.45 & -1005 & -261.1 \end{bmatrix} \qquad \boldsymbol{K}_{25} = \begin{bmatrix} 618.6 & 184.2 & 459.4 & 3.857 \\ 813.5 & 88.44 & 971 & 239.2 \end{bmatrix}$$

$$\boldsymbol{K}_{19} = \begin{bmatrix} -751.2 & -225 & -550.2 & -25.48 \\ -908.9 & -111 & -1067 & -272.7 \end{bmatrix} \qquad \boldsymbol{K}_{29} = \begin{bmatrix} 678.7 & 185.7 & 543.1 & 29.18 \\ 872.4 & 99.12 & 1031 & 250 \end{bmatrix}$$

It is observed that increase of the weightage on $\boldsymbol{R}$ will decrease the controller gain $\boldsymbol{K}_{1j}$ and $\boldsymbol{K}_{2j}$ and hence lower the magnitude of the control input $\boldsymbol{u}(t) = \sum_{j=1}^{r} \mu_j (\boldsymbol{K}_{1j}\boldsymbol{x}(t) + \boldsymbol{K}_{2j}\boldsymbol{x}_r(t))$. The simulation results for both the above cases are plotted in Figs. 3.4 – 3.6. Results for $\boldsymbol{Q} = 25 \times 10^3 \boldsymbol{I}$, $\boldsymbol{R} = \text{diag}[0.5 \;\; 0.5]$ are shown by dashed lines and results for $\boldsymbol{Q} = 10 \times 10^3 \boldsymbol{I}$, $\boldsymbol{R} = \text{diag}[2 \;\; 2]$ are shown by dotted lines. Fig. 3.4 compares the actual trajectory for these two cases (dashed and dotted lines) with the reference trajectory (solid line) for the state variables $x_1(t)$, $x_2(t)$, $x_3(t)$ and $x_4(t)$. Fig. 3.5 compares the error between the reference trajectory and the actual trajectory for both these cases for all the states $x_1(t)$, $x_2(t)$, $x_3(t)$ and $x_4(t)$. Fig. 3.6 shows the control inputs $u_1(t)$ and $u_2(t)$ for the given performance measure. Above simulation results illustrate the effectiveness of the proposed controller design. It is observed from the Figs. 3.4 – 3.6 that the desired control result can be obtained by suitable selection of $\boldsymbol{Q}$ and $\boldsymbol{R}$ matrices.

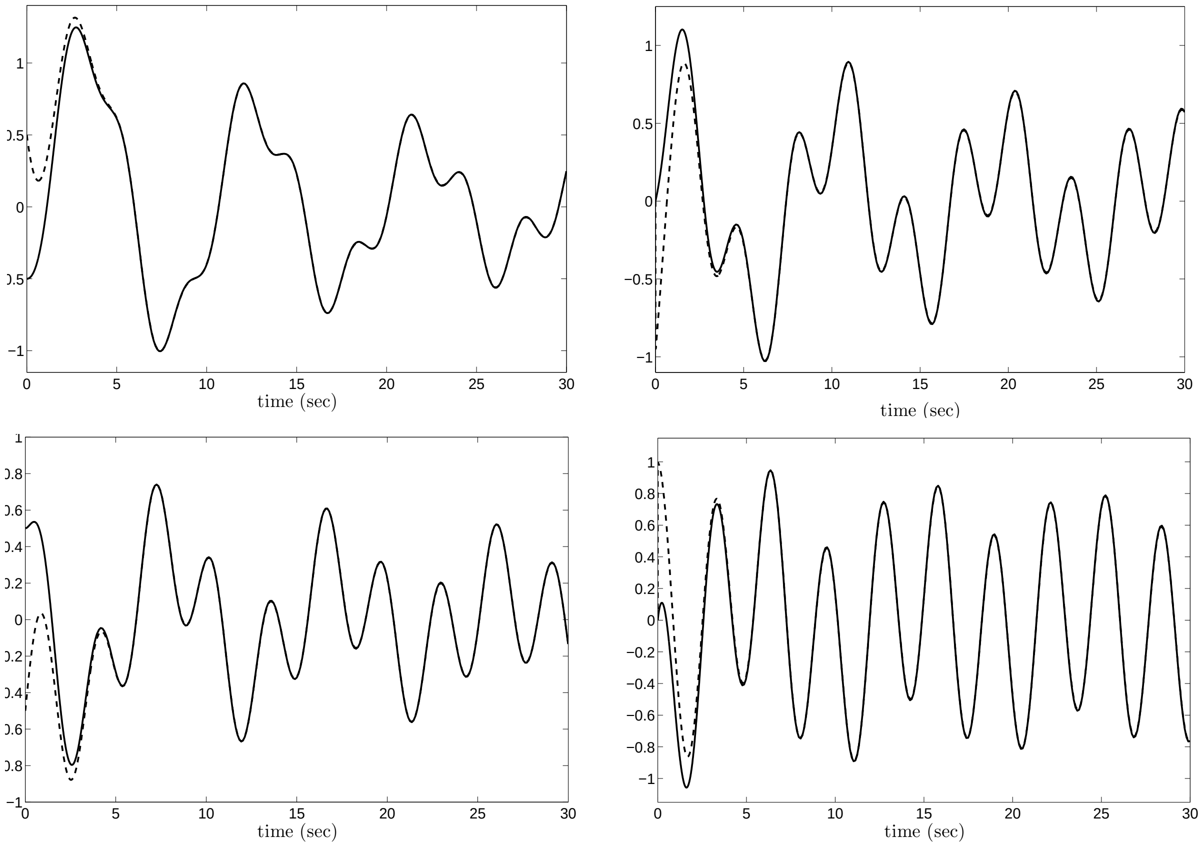


Fig. 3.7: Trajectories of state variables $\boldsymbol{x}(t)$ (dashed line) and the reference trajectories $\boldsymbol{x}_r(t)$ (solid line) with $\boldsymbol{A}_r$, $\boldsymbol{r}(t)$ as defined in (3.56) and (3.57)

Now the reference model (3.4) with different value of $\boldsymbol{A}_r$ and $\boldsymbol{r}(t)$ are considered as given below:

$$\boldsymbol{A}_r = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & -3 & -4 \end{bmatrix} \tag{3.56}$$

and

$$\boldsymbol{r}(t) = [0, \quad 0.5sin(2t), \quad 0, \quad 2cos(2t)]^T. \tag{3.57}$$

The feedback gain matrices are obtained with the reference model parameters as given in (3.56) and (3.57) and $\boldsymbol{Q} = 25 \times 10^3 \boldsymbol{I}$, $\boldsymbol{R} = \text{diag}[0.5 \ \ 0.5]$. The simulation results are shown in Fig. 3.7 and in this case also the system states closely track the reference trajectory.

3.5.2 Robust Optimal Fuzzy Guaranteed Cost Control

Let us consider the nonlinear equation with parameter uncertainties representing the equation of motion of an inverted pendulum on a cart as given below (Fig. 3.8) [66, 109].

$$\begin{aligned} \dot{x_1}(t) &= x_2(t) \\ \dot{x_2}(t) &= \frac{g_r \sin(x_1(t)) - amlx_2^2(t)\sin(2x_1(t))/2 - a\cos(x_1(t))u(t)}{4l/3 - aml\cos^2(x_1(t))} \\ y(t) &= x_1(t). \end{aligned} \tag{3.58}$$

Here $x_1(t)$ and $x_2(t)$ represent the angular displacement about the vertical axis (in rad) and the angular velocity (in rad/sec) respectively, $g_r = 9.8$ m/s^2 is the acceleration due to gravity, $a = 1/(m+M)$, $m \in [m_{\min}\ \ m_{\max}] = [2\ \ 3]$ kg is the mass of the pendulum, $M \in [M_{\min}\ \ M_{\max}] = [8\ \ 10]$ kg is the mass of the cart, $2l = 1$ m is the length of the pendulum and $u(t)$ is the force applied on the cart (in Newton).

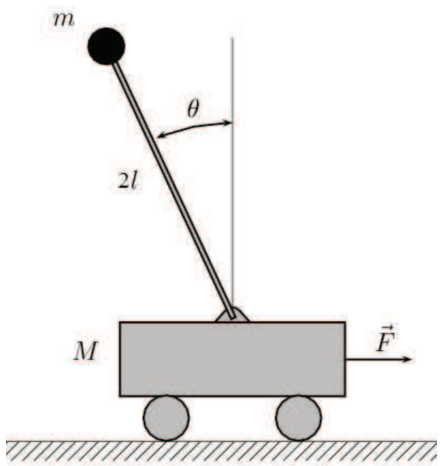


Fig. 3.8: Configuration of an inverted pendulum

The fuzzy model of this system is derived in Chapter 2 and it is used to design the controller. The model is described by the following rules:

Plant rule i:

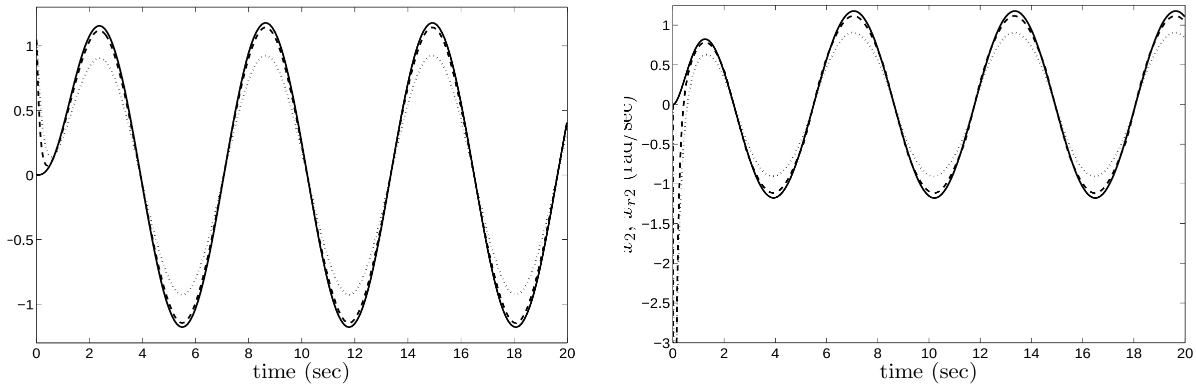
IF x_1 is about N_i *THEN*

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= (\boldsymbol{A}_i + \Delta\boldsymbol{A}_i(t))\boldsymbol{x}(t) + (\boldsymbol{B}_i + \Delta\boldsymbol{B}_i(t))u(t) \\ y(t) &= \boldsymbol{C}_i\boldsymbol{x}(t), \end{aligned}$$

where N_i is the triangular fuzzy set of x_1 about $0, \pm\pi/12, \pm\pi/6, \pm\pi/4, \pm\pi/3$ for $i = 1, 2, ..., 5$

Table. 3.1: Parameters of the feedback gain matrices $\boldsymbol{K}_i$ for robust fuzzy guaranteed cost controller – Inverted pendulum

		$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
$\boldsymbol{Q}=10^5\boldsymbol{I}$, $\boldsymbol{R}=1$	K_{11i}	15213	15862	14646	12155	5978.4
	K_{12i}	1789.5	1866.4	1723.4	1429.5	695.85
	K_{21i}	-14252	-14864	-13725	-11385	-5545.2
	K_{22i}	-1777.9	-1854.4	-1712.3	-1420.3	-691.53
$\boldsymbol{Q}=5\times10^4\boldsymbol{I}$, $\boldsymbol{R}=2$	K_{11i}	5382.8	5611.4	5180.7	4300.6	2187.1
	K_{12i}	1070.6	1117.2	1031.5	855.04	422.08
	K_{21i}	-4014.4	-4189	-3867.6	-3205.7	-1586
	K_{22i}	-846.74	-883.58	-815.79	-676.21	-334.03



(a) $x_1(t)$ and $x_{r1}(t)$ (b) $x_2(t)$ and $x_{r2}(t)$

Fig. 3.9: Trajectories of state variables $\boldsymbol{x}(t)$ (dashed line: $\boldsymbol{Q}=10^5\boldsymbol{I}$ and $\boldsymbol{R}=1$, dotted line: $\boldsymbol{Q}=5\times10^4\boldsymbol{I}$ and $\boldsymbol{R}=2$) and the reference trajectories $\boldsymbol{x}_r(t)$ (solid line).

respectively. The matrices $\boldsymbol{A}_i$, $\boldsymbol{B}_i$, $\Delta\boldsymbol{A}_i$, and $\Delta\boldsymbol{B}_i$ take the following forms,

$$\boldsymbol{A}_i=\begin{bmatrix}0 & 0\\ a_{i21} & a_{i22}\end{bmatrix},\quad \boldsymbol{B}_i=\begin{bmatrix}0\\ b_{i21}\end{bmatrix},\quad \Delta\boldsymbol{A}_i(t)=\begin{bmatrix}0 & 0\\ \Delta a_{i21}(t) & \Delta a_{i22}(t)\end{bmatrix},\quad \Delta\boldsymbol{B}_i(t)=\begin{bmatrix}0\\ \Delta a_{i21}(t)\end{bmatrix},$$

$$\boldsymbol{C}_i=\begin{bmatrix}1 & 0\end{bmatrix}.$$

where the parameters of $\boldsymbol{A}_i$, $\boldsymbol{B}_i$ and the bounds of $\Delta\boldsymbol{A}_i(t)$, $\Delta\boldsymbol{B}_i(t)$ are shown in Table 2.1 (Chapter 2).

For the tracking control problem, the reference model and the reference input given in [66] are considered as shown below:

$$\begin{bmatrix}\dot{x}_{r1}(t)\\ \dot{x}_{r2}(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -4 & -3\end{bmatrix}\begin{bmatrix}x_{r1}(t)\\ x_{r2}(t)\end{bmatrix}+\begin{bmatrix}0\\ 5\sin(t)\end{bmatrix}. \quad (3.59)$$

Similar to the example shown in the previous subsection, two cases, viz (i) $\boldsymbol{Q}=10^5\boldsymbol{I}$ and $\boldsymbol{R}=1$ and (ii) $\boldsymbol{Q}=5\times10^4\boldsymbol{I}$ and $\boldsymbol{R}=2$ are considered. With the LMI optimization problem described in Theorem 3.4, the parameters of feedback gain matrices $\boldsymbol{K}_{1i}=[K_{11i}\ \ K_{12i}]$, $\boldsymbol{K}_{2i}=[K_{21i}\ \ K_{22i}]$ are obtained as shown in Table 3.1. The parameters of $\boldsymbol{P}$ is The upper bounds of the cost function J_0 are obtained as 1.26×10^9 and 6.65×10^8 for the first and second case, respectively. Mass of the pendulum and the cart are assumed to vary as $m + \Delta m(t) = 2.5 + 0.5\sin^2(5t)$ and $M + \Delta M(t) = 9 + \sin(4t)$ respectively. The simulation results with initial condition $x(0) = [\pi/3 \ \ 0]^T$ and $x_r(0) = [0 \ \ 0]^T$ are shown in Fig. 3.9. It is observed from Fig. 3.9 that using the proposed controller, the closed loop system is able to track the reference trajectory closely even in the presence of uncertainties.

3.6 Summary

In this chapter, a T-S model based robust guaranteed cost controller is proposed for trajectory tracking in nonlinear systems. The problem of performance function minimization is formulated with the sufficient conditions in terms of polynomial matrix inequalities (PMIs). The matrix inequalities are then recast into standard LMIs which can be solved using the efficient convex optimization algorithm. To show the effectiveness of our proposed controller design, simulation is carried out for two examples: a two-link robotic manipulator and an inverted pendulum on a cart. Simulation results show that our proposed controller maintains an upper bound on a given cost function while closely tracking the reference trajectory.

Chapter 4

Robust Fuzzy Control of Uncertain Nonlinear Systems via Parametric Lyapunov Function

4.1 Introduction

Fuzzy logic based control has proven to be a successful approach for controlling nonlinear systems [8, 40, 66, 104, 120]. The fuzzy-model proposed by Takagi and Sugeno [22], known as the T-S fuzzy model, is a popular type of model representation. There are many successful applications of the T-S fuzzy model based approach in nonlinear control systems. Linear matrix inequality (LMI) based T–S fuzzy control is one important and successful approach used in nonlinear control. Today adequate literature is available that discusses linear matrix inequality (LMI) based T-S fuzzy control system design using the fixed Lyapunov function (e.g., [2, 62, 115, 121–123]). Though LMI-based approach gained popularity and great success, conservatism is still dominant in fixed quadratic Lyapunov function based approach due to the limited choice of Lyapunov function [108].

To reduce the conservatism in the stabilization problems, different types of Lyapunov functions such as piecewise Lyapunov function and fuzzy Lyapunov function are used in place of parameter independent or fixed Lyapunov function. These Lyapunov functions reduce the conservatism by allowing the Lyapunov function to vary across the different regions. In uncertain nonlinear systems the presence of uncertainty introduces conservatism in the design. For systems with severe uncertainties, the results may be highly conservative. To reduce the conservatism introduced by the uncertain terms, a richer class of Lyapunov function called parametric Lyapunov function is considered in the proposed approach. This design is mainly aimed at nonlinear systems with severe uncertainties.

The stability of uncertain linear systems using the parametric Lyapunov function was analyzed in [124, 125]. It is also reported that the parametric Lyapunov function is a richer class of Lyapunov function candidate suitable for systems with uncertainties. It has been observed that the design with parametric Lyapunov function is less conservative than the fixed Lyapunov function based design [124, 125]. The fixed Lyapunov function is a special case of general parametric Lyapunov function. Apart from being used for stability analysis, parametric Lyapunov function is also used in controller design for linear systems with parameterically varying uncertainties [126].

Motivated from the results of [124,125], this chapter discusses the robust fuzzy control system design using parametric Lyapunov function for nonlinear systems with slowly varying uncertainties. At first, the sufficient conditions for basic quadratic stabilization with parametric Lyapunov function are derived. Next, robust controller design methods are discussed for H_∞ stabilization and H_∞ tracking control. The results obtained for robust H_∞ stabilization with parametric Lyapunov function are presented after reviewing the results from literature for fixed Lyapunov function based robust H_∞ stabilization. In the case of tracking control problem, the design with fixed Lyapunov function based approach proposed by [66, 92] is in bilinear form and it is solved by a two-step procedure. In this thesis, an LMI based robust H_∞ controller with fixed Lyapunov function is discussed for tracking the states of a reference model. Finally, robust H_∞ tracking controller design with parametric Lyapunov function is presented.

4.2 T-S Fuzzy Model and Constant Lyapunov Function based Stability Conditions

This section starts with introduction to T-S fuzzy model and then stability conditions with a constant Lyapunov function are summarized. The continuous fuzzy model proposed by Takagi and Sugeno [22] represents the dynamics of a nonlinear system using fuzzy IF-THEN rules. Let us consider the fuzzy model of an uncertain nonlinear system in the following form:

Plant rule i:

IF $z_1(t)$ is N_{i1} and $z_2(t)$ is N_{i2} and....$z_p(t)$ is N_{ip} *THEN*

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= (\boldsymbol{A}_i + \Delta\boldsymbol{A}_i(t))\boldsymbol{x}(t) + (\boldsymbol{B}_i + \Delta\boldsymbol{B}_i(t))\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}_i\boldsymbol{x}(t), \qquad\qquad i = 1, 2, ...r \end{aligned} \tag{4.1}$$

where $z_1(t), ..., z_p(t)$ are premise variables, p is the number of premise variables, N_{ij} $(j = 1...p)$ is the fuzzy set and r is the number of rules. Here, $\boldsymbol{x}(t) \in \mathbb{R}^{n\times 1}$ is the state vector, $\boldsymbol{y}(t) \in \mathbb{R}^{n_y \times 1}$ is the controlled output and $\boldsymbol{u}(t) \in \mathbb{R}^{m\times 1}$ is the input vector. $\boldsymbol{A}_i \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}_i \in \mathbb{R}^{n\times m}$, $\boldsymbol{C}_i \in \mathbb{R}^{n_y\times n}$ are constant real matrices and $\Delta\boldsymbol{A}_i(t)$, $\Delta\boldsymbol{B}_i(t)$, are time varying matrices of appropriate dimensions, which represent parametric uncertainties.

Given a pair of input and output $(\boldsymbol{x}(t), \boldsymbol{u}(t))$, the final output of the fuzzy system is inferred as follows:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \sum_{i=1}^{r} \mu_i(z(t))\{(\boldsymbol{A}_i + \Delta\boldsymbol{A}_i(t))\boldsymbol{x}(t) + (\boldsymbol{B}_i + \Delta\boldsymbol{B}_i(t))\boldsymbol{u}(t)\} \\ \boldsymbol{y}(t) &= \sum_{i=1}^{r} \mu_i(z(t))\boldsymbol{C}_i\boldsymbol{x}(t) \end{aligned} \tag{4.2}$$

where

$$\mu_i(z(t)) = \frac{\zeta_i(z(t))}{\sum_{j=1}^{r}\zeta_j(z(t))}, \quad \zeta_i(z(t)) = \prod_{j=1}^{p} N_{ij}(z_j(t))$$

and $N_{ij}(z_j(t))$ is the degree of membership of $z_j(t)$ in the fuzzy set N_{ij}.

The uncertain matrices $\Delta\boldsymbol{A}_i(t)$, $\Delta\boldsymbol{B}_i(t)$ are assumed to be norm bounded and are described by [2]:

$$\begin{bmatrix} \Delta\boldsymbol{A}_i(t) & \Delta\boldsymbol{B}_i(t) \end{bmatrix} = \sum_{l=1}^{L} \boldsymbol{M}_{il}\Delta_{il}(t) \begin{bmatrix} \boldsymbol{N}_{i1l} & \boldsymbol{N}_{i2l} \end{bmatrix}, \tag{4.3}$$

where $\boldsymbol{M}_{il}$, $\boldsymbol{N}_{i1l}$ and $\boldsymbol{N}_{i2l}$ are known real constant matrices of appropriate dimension and $\Delta_{il}(t)$ is a time–varying function, satisfying $|\Delta_{il}(t)| < 1, \quad \forall \quad t > 0$.

Let us consider the Parallel Distributed Compensation(PDC) fuzzy controller [8],

$$\boldsymbol{u}(t) = \sum_{i=1}^{r} \mu_i \boldsymbol{K}_i \boldsymbol{x}(t). \tag{4.4}$$

By employing a constant quadratic Lyapunov function $\boldsymbol{x}^T(t)\boldsymbol{P}\boldsymbol{x}(t)$, the stabilization conditions for a closed loop fuzzy system are given by the following theorem.

Theorem 4.1. [2]: Let us consider the fuzzy model (4.2) with the T-S state feedback control law (4.4). If there exists a symmetric and positive definite matrix $\boldsymbol{Y}$, some matrices $\boldsymbol{W}_j$, $(j = 1, 2, ..., r)$ and ϵ_l,$(l = 1, 2, ..., L)$ such that the following matrix inequality is satisfied:

$$\boldsymbol{\phi}_{ii} < 0, \quad i = 1, 2, \ldots, r \tag{4.5}$$

$$\frac{1}{r-1}\boldsymbol{\phi}_{ii} + \frac{1}{2}(\boldsymbol{\phi}_{ij} + \boldsymbol{\phi}_{ji}) < 0, \quad 1 \leq i \neq j \leq r \tag{4.6}$$

where

$$\phi_{ij} = \left[\begin{array}{c|ccc|ccc} \boldsymbol{Y}\boldsymbol{A}_i^T + \boldsymbol{B}_i^T\boldsymbol{W}_j^T + (*) & * & \dots & * & * & \dots & * \\ \hline \epsilon_1 \boldsymbol{M}_{i1}^T & \epsilon_1 & & & & & \\ \vdots & & \ddots & & & \boldsymbol{0} & \\ \epsilon_l \boldsymbol{M}_{il}^T & & & \epsilon_l & & & \\ \hline \boldsymbol{N}_{i11}\boldsymbol{Y} + \boldsymbol{N}_{i21}\boldsymbol{W}_j & & & & \epsilon_1 & & \\ \vdots & & \boldsymbol{0} & & & \ddots & \\ \boldsymbol{N}_{i1l}\boldsymbol{Y} + \boldsymbol{N}_{i2l}\boldsymbol{W}_j & & & & & & \epsilon_l \end{array}\right],$$

and $\boldsymbol{W}_j = \boldsymbol{K}_j\boldsymbol{Y}$, then the uncertain nonlinear system represented by (4.2) with control law (4.4) is globally asymptotically stable.

Proof: The proof is given in Appendix A. □

4.3 Robust Stabilization with Parametric-Lyapunov Function

In the previous section, the results obtained from robust stabilization conditions based on fixed quadratic Lyapunov function have been summarized. In order to reduce the conservatism existing in the results obtained by using fixed Lyapunov function, the parametric-Lyapunov function proposed by Barmish [127] is considered for deriving the conditions for robust stabilization of fuzzy systems.

4.3.1 T-S Fuzzy Model with Uncertainty

Let us consider the fuzzy model of an uncertain nonlinear system in the following form:

Plant rule i:

IF $z_1(t)$ is N_{i1} and $z_2(t)$ is N_{i2} and....$z_p(t)$ is N_{ip} *THEN*

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \left(\boldsymbol{A}_{i0} + \sum_{l=1}^{L}\theta_l(t)\boldsymbol{A}_{il}\right)\boldsymbol{x}(t) + \left(\boldsymbol{B}_{i0} + \sum_{l=1}^{L}\theta_l(t)\boldsymbol{B}_{il}\right)\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}_i\boldsymbol{x}(t), \qquad i = 1, 2, ...r, \end{aligned} \tag{4.7}$$

where $z_1(t), ..., z_p(t)$ are premise variables, p is the number of premise variables, N_{ij} $(j = 1...p)$ is the fuzzy set and r is the number of rules. Here, $\boldsymbol{x}(t) \in \mathbb{R}^{n\times 1}$ is the state vector, $\boldsymbol{y}(t) \in \mathbb{R}^{n_y\times 1}$ is the controlled output and $\boldsymbol{u}(t) \in \mathbb{R}^{m\times 1}$ is the input vector. $\boldsymbol{A}_{i0} \in \mathbb{R}^{n\times n}$, $\boldsymbol{A}_{il} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}_{i0} \in \mathbb{R}^{n\times m}$, $\boldsymbol{B}_{il} \in \mathbb{R}^{n\times m}$, $\boldsymbol{C}_i \in \mathbb{R}^{n_y\times n}$ are constant real matrices and $\theta_l(t)$ represents time varying parametric uncertainties with known lower and upper bounds of uncertainty.

The final output of the fuzzy system can be represented as

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \sum_{i=1}^{r} \mu_i(z(t))\{\boldsymbol{A}_i(\boldsymbol{\theta}(t))\boldsymbol{x}(t) + \boldsymbol{B}_i(\boldsymbol{\theta}(t))\boldsymbol{u}(t)\} \\ \boldsymbol{y}(t) &= \sum_{i=1}^{r} \mu_i(z(t))\boldsymbol{C}_i\boldsymbol{x}(t). \end{aligned} \tag{4.8}$$

where $\boldsymbol{A}_i(\boldsymbol{\theta}(t)) = \boldsymbol{A}_{i0} + \sum_{l=1}^{L}\theta_l(t)\boldsymbol{A}_{il}$ and $\boldsymbol{B}_i(\boldsymbol{\theta}(t)) = \boldsymbol{B}_{i0} + \sum_{l=1}^{L}\theta_l(t)\boldsymbol{B}_{il}$. For simplicity $\boldsymbol{A}_i(\boldsymbol{\theta}(t))$ and $\boldsymbol{B}_i(\boldsymbol{\theta}(t))$ are denoted as $\boldsymbol{A}_i(\boldsymbol{\theta})$ and $\boldsymbol{B}_i(\boldsymbol{\theta})$.

4.3.2 Parametric Lyapunov Function

The fixed Lyapunov function will guard against arbitrarily fast parameter variations. Hence,the results will be conservative for systems with slowly-varying uncertainties [124]. To reduce the conservatism, a class of parameter-dependent Lyapunov functions or parametric Lyapunov function is considered as given by

$$V(t,\boldsymbol{\theta}) = \boldsymbol{x}^T(t)\boldsymbol{P}(\boldsymbol{\theta}(t))\boldsymbol{x}(t) \tag{4.9}$$

where

$$\boldsymbol{P}(\boldsymbol{\theta}(t)) = \boldsymbol{P}_0 + \sum_{l=1}^{L}\theta_l(t)\boldsymbol{P}_l.$$

For uncertain linear systems Gahinet et al. [124] presented the LMI conditions with parametric Lyapunov function for robust stability analysis and showed that results are less conservative than the fixed Lyapunov function. For simplicity $\boldsymbol{P}(\boldsymbol{\theta}(t))$ is denoted as $\boldsymbol{P}(\boldsymbol{\theta})$.

4.3.3 Robust Stabilization

This subsection discusses the conditions for robust stabilization of uncertain fuzzy systems with parametric Lyapunov function.

Theorem 4.2. Let us consider the fuzzy system represented by (4.8) with the PDC control law (4.4). If there exist symmetric matrices $\boldsymbol{P}_0$, $\boldsymbol{P}_l$, $(l = 1, ..., L)$ and some matrices $\boldsymbol{K}_j$, $(j = 1, ..., r)$ such that

$$\boldsymbol{P}_0 + \sum_{l=1}^{L}\theta_l(t)\boldsymbol{P}_l \quad > \quad 0 \tag{4.10}$$

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\left((\boldsymbol{A}_i(\boldsymbol{\theta}) + \boldsymbol{B}_i(\boldsymbol{\theta})\boldsymbol{K}_j)^T\boldsymbol{P}(\boldsymbol{\theta}) + (*) + (\boldsymbol{P}(\dot{\boldsymbol{\theta}}) - \boldsymbol{P}_0)\right) \quad < \quad 0 \tag{4.11}$$

hold for all admissible trajectories of the parameter vector $\boldsymbol{\theta}$, then the uncertain nonlinear system represented by (4.8) with control law (4.4) is asymptotically stable.

Proof: Let us consider the following parametric-Lyapunov function

$$V(t,\boldsymbol{\theta}) = \boldsymbol{x}^T(t)\boldsymbol{P}(\boldsymbol{\theta})\boldsymbol{x}(t) \tag{4.12}$$

where

$$\boldsymbol{P}(\boldsymbol{\theta}) = \boldsymbol{P}_0 + \sum_{l=1}^{L}\theta_l(t)\boldsymbol{P}_l > 0.$$

The time derivative of (4.12) along the trajectory of (4.8) is

$$\begin{aligned}
\dot{V}(t,\boldsymbol{\theta}) &= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\boldsymbol{x}^T(t)\left((\boldsymbol{A}_i(\boldsymbol{\theta})+\boldsymbol{B}_i(\boldsymbol{\theta})\boldsymbol{K}_j)^T\boldsymbol{P}(\boldsymbol{\theta}) + (*) + \frac{d}{dt}\boldsymbol{P}(\boldsymbol{\theta})\right)\boldsymbol{x}(t) && (4.13)\\
&= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\boldsymbol{x}^T(t)\left((\boldsymbol{A}_i(\boldsymbol{\theta})+\boldsymbol{B}_i(\boldsymbol{\theta})\boldsymbol{K}_j)^T\boldsymbol{P}(\boldsymbol{\theta}) + (*) + \dot{\boldsymbol{P}}(\boldsymbol{\theta})\right)\boldsymbol{x}(t) && (4.14)\\
&= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\boldsymbol{x}^T(t)\left((\boldsymbol{A}_i(\boldsymbol{\theta})+\boldsymbol{B}_i(\boldsymbol{\theta})\boldsymbol{K}_j)^T\boldsymbol{P}(\boldsymbol{\theta}) + (*) + \left(\boldsymbol{P}(\dot{\boldsymbol{\theta}}) - \boldsymbol{P}_0\right)\right)\boldsymbol{x}(t) && (4.15)
\end{aligned}$$

From (4.15), $\dot{V}(t,\boldsymbol{\theta}) < 0$ if the following inequality is satisfied:

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\left((\boldsymbol{A}_i(\boldsymbol{\theta})+\boldsymbol{B}_i(\boldsymbol{\theta})\boldsymbol{K}_j)^T\boldsymbol{P}(\boldsymbol{\theta}) + (*) + \left(\boldsymbol{P}(\dot{\boldsymbol{\theta}}) - \boldsymbol{P}_0\right)\right) < 0 \tag{4.16}$$

Hence proved. □

4.3.4 Reduction to Finite Dimensional Matrix Inequalities

The matrix inequalities in the Theorem 4.2 affinely depend on the parameter vector $\boldsymbol{\theta}$. Using the concept of multiconvexity and some assumptions, this parameter dependence can be avoided. As in [124] and [125], it is assumed that the lower and upper bounds of the uncertain parameter and their rates of variation are known. Specifically:

1. Each parameter θ_l ranges between the known lower and upper bounds $\underline{\theta}_l$ and $\overline{\theta}_l$ respectively, i.e.,

$$\theta_l \in [\underline{\theta}_l \;\; \overline{\theta}_l], \tag{4.17}$$

2. The rate of variation $\dot{\theta}_l$ is well defined at all times and satisfies

$$\dot{\theta}_l \in [\underline{v}_l \;\; \overline{v}_l], \tag{4.18}$$

   where $\underline{v}_l$ and $\overline{v}_l$ are known lower and upper bounds of $\dot{\theta}_l$

With these assumptions, the parameter vector θ_l takes values within the hyper-rectangle called parameter box and the rate vector $\dot{\theta}_l$ takes values in another hyper-rectangle called rate box. It is denoted as,

$$\mathcal{V} := \{(\nu_1, \nu_2, ..., \nu_L)^T : \nu_l \in \{\underline{\theta}_l \ \ \overline{\theta}_l\}\}, \tag{4.19}$$

$$\mathcal{W} := \{(\omega_1, \omega_2, ..., \omega_L)^T : \omega_l \in \{\underline{v}_l \ \ \overline{v}_l\}\}, \tag{4.20}$$

which are the set of 2^L vertices of the parameter box and the rate box respectively.

Theorem 4.3. Consider the fuzzy system represented by (4.8) with the PDC control law (4.4). If there exist symmetric matrices $\boldsymbol{P}_0$, $\boldsymbol{P}_l$, $(l = 1, ..., L)$ and some matrices $\boldsymbol{K}_j$, $(j = 1, ..., r)$ such that

$$\boldsymbol{P}_0 + \sum_{l=1}^{L} \nu_l \boldsymbol{P}_l > 0, \quad \forall \ \ \nu \in \mathcal{V} \tag{4.21}$$

$$\boldsymbol{\Gamma}_{ii}(\boldsymbol{\nu}, \boldsymbol{\omega}) < 0, \quad \forall \ \ (\nu, \omega) \in \mathcal{V} \times \mathcal{W}, \quad i = 1, 2, \ldots, r \tag{4.22}$$

$$\frac{1}{r-1}\boldsymbol{\Gamma}_{ii}(\boldsymbol{\nu}, \boldsymbol{\omega}) + \frac{1}{2}(\boldsymbol{\Gamma}_{ij}(\boldsymbol{\nu}, \boldsymbol{\omega}) + \boldsymbol{\Gamma}_{ji}(\boldsymbol{\nu}, \boldsymbol{\omega})) < 0, \quad \forall \ \ (\nu, \omega) \in \mathcal{V} \times \mathcal{W}, \quad 1 \le i \ne j \le r \tag{4.23}$$

$$\boldsymbol{\Pi}_{lii} \ge 0, \quad l = 1, ..., L, \quad i = 1, 2, \ldots, r \tag{4.24}$$

$$\frac{1}{r-1}\boldsymbol{\Pi}_{lii} + \frac{1}{2}(\boldsymbol{\Pi}_{lij} + \boldsymbol{\Pi}_{lji}) \ge 0, \quad l = 1, ..., L, \quad 1 \le i \ne j \le r \tag{4.25}$$

where $\boldsymbol{\Gamma}_{ij}(\boldsymbol{\nu}, \boldsymbol{\omega}) = (\boldsymbol{A}_i(\boldsymbol{\nu}) + \boldsymbol{B}_i(\boldsymbol{\nu})\boldsymbol{K}_j)^T \boldsymbol{P}(\boldsymbol{\nu}) + (*) + (\boldsymbol{P}(\boldsymbol{\omega}) - \boldsymbol{P}_0)$ and $\boldsymbol{\Pi}_{lij} = (\boldsymbol{A}_{il} + \boldsymbol{B}_{il}\boldsymbol{K}_j)^T \boldsymbol{P}_l + (*)$ hold, then the uncertain nonlinear system represented by (4.8) with control law (4.4) is asymptotically stable for all admissible trajectories of the parameter vector $\boldsymbol{\theta}(t)$.

Proof: Applying the multi-convexity concept [124] for the conditions in Theorem 4.2, the closed loop system will be stable in the uncertainly domain (for all admissible values of θ_l and $\dot{\theta}_l$) if the following conditions are satisfied:

(i) The inequalities (4.10) and (4.11) are satisfied in the vertices $\mathcal{V} \times \mathcal{W}$

and

(ii) the following inequality holds

$$\frac{\partial^2}{\partial \theta_l^2}\left(\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j \left((\boldsymbol{A}_i(\boldsymbol{\theta}) + \boldsymbol{B}_i(\boldsymbol{\theta})\boldsymbol{K}_j)^T \boldsymbol{P}(\boldsymbol{\theta}) + (*) + (\boldsymbol{P}(\dot{\boldsymbol{\theta}}) - \boldsymbol{P}_0)\right)\right) \ge 0, \quad \forall \ \ l = 1, ..., L. \tag{4.26}$$

The above condition (i) hold for all admissible values of θ_l and $\dot{\theta}_l$ if the following inequality (4.27) and (4.28) are satisfied.

$$\boldsymbol{P}_0 + \sum_{l=1}^{L} \nu_l \boldsymbol{P}_l > 0, \quad \forall \ \ \nu \in \mathcal{V} \tag{4.27}$$

$$\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j \left((\boldsymbol{A}_i(\boldsymbol{\nu}) + \boldsymbol{B}_i(\boldsymbol{\nu})\boldsymbol{K}_j)^T \boldsymbol{P}(\boldsymbol{\nu}) + (*) + (\boldsymbol{P}(\boldsymbol{\omega}) - \boldsymbol{P}_0)\right) < 0, \quad \forall \ \ (\nu, \omega) \in \mathcal{V} \times \mathcal{W} \tag{4.28}$$

From condition (ii), the following inequality (4.29) is obtained.

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\left((\boldsymbol{A}_{il}+\boldsymbol{B}_{il}\boldsymbol{K}_j)^T\boldsymbol{P}_l+(*)\right)\geq 0, \quad \forall \ \ l=1,...,L \tag{4.29}$$

Applying Lemma A.1 (Appendix) to (4.28) and (4.29), the inequalities (4.22) to (4.25) are obtained. Hence proved. □

The inequalities in Theorem 4.3 are not standard LMIs with respect to $\boldsymbol{P}_0$, $\boldsymbol{P}_l$ and $\boldsymbol{K}_j$ and it is difficult to solve them simultaneously. It is easily observed that if the matrices $\boldsymbol{K}_j$ are fixed then the inequalities in Theorem 4.3 will become standard LMIs and these can be easily solved. For finding the value of $\boldsymbol{K}_j$ the constant Lyapunov function based approach discussed in Section 4.2 can be used. But the results with the constant Lyapunov function based approach may be conservative than the parametric Lyapunov function based approach. Hence instead of finding the value of $\boldsymbol{K}_j$ with entire range of uncertainty, a fraction of the uncertainty can be considered for designing $\boldsymbol{K}_j$ for the fixed Lyapunov function based approach. If this value of $\boldsymbol{K}_j$ is feasible in the entire range of uncertainty with the parametric Lyapunov function based approach then the controller will stabilize the fuzzy system (2.2) over the entire range of the uncertainty. Based on this, the following iterative LMI (ILMI) based algorithm is proposed for solving the inequalities in Theorem 4.3.

Algorithm 4.1:

Step 1: Derive the fuzzy model for the given uncertain nonlinear system.

Step 2: Set counter, c = 1. Choose σ a small positive fraction say 0.05.

Step 3: By the fixed Lyapunov function based approach solve for feedback gain matrices $\boldsymbol{K}_j$. If the LMIs are infeasible, goto Step 5.

Step 4: Substitute the value of $\boldsymbol{K}_j$ in the inequalities given by Theorem 4.3 and solve for $\boldsymbol{P}_0$, $\boldsymbol{P}_l$. If the inequalities are infeasible, goto Step 6.

Step 5: If $(1-c\sigma)>0$, reduce the bounds of the uncertainty range $\overline{\theta}_l$ and $\underline{\theta}_l$ by $(1-c\sigma)\overline{\theta}_l$ and $(1-c\sigma)\underline{\theta}_l$ for $l=1,...,L$ and derive the fuzzy model with the new range of uncertainty. Increment the counter c and goto Step 3.

Step 6: Choose the value of $\boldsymbol{K}_j$ with the latest feasible solution obtained during the previous iterations in Step 4 and Stop. If the conditions in Step 4 are infeasible during all the iterations, then controller cannot be designed with the proposed method.

4.3.5 Example

Let us consider the following nonlinear system and its fuzzy model from [2, 40]. The system is open-loop unstable. The nonlinear system is shown below after including a few additional uncertain terms.

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) + (1+d_2)\sin x_3(t) + [(1+d_1)x_1^2 + 1]u(t) \\
\dot{x}_2(t) &= (1+0.5d_1)x_1(t) + (2+d_1)x_2(t) + 0.75d_1x_3(t) + 0.1w_1(t) \\
\dot{x}_3(t) &= (1+d_1)x_1^2(t)x_2(t) + x_1(t) \\
\dot{x}_4(t) &= (1+d_2)\sin x_3(t) + 0.1w_2(t) \\
y(t) &= 0.1x_2(t) + 0.1x_4(t)
\end{aligned} \tag{4.30}$$

where $x_1(t) \in [-a \quad a]$, $x_2(t) \in [-b \quad b]$ with $a = 0.8$ and $b = 0.6$. The uncertainties are $d_1 = a_{d1}\sin(t/2)/\sqrt{8}$ and $d_2 = a_{d2}\cos(t/2)/2$. Here the stabilization problem is considered and hence the disturbance input is set to $\boldsymbol{w}(t) = 0$. This disturbance inputs will be considered later.

The fuzzy model of the nonlinear system is given by:

Plant rule i:

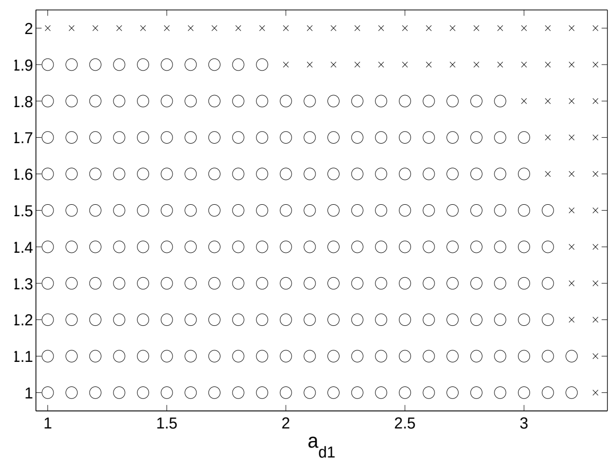
IF $x_1(t)$ is F_1^i and $x_3(t)$ is F_3^i *THEN*

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \left(\boldsymbol{A}_{i0} + \sum_{l=1}^{L}\theta_l\boldsymbol{A}_{il}\right)\boldsymbol{x}(t) + \left(\boldsymbol{B}_{i0} + \sum_{l=1}^{L}\theta_l\boldsymbol{B}_{il}\right)\boldsymbol{u}(t) \\
y(t) &= \boldsymbol{C}\boldsymbol{x}(t), \qquad\qquad i = 1, 2, ...4
\end{aligned} \tag{4.31}$$

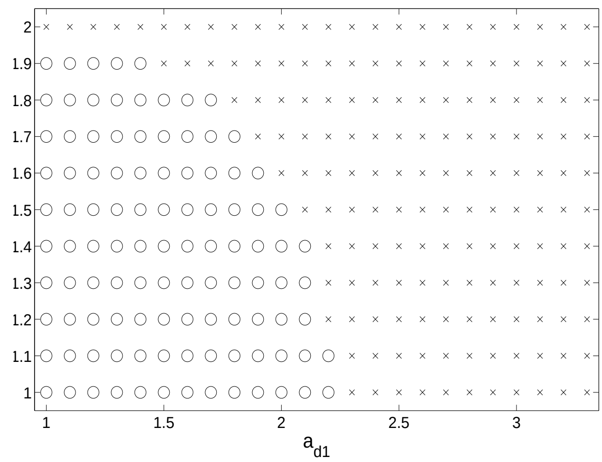
where the fuzzy sets F_1^i, F_3^i, and the parameters of the constant matrices $\boldsymbol{A}_{i0}$, $\boldsymbol{B}_{i0}$, $i = 1, ..., 4$ can be found in [40]. The matrices related to the uncertainty $\theta_1(t) = d_1(t)$ and $\theta_2(t) = d_2(t)$ are given by

$$\boldsymbol{A}_{11} = \boldsymbol{A}_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0.75 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{A}_{31} = \boldsymbol{A}_{41} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0.75 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{A}_{12} = \boldsymbol{A}_{32} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{A}_{22} = \boldsymbol{A}_{42} = \begin{bmatrix} 0 & 0 & \frac{\sin b}{b} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sin b}{b} & 0 \end{bmatrix}$$



(a) With parametric Lyapunov function (proposed)



(b) With Constant Lyapunov function [2]

Fig. 4.1: Stabilization region (o – Feasible point, × – Infeasible point).

$$\boldsymbol{B}_{11} = \boldsymbol{B}_{21} = \begin{bmatrix} a^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{B}_{31} = \boldsymbol{B}_{41} = \boldsymbol{0}_{4\times 1}, \boldsymbol{B}_{12} = ... = \boldsymbol{B}_{42} = \boldsymbol{0}_{4\times 1}, \quad \boldsymbol{C} = \begin{bmatrix} 0 & 0.1 & 0 & 0.1 \end{bmatrix}$$

In the case of fixed Lyapunov function based approach, the uncertainties are expressed in the form given in (4.3). The uncertain matrices $\Delta\boldsymbol{A}_1$ and $\Delta\boldsymbol{B}_1$ are given by

$$\Delta\boldsymbol{A}_1(t) \quad = \quad \boldsymbol{M}_{11}\Delta_{11}(t)\boldsymbol{N}_{111} + \boldsymbol{M}_{12}\Delta_{12}(t)\boldsymbol{N}_{112}$$

$$= \begin{bmatrix} 0 \\ \frac{1}{a} \\ a \\ 0 \end{bmatrix} d_1(t) \begin{bmatrix} \frac{a}{2} & a & \frac{3a}{4} & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} d_2(t) \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Delta \boldsymbol{B}_1(t) = \boldsymbol{M}_{11}\Delta_{11}(t)\boldsymbol{N}_{121} = \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \end{bmatrix} d_1(t) \begin{bmatrix} a \end{bmatrix}.$$

The other matrices related to the uncertain terms are also expressed in a similar form. The matrices $\boldsymbol{M}_{il}$, $\boldsymbol{N}_{i1l}$ and $\boldsymbol{N}_{12l}$ are multiplied by suitable constants depending upon the values of a_{d1} and a_{d2} for modeling the uncertainties and satisfying the condition $|d_l(t)| < 1$. Feasibility arising from the conditions of Theorem 4.3 are evaluated for different values of a_{d1} and a_{d2} and the feasible points are shown in Fig. 4.1. The feasible and infeasible points are marked by 'o' and '$\times$' respectively. Results from Fig. 4.1 illustrate that the parametric Lyapunov function based approach is less conservative than the fixed Lyapunov function based approach. In the above form of expressing $\Delta \boldsymbol{A}_1(t)$, the term $\boldsymbol{M}_{11}\Delta_{111}(t)\boldsymbol{N}_{111}$ is conservative and it includes additional uncertainty. Hence the feasibility points for different uncertain terms with $\Delta \boldsymbol{A}_1(t) = \boldsymbol{M}_{11}\Delta_{111}(t)\boldsymbol{N}_{111}$ are checked and in this case also the proposed parametric Lyapunov function based approach yields less conservative result.

4.4 Robust H_∞ Control

The robust stabilization conditions with a parametric Lyapunov function are presented in the previous section. This section proposes robust H_∞ controller design methods for stabilization and tracking control problems.

4.4.1 Robust H_∞ Stabilization

Here the H_∞ controller design that stabilizes an uncertain fuzzy system is considered. The performance criterion under zero initial condition assumption is defined as

$$\int_0^{t_f} \boldsymbol{y}^T(t)\boldsymbol{Q}\boldsymbol{y}(t)dt \leq \gamma^2 \int_0^{t_f} \boldsymbol{w}^T(t)\boldsymbol{w}(t)dt, \tag{4.32}$$

where $\boldsymbol{Q}$ is a symmetric positive definite weighting matrix, $\boldsymbol{w}(t)$ is the external disturbance, t_f is the terminal time of control and γ is the prescribed disturbance attenuation level.

Design with Fixed Lyapunov Function

Let us cosider the fuzzy system (4.2) with external disturbance $\boldsymbol{w}(t)$. The inferred fuzzy model is described by the following fuzzy system:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \sum_{i=1}^{r}\mu_i(z(t))\{(\boldsymbol{A}_i+\Delta\boldsymbol{A}_i(t))\boldsymbol{x}(t)+(\boldsymbol{B}_i+\Delta\boldsymbol{B}_i(t))\boldsymbol{u}(t)+\boldsymbol{D}_i\boldsymbol{w}(t)\}\\ \boldsymbol{y}(t) &= \sum_{i=1}^{r}\mu_i(z(t))\boldsymbol{C}_i\boldsymbol{x}(t)\end{aligned} \tag{4.33}$$

The robust H_∞ fuzzy stabilization results from [2] are summarized in the following theorem:

Theorem 4.4. [2]: Let us consider the fuzzy system (4.33) with the T-S state feedback control law (4.4). With the given disturbance attenuation level γ, if there exist a symmetric and positive definite matrix $\boldsymbol{Y}$, some matrices $\boldsymbol{W}_j$, $(j = 1, 2, ..., r)$ and ϵ_l,$(l = 1, 2, ..., L)$ such that the following matrix inequality is satisfied:

$$\bar{\boldsymbol{\phi}}_{ii} < 0, \quad i = 1, 2, \ldots, r \tag{4.34}$$

$$\frac{1}{r-1}\bar{\boldsymbol{\phi}}_{ii}+\frac{1}{2}(\bar{\boldsymbol{\phi}}_{ij}+\bar{\boldsymbol{\phi}}_{ji}) < 0, \quad 1 \le i \ne j \le r \tag{4.35}$$

where

$$\bar{\boldsymbol{\phi}}_{ij} = \left[\begin{array}{ccc|ccc|ccc}
\boldsymbol{Y}\boldsymbol{A}_i^T+\boldsymbol{W}_j^T\boldsymbol{B}_i^T+(*) & * & * & * & \ldots & * & * & \ldots & * \\
\boldsymbol{D}_i^T & -\gamma^2\boldsymbol{I} & * & * & \ldots & * & * & \ldots & * \\
\boldsymbol{C}_i\boldsymbol{Y} & \boldsymbol{0} & -\boldsymbol{Q}^{-1} & * & \ldots & * & * & \ldots & * \\
\hline
\epsilon_1\boldsymbol{M}_{i1}^T & 0 & 0 & \epsilon_1 & & & & & \\
\vdots & \vdots & \vdots & & \ddots & & & \boldsymbol{0} & \\
\epsilon_l\boldsymbol{M}_{il}^T & 0 & 0 & & & \epsilon_l & & & \\
\hline
\boldsymbol{N}_{i11}^T+\boldsymbol{W}_j^T\boldsymbol{N}_{i21}^T & 0 & 0 & & & & \epsilon_1 & & \\
\vdots & \vdots & \vdots & & \boldsymbol{0} & & & \ddots & \\
\boldsymbol{N}_{i1l}^T+\boldsymbol{W}_j^T\boldsymbol{N}_{i2l}^T & 0 & 0 & & & & & & \epsilon_l
\end{array}\right]$$

and $\boldsymbol{W}_j = \boldsymbol{K}_j\boldsymbol{Y}$, then the uncertain fuzzy system represented by (4.33) is asymptotically stable with the control law (4.4) and satisfies the performance criteria (4.32).

Design with Parametric Lyapunov Function

In this section, robust H_∞ controller design with parametric Lyapunov function is discussed . Let us consider the fuzzy system (4.8) with some external disturbance $\boldsymbol{w}$(t). The final output of the fuzzy system can be represented as

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \sum_{i=1}^{r} \mu_i(z(t))\{\boldsymbol{A}_i(\boldsymbol{\theta})\boldsymbol{x}(t) + \boldsymbol{B}_i(\theta)\boldsymbol{u}(t) + \boldsymbol{D}_i\boldsymbol{w}(t)\} \\ \boldsymbol{y}(t) &= \sum_{i=1}^{r} \mu_i(z(t))\boldsymbol{C}_i\boldsymbol{x}(t). \end{aligned} \tag{4.36}$$

The result of the H_∞ controller design for the T-S fuzzy model with parametric uncertainties is summarized in the following theorem.

Theorem 4.5. Let us consider the fuzzy system represented by (4.36) with the PDC control law (4.4). If there exists symmetric matrices $\boldsymbol{P}_0$, $\boldsymbol{P}_l$, $(l = 1, ..., L)$ and some matrices $\boldsymbol{K}_j$, $(j = 1, ..., r)$ such that

$$\boldsymbol{P}_0 + \sum_{l=1}^{L} \nu_l \boldsymbol{P}_l > 0, \ \forall \ \nu \in \mathcal{V} \tag{4.37}$$

$$\bar{\boldsymbol{\Gamma}}_{ii}(\boldsymbol{\nu}, \boldsymbol{\omega}) < 0, \ \forall \ (\nu, \omega) \in \mathcal{V} \times \mathcal{W}, \ \ i = 1, 2, \ldots, r \tag{4.38}$$

$$\frac{1}{r-1}\bar{\boldsymbol{\Gamma}}_{ii}(\boldsymbol{\nu}, \boldsymbol{\omega}) + \frac{1}{2}(\bar{\boldsymbol{\Gamma}}_{ij}(\boldsymbol{\nu}, \boldsymbol{\omega}) + \bar{\boldsymbol{\Gamma}}_{ji}(\boldsymbol{\nu}, \boldsymbol{\omega})) < 0, \ \forall \ (\nu, \omega) \in \mathcal{V} \times \mathcal{W}, \ \ 1 \le i \ne j \le r \tag{4.39}$$

$$\bar{\boldsymbol{\Pi}}_{lii} \ge 0, \ \ l = 1, ..., L, \ \ i = 1, 2, \ldots, r \tag{4.40}$$

$$\frac{1}{r-1}\bar{\boldsymbol{\Pi}}_{lii} + \frac{1}{2}(\bar{\boldsymbol{\Pi}}_{lij} + \bar{\boldsymbol{\Pi}}_{lji}) \ge 0, \ \ l = 1, ..., L, \ \ 1 \le i \ne j \le r \tag{4.41}$$

where

$$\bar{\boldsymbol{\Gamma}}_{ij}(\boldsymbol{\nu}, \boldsymbol{\omega}) = \begin{bmatrix} \bar{\boldsymbol{\Lambda}}_{ij}(\nu, \omega) & * \\ \boldsymbol{D}_i^T \boldsymbol{P}(\boldsymbol{\nu}) & -\gamma^2 \boldsymbol{I} \end{bmatrix}$$

and $\bar{\boldsymbol{\Pi}}_{lij} = (\boldsymbol{A}_{il} + \boldsymbol{B}_{il}\boldsymbol{K}_j)^T\boldsymbol{P}_l + (*)$ and $\bar{\boldsymbol{\Lambda}}_{ij}(\nu, \omega) = (\boldsymbol{A}_i(\boldsymbol{\nu}) + \boldsymbol{B}_i(\boldsymbol{\nu})\boldsymbol{K}_j)^T\boldsymbol{P}(\boldsymbol{\nu}) + (*) + (\boldsymbol{P}(\boldsymbol{\omega}) - \boldsymbol{P}_0) + \boldsymbol{C}_i^T\boldsymbol{Q}\boldsymbol{C}_i$, then the H_∞ performance given by (4.32) is guaranteed for the overall fuzzy system.

Proof: Let us consider the following parametric Lyapunov function,

$$V(t, \boldsymbol{\theta}) = \boldsymbol{x}^T(t)\boldsymbol{P}(\boldsymbol{\theta})\boldsymbol{x}(t). \tag{4.42}$$

Then from the time derivative of $V(t, \boldsymbol{\theta})$ along the trajectory of the fuzzy system (4.36), it follows

$$\begin{aligned} &\dot{V}(t, \boldsymbol{\theta}) + \boldsymbol{y}^T(t)\boldsymbol{Q}\boldsymbol{y}(t) - \gamma^2\boldsymbol{w}^T(t)\boldsymbol{w}(t) \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j \boldsymbol{x}^T(t)\left((\boldsymbol{A}_i(\boldsymbol{\theta}) + \boldsymbol{B}_i(\boldsymbol{\theta})\boldsymbol{K}_j)^T\boldsymbol{P}(\boldsymbol{\theta}) + (*) + \frac{d}{dt}\boldsymbol{P}(\boldsymbol{\theta}) + \boldsymbol{C}_i^T\boldsymbol{Q}\boldsymbol{C}_i\right)\boldsymbol{x}(t) \\ &\quad + \boldsymbol{x}^T(t)\boldsymbol{P}(\boldsymbol{\theta})\boldsymbol{D}_i\boldsymbol{w}(t) + \boldsymbol{w}^T(t)\boldsymbol{D}_i^T\boldsymbol{P}(\boldsymbol{\theta})\boldsymbol{x}(t) - \gamma^2\boldsymbol{w}^T(t)\boldsymbol{w}(t) \end{aligned} \tag{4.43}$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{w}(t) \end{bmatrix}^T \begin{bmatrix} \bar{\boldsymbol{\Lambda}}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) & * \\ \boldsymbol{D}_i^T\boldsymbol{P}(\boldsymbol{\theta}) & -\gamma^2\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{w}(t) \end{bmatrix} \tag{4.44}$$

where

$$\bar{\boldsymbol{\Lambda}}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \Big((\boldsymbol{A}_i(\boldsymbol{\theta}) + \boldsymbol{B}_i(\boldsymbol{\theta})\boldsymbol{K}_j)^T\boldsymbol{P}(\boldsymbol{\theta}) + (*) + (\boldsymbol{P}(\dot{\boldsymbol{\theta}}) - \boldsymbol{P}_0) + \boldsymbol{C}_i^T\boldsymbol{Q}\boldsymbol{C}_i\Big)$$

Based on Lemma A.1 (Appendix) and the multi-convexity and the concepts explained in Theorems 4.2 and 4.3, if the conditions (4.38) to (4.41) are satisfied, then the following inequality holds,

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j \begin{bmatrix} \bar{\boldsymbol{\Lambda}}(\theta,\dot{\theta}) & * \\ \boldsymbol{D}_i^T\boldsymbol{P}(\boldsymbol{\theta}) & -\gamma^2\boldsymbol{I} \end{bmatrix} < 0 \tag{4.45}$$

From (4.44) and (4.45), the following condition is obtained

$$\dot{V}(t,\boldsymbol{\theta}) + \boldsymbol{y}^T(t)\boldsymbol{Q}\boldsymbol{y}(t) - \gamma^2\boldsymbol{w}^T(t)\boldsymbol{w}(t) < 0 \tag{4.46}$$

Integrating the above inequality from 0 to ∞ yields

$$V(\infty) - V(0) + \int_0^\infty \boldsymbol{y}^T(t)\boldsymbol{Q}\boldsymbol{y}(t) - \int_0^\infty \gamma^2\boldsymbol{w}^T(t)\boldsymbol{w}(t) < 0 \tag{4.47}$$

With zero initial condition, $V(0) = 0$ and hence $\int_0^\infty \boldsymbol{y}^T(t)\boldsymbol{Q}\boldsymbol{y}(t) < \int_0^\infty \gamma^2\boldsymbol{w}^T(t)\boldsymbol{w}(t)$. Thus the proof is complete. □

Similar to the steps given in Algorithm 4.1, the robust H_∞ controller can be designed as follows:

Algorithm 4.2:

Step 1: Derive the fuzzy model for the given uncertain nonlinear system.

Step 2: Given the disturbance attenuation level γ^2, set counter c = 1. Choose σ, a small positive fraction say 0.05.

Step 3: By fixed Lyapunov function based H_∞ design approach solve for the feedback gain matrices $\boldsymbol{K}_j$. If the LMIs are infeasible, goto Step 5.

Step 4: Substitute the value of $\boldsymbol{K}_j$ in the inequalities given by Theorem 4.5 and solve for $\boldsymbol{Y}_0$, $\boldsymbol{Y}_l$. If the inequalities are infeasible, goto Step 6.

Step 5: If $(1 - c\sigma) > 0$, reduce the bounds of the uncertainty range $\overline{\theta}_l$ and $\underline{\theta}_l$ by $(1 - c\sigma)\overline{\theta}_l$ and $(1 - c\sigma)\underline{\theta}_l$ for $l = 1, ..., L$ and derive the fuzzy model with the new range of uncertainty. Increment the counter c and goto Step 3.

Step 6: Choose the value of $\boldsymbol{K}_j$ with the latest feasible solution obtained during the previous iterations in Step 4 and Stop. If the conditions in Step 4 are infeasible during all the iterations, then controller cannot be designed with the proposed method.

4.4.2 Robust H_∞ Tracking Control

This subsection discusses the design of robust H_∞ tracking controller to track the states of a given reference model and satisfy the given performance measure.

Let us consider the uncertain fuzzy system (4.36) with external disturbance $\tilde{\boldsymbol{w}}(t)$ and a reference model in the form:

$$\dot{\boldsymbol{x}}_r(t) = \boldsymbol{A}_r\boldsymbol{x}_r(t) + \boldsymbol{D}_r\boldsymbol{r}(t) \tag{4.48}$$

where $\boldsymbol{x}_r(t)$ is the reference state, $\boldsymbol{A}_r$ is a specific asymptotically stable matrix, $\boldsymbol{D}_r$ is a constant matrix and $\boldsymbol{r}(t)$ is a bounded reference input.

The tracking error is defined as

$$\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_r(t) \tag{4.49}$$

Let us consider the H_∞ tracking performance related to the tracking error $\boldsymbol{e}(t)$ as follows [66]:

$$\int_0^{t_f} \boldsymbol{e}^T(t)\boldsymbol{Q}\boldsymbol{e}(t)dt \le \rho^2 \int_0^{t_f} \tilde{\boldsymbol{w}}^T(t)\tilde{\boldsymbol{w}}(t)dt \tag{4.50}$$

where $\boldsymbol{Q}$ is a positive definite weight matrix, t_f is the terminal time of control and ρ is the prescribed disturbance attenuation level.

Design with Fixed Lyapunov Function

The design with fixed Lyapunov function based approach proposed by [66, 92] is in bilinear form and it is solved by a two-step procedure. Hence, an LMI formulation of robust H_∞ controller design method for a fixed Lyapunov function based approach is proposed in this subsection to track the states of a reference model.

Let us consider the fuzzy system represented by (4.33) and the reference model (4.48). Suppose the following fuzzy control rule is employed to deal with the design of a fuzzy controller for the system represented by (4.33).

Control Rule i:

IF $z_1(t)$ is N_{i1} and $z_2(t)$ is N_{i2} and....$z_p(t)$ is N_{ip} *THEN*

$$\boldsymbol{u}(t) = \boldsymbol{K}_{1i}\boldsymbol{e}(t) + \boldsymbol{K}_{2i}\boldsymbol{x}_r(t),\ i = 1, 2, \ldots, r \tag{4.51}$$

Then, the overall fuzzy control law is represented by

$$\boldsymbol{u}(t) = \sum_{i=1}^{r} \mu_i(\boldsymbol{K}_{1i}\boldsymbol{e}(t) + \boldsymbol{K}_{2i}\boldsymbol{x}_r(t)) \tag{4.52}$$

where $\boldsymbol{K}_{1i}$ and $\boldsymbol{K}_{2i}$ are the controller gains. A fuzzy controller is to be designed with the feedback gains $\boldsymbol{K}_{1i}$ and $\boldsymbol{K}_{2i}$ $(i = 1, 2, ..., r)$ such that the resulting closed-loop system is asymptotically stable and also satisfies H_∞ performance given in (4.50). With the control law given by (4.52), the overall closed-loop system can be written as

$$\begin{aligned} \dot{\boldsymbol{e}}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\{(\boldsymbol{A}_i + \Delta\boldsymbol{A}_i(t))(\boldsymbol{e}(t) + \boldsymbol{x}_r(t)) - \boldsymbol{A}_r\boldsymbol{x}_r(t) \\ &\qquad + (\boldsymbol{B}_i + \Delta\boldsymbol{B}_i(t))(\boldsymbol{K}_{1j}\boldsymbol{e}(t) + \boldsymbol{K}_{2j}\boldsymbol{x}_r(t)) + \boldsymbol{D}_i\boldsymbol{w}(t) - \boldsymbol{D}_r\boldsymbol{r}(t)\} \end{aligned} \tag{4.53}$$

Combining (4.53) and (4.48), the augmented system can be expressed as

$$\dot{\tilde{\boldsymbol{x}}}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\{(\tilde{\boldsymbol{A}}_i + \Delta\tilde{\boldsymbol{A}}_i(t) + (\tilde{\boldsymbol{B}}_i + \Delta\tilde{\boldsymbol{B}}_i(t))\tilde{\boldsymbol{K}}_j)\tilde{\boldsymbol{x}}(t) + \tilde{\boldsymbol{D}}_i\tilde{\boldsymbol{w}}(t) \tag{4.54}$$

where

$$\tilde{\boldsymbol{x}}(t) = \left[\begin{array}{c} \boldsymbol{e}(t) \\ \hline \boldsymbol{x}_r(t) \end{array}\right], \tilde{\boldsymbol{w}}(t) = \left[\begin{array}{c} \boldsymbol{w}(t) \\ \hline \boldsymbol{r}(t) \end{array}\right],$$

$$\tilde{\boldsymbol{A}}_i = \left[\begin{array}{c|c} \boldsymbol{A}_i & \boldsymbol{A}_i - \boldsymbol{A}_r \\ \hline \boldsymbol{0} & \boldsymbol{A}_r \end{array}\right], \ \Delta\tilde{\boldsymbol{A}}_i(t) = \left[\begin{array}{c|c} \Delta\boldsymbol{A}_i(t) & \Delta\boldsymbol{A}_i(t) \\ \hline \boldsymbol{0} & \boldsymbol{0} \end{array}\right], \ \tilde{\boldsymbol{B}}_i = \left[\begin{array}{c} \boldsymbol{B}_i \\ \hline \boldsymbol{0} \end{array}\right], \ \Delta\tilde{\boldsymbol{B}}_i(t) = \left[\begin{array}{c} \Delta\boldsymbol{B}_i(t) \\ \hline \boldsymbol{0} \end{array}\right],$$

$$\tilde{\boldsymbol{D}}_i = \left[\begin{array}{c|c} \boldsymbol{D}_i & -\boldsymbol{D}_r \\ \hline \boldsymbol{0} & \boldsymbol{D}_r \end{array}\right], \ \tilde{\boldsymbol{K}}_j = \left[\begin{array}{c|c} \boldsymbol{K}_{1j} & \boldsymbol{K}_{2j} \end{array}\right]$$

With the augmented system represented by (4.54), the results of T-S fuzzy model based H_∞ tracking controller design are summarized in the following theorem.

Theorem 4.6. Let us consider the fuzzy system (4.33) and the reference model (4.48) with the T-S control law (4.52). With the given disturbance attenuation level ρ^2, if there exist a symmetric and positive definite matrix $\boldsymbol{Y}$, certain matrices $\boldsymbol{X}_{1j}$, $\boldsymbol{X}_{2j}$, $(j = 1, 2, ..., r)$ and scalars ϵ_{il},$(l = 1, 2, ..., L)$ such that the following matrix inequality is satisfied:

$$\begin{aligned} \tilde{\boldsymbol{\phi}}_{ii} &< 0, \quad i = 1, 2, \ldots, r & (4.55) \\ \frac{1}{r-1}\tilde{\phi}_{ii} + \frac{1}{2}(\tilde{\phi}_{ij} + \tilde{\phi}_{ji}) &< 0, \quad 1 \leq i \neq j \leq r & (4.56) \end{aligned}$$

where

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{Y}_{11} & * \\ \boldsymbol{Y}_{21} & \boldsymbol{Y}_{22} \end{bmatrix}, \ \tilde{\boldsymbol{\phi}}_{ij} = \left[\begin{array}{c|c|c} \boldsymbol{\mathcal{A}}_{ij}^{11} & * & * \\ \hline \boldsymbol{\mathcal{A}}_{ij}^{21} & -\boldsymbol{\epsilon}_i & * \\ \hline \boldsymbol{\mathcal{A}}_{ij}^{31} & \boldsymbol{0} & -\boldsymbol{\epsilon}_i \end{array}\right],$$

$$\boldsymbol{\mathcal{A}}_{ij}^{11} = \begin{bmatrix} \boldsymbol{H}_{ij}^{11} & * & * & * & * \\ \boldsymbol{H}_{ij}^{21} & \boldsymbol{H}_{ij}^{22} & * & * & * \\ \boldsymbol{D}_i^T & \boldsymbol{0} & -\rho^2\boldsymbol{I} & * & * \\ -\boldsymbol{D}_r^T & \boldsymbol{D}_r^T & \boldsymbol{0} & -\rho^2\boldsymbol{I} & * \\ \boldsymbol{Y}_{11} & \boldsymbol{Y}_{21}^T & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{Q}^{-1} \end{bmatrix}, \ \boldsymbol{\mathcal{A}}_{ij}^{21} = \begin{bmatrix} \epsilon_1\boldsymbol{M}_{i1}^T & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \epsilon_l\boldsymbol{M}_{il}^T & 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{\mathcal{A}}_{ij}^{31} = \begin{bmatrix} \boldsymbol{N}_{i11}(\boldsymbol{Y}_{11}+\boldsymbol{Y}_{21}) + \boldsymbol{N}_{i21}\boldsymbol{X}_{1j} & \boldsymbol{N}_{i11}(\boldsymbol{Y}_{21}^T+\boldsymbol{Y}_{22}) + \boldsymbol{N}_{i21}\boldsymbol{X}_{2j} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{N}_{i1l}(\boldsymbol{Y}_{11}+\boldsymbol{Y}_{21}) + \boldsymbol{N}_{i2l}\boldsymbol{X}_{1j} & \boldsymbol{N}_{i1l}(\boldsymbol{Y}_{21}^T+\boldsymbol{Y}_{22}) + \boldsymbol{N}_{i2l}\boldsymbol{X}_{2j} & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} \boldsymbol{H}_{ij}^{11} &= \Big((\boldsymbol{Y}_{11}+\boldsymbol{Y}_{21}^T)\boldsymbol{A}_i^T - \boldsymbol{Y}_{21}^T\boldsymbol{A}_r^T + \boldsymbol{X}_{1j}^T\boldsymbol{B}_i^T)\Big) + (\,*\,) \\ \boldsymbol{H}_{ij}^{21} &= (\boldsymbol{Y}_{21}+\boldsymbol{Y}_{22})\boldsymbol{A}_i^T - \boldsymbol{Y}_{22}\boldsymbol{A}_r^T + \boldsymbol{X}_{2j}^T\boldsymbol{B}_i^T + \boldsymbol{A}_r\boldsymbol{Y}_{21} \\ \boldsymbol{H}_{ij}^{22} &= (\boldsymbol{Y}_{22}\boldsymbol{A}_r^T) + (*) \\ \boldsymbol{\epsilon}_i &= \mathrm{diag}(\epsilon_{i1}, ..., \epsilon_{il}) \end{aligned}$$

and $\boldsymbol{X}_{1j} = \boldsymbol{K}_{1j}\boldsymbol{Y}_{11} + \boldsymbol{K}_{2j}\boldsymbol{Y}_{21}$, $\boldsymbol{X}_{2j} = \boldsymbol{K}_{1j}\boldsymbol{Y}_{21}^T + \boldsymbol{K}_{2j}\boldsymbol{Y}_{22}$, then the uncertain nonlinear system represented by (4.2) is globally asymptotically stable and satisfies the performance criteria (4.50).

Proof: Applying Lemma A.1 (Appendix) to the inequalities (4.55) and (4.56), the following parameterized inequality is obtained,

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\tilde{\phi}_{ij} < 0. \tag{4.57}$$

From the above equation, it follows that

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j \begin{bmatrix} \boldsymbol{H}_{ij}^{11} + \Delta\boldsymbol{H}_{ij}^{11}(t) & * & * & * & * \\ \boldsymbol{H}_{ij}^{21} + \Delta\boldsymbol{H}_{ij}^{21}(t) & \boldsymbol{H}_{ij}^{22} & * & * & * \\ \boldsymbol{D}_i^T & \boldsymbol{0} & -\rho^2\boldsymbol{I} & * & * \\ -\boldsymbol{D}_r^T & \boldsymbol{D}_r^T & \boldsymbol{0} & -\rho^2\boldsymbol{I} & * \\ \boldsymbol{Y}_{11} & \boldsymbol{Y}_{21}^T & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{Q}^{-1} \end{bmatrix} < 0 \tag{4.58}$$

where

$$\begin{aligned} \Delta\boldsymbol{H}_{ij}^{11}(t) &= \Big((\boldsymbol{Y}_{11}+\boldsymbol{Y}_{21}^T)\Delta\boldsymbol{A}_i^T(t) + \boldsymbol{X}_{1j}^T\Delta\boldsymbol{B}_i^T(t)\Big) + (\,*\,) \\ \Delta\boldsymbol{H}_{ij}^{21}(t) &= (\boldsymbol{Y}_{21}+\boldsymbol{Y}_{22})\Delta\boldsymbol{A}_i^T(t) + \boldsymbol{X}_{2j}^T\Delta\boldsymbol{B}_i^T(t) \end{aligned}$$

Using the augmented system represented in (4.54), the inequality in (4.58) can be written as

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\begin{bmatrix}\tilde{\boldsymbol{\Omega}}_{ij}(t)+\boldsymbol{Y}\tilde{\boldsymbol{Q}}\boldsymbol{Y} & * \\ \tilde{\boldsymbol{D}}_i^T & -\rho^2\boldsymbol{I}\end{bmatrix}<0 \tag{4.59}$$

where $\tilde{\boldsymbol{\Omega}}_{ij}(t)=\boldsymbol{Y}(\tilde{\boldsymbol{A}}_i+\Delta\tilde{\boldsymbol{A}}_i(t)+(\tilde{\boldsymbol{B}}_i+\Delta\tilde{\boldsymbol{B}}_i(t))\tilde{\boldsymbol{K}}_j)^T+(*)$ and $\tilde{\boldsymbol{Q}}=\text{diag}(\boldsymbol{Q},\boldsymbol{0})$.

With $\boldsymbol{P}=\boldsymbol{Y}^{-1}$, pre-multiplying and post-multiplying by diag$(\boldsymbol{P},\boldsymbol{I})$, the following inequality is obtained

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\begin{bmatrix}\boldsymbol{P}\tilde{\boldsymbol{\Omega}}_{ij}(t)\boldsymbol{P}+\tilde{\boldsymbol{Q}} & * \\ \tilde{\boldsymbol{D}}_i^T\boldsymbol{P} & -\rho^2\boldsymbol{I}\end{bmatrix}<0 \tag{4.60}$$

Let us consider a Lyapunov function $V(t)=\tilde{\boldsymbol{x}}^T(t)\boldsymbol{P}\tilde{\boldsymbol{x}}(t)$ for the closed loop system given by (4.54). Then from the derivative of the Lyapunov function, it follows that

$$\begin{aligned}&\dot{V}(t)+\tilde{\boldsymbol{x}}^T(t)\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{x}}(t)-\rho^2\tilde{\boldsymbol{w}}^T(t)\tilde{\boldsymbol{w}}(t)\\ &=\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big\{\tilde{\boldsymbol{x}}^T(t)\Big((\tilde{\boldsymbol{A}}_i+\Delta\tilde{\boldsymbol{A}}_i(t)+(\tilde{\boldsymbol{B}}_i+\Delta\tilde{\boldsymbol{B}}_i(t))\tilde{\boldsymbol{K}}_j)^T\boldsymbol{P}+(*)+\tilde{\boldsymbol{Q}}\Big)\tilde{\boldsymbol{x}}(t)\Big\}\\ &\qquad+\tilde{\boldsymbol{x}}^T(t)\boldsymbol{P}\tilde{\boldsymbol{D}}_i\tilde{\boldsymbol{w}}(t)+\tilde{\boldsymbol{w}}^T(t)\tilde{\boldsymbol{D}}_i^T\boldsymbol{P}\tilde{\boldsymbol{x}}(t)-\rho^2\tilde{\boldsymbol{w}}^T(t)\tilde{\boldsymbol{w}}(t) &(4.61)\\ &=\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\begin{bmatrix}\tilde{\boldsymbol{x}}(t)\\ \tilde{\boldsymbol{w}}(t)\end{bmatrix}^T\begin{bmatrix}\boldsymbol{P}\tilde{\boldsymbol{\Omega}}_{ij}(t)\boldsymbol{P}+\tilde{\boldsymbol{Q}} & * \\ \tilde{\boldsymbol{D}}_i^T\boldsymbol{P} & -\rho^2\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{x}}(t)\\ \tilde{\boldsymbol{w}}(t)\end{bmatrix} &(4.62)\end{aligned}$$

From (4.62) and (4.60), it follows that

$$\dot{V}(t)+\tilde{\boldsymbol{x}}^T(t)\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{x}}(t)-\rho^2\tilde{\boldsymbol{w}}^T(t)\tilde{\boldsymbol{w}}(t)<0 \tag{4.63}$$

Integrating the above inequality from 0 to ∞, on both sides, yields

$$V(\infty)-V(0)+\int_0^\infty\tilde{\boldsymbol{x}}^T(t)\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{x}}(t)-\int_0^\infty\rho^2\tilde{\boldsymbol{w}}^T(t)\tilde{\boldsymbol{w}}(t)<0 \tag{4.64}$$

With zero initial condition, $V(0)=0$ and hence

$$\int_0^\infty\tilde{\boldsymbol{x}}^T(t)\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{x}}(t) \;<\; \int_0^\infty\rho^2\tilde{\boldsymbol{w}}^T(t)\tilde{\boldsymbol{w}}(t) \tag{4.65}$$

$$\int_0^\infty\boldsymbol{e}^T(t)\boldsymbol{Q}\boldsymbol{e}(t) \;<\; \int_0^\infty\rho^2\tilde{\boldsymbol{w}}^T(t)\tilde{\boldsymbol{w}}(t) \tag{4.66}$$

Thus the proof is complete. □

Design with Parametric Lyapunov Function

This sub-subsection presents the design methodology for a robust H_∞ tracking controller with parametric Lyapunov function. Here the uncertain fuzzy system (4.36) and the reference model (4.48) are considered with the control law (4.52). The augmented system can be expressed as

$$\dot{\tilde{\boldsymbol{x}}}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\{(\tilde{\boldsymbol{A}}_i(\theta) + \tilde{\boldsymbol{B}}_i(\theta)\tilde{\boldsymbol{K}}_j)\tilde{\boldsymbol{x}}(t) + \tilde{\boldsymbol{D}}_i\tilde{\boldsymbol{w}}(t)\} \tag{4.67}$$

where

$$\tilde{\boldsymbol{x}}(t) = \begin{bmatrix} \boldsymbol{e}(t) \\ \hline \boldsymbol{x}_r(t) \end{bmatrix}, \tilde{\boldsymbol{w}}(t) = \begin{bmatrix} \boldsymbol{w}(t) \\ \hline \boldsymbol{r}(t) \end{bmatrix},$$

$$\tilde{\boldsymbol{A}}_i(\theta) = \tilde{\boldsymbol{A}}_{0i} + \sum_{l=1}^{L}\theta_l\tilde{\boldsymbol{A}}_{li}, \tilde{\boldsymbol{B}}_i(\theta) = \tilde{\boldsymbol{B}}_{0i} + \sum_{l=1}^{L}\theta_l\tilde{\boldsymbol{B}}_{li}$$

$$\tilde{\boldsymbol{A}}_{0i} = \left[\begin{array}{c|c} \boldsymbol{A}_{0i} & \boldsymbol{A}_{0i} - \boldsymbol{A}_r \\ \hline \boldsymbol{0} & \boldsymbol{A}_r \end{array}\right],\ \tilde{\boldsymbol{A}}_{li} = \left[\begin{array}{c|c} \boldsymbol{A}_{li} & \boldsymbol{A}_{li} \\ \hline \boldsymbol{0} & \boldsymbol{0} \end{array}\right],\ \tilde{\boldsymbol{B}}_{0i} = \begin{bmatrix} \boldsymbol{B}_{0i} \\ \hline \boldsymbol{0} \end{bmatrix},\ \tilde{\boldsymbol{B}}_{li} = \begin{bmatrix} \boldsymbol{B}_{li} \\ \hline \boldsymbol{0} \end{bmatrix},$$

and the other matrices $\tilde{\boldsymbol{K}}_j$ and $\tilde{\boldsymbol{D}}_i$ are the same as discussed in the previous sub-subsection.

With the augmented system represented by (4.67), the results of the T-S model based H_∞ tracking controller design with parametric Lyapunov function are summarized in the following theorem, followed by the proof.

Theorem 4.7. Let us consider the augmented fuzzy system represented by (4.67) with the PDC control law (4.52). For a given disturbance attenuation level ρ^2, if there exist symmetric matrices $\boldsymbol{P}_0$, $\boldsymbol{P}_l$, $(l = 1, ..., L)$ and some matrices $\boldsymbol{K}_j$, $(j = 1, ..., r)$ such that

$$\begin{aligned} \boldsymbol{P}_0 + \sum_{l=1}^{L}\nu_l\boldsymbol{P}_l &> 0, \ \forall \ \nu \in \mathcal{V} && (4.68) \\ \tilde{\boldsymbol{\Gamma}}_{ii}(\boldsymbol{\nu}, \boldsymbol{\omega}) &< 0, \ \forall \ (\nu, \omega) \in \mathcal{V}\times\mathcal{W}, \ \ i = 1, 2, \dots, r && (4.69) \\ \frac{1}{r-1}\tilde{\boldsymbol{\Gamma}}_{ii}(\boldsymbol{\nu}, \boldsymbol{\omega}) + \frac{1}{2}(\tilde{\boldsymbol{\Gamma}}_{ij}(\boldsymbol{\nu}, \boldsymbol{\omega}) + \tilde{\boldsymbol{\Gamma}}_{ji}(\boldsymbol{\nu}, \boldsymbol{\omega})) &< 0, \ \forall \ (\nu, \omega) \in \mathcal{V}\times\mathcal{W}, \ \ 1 \le i \ne j \le r && (4.70) \\ \tilde{\boldsymbol{\Pi}}_{lii} &\ge 0, \ \ l = 1, ..., L, \ \ i = 1, 2, \dots, r && (4.71) \\ \frac{1}{r-1}\tilde{\boldsymbol{\Pi}}_{lii} + \frac{1}{2}(\tilde{\boldsymbol{\Pi}}_{lij} + \tilde{\boldsymbol{\Pi}}_{lji}) &\ge 0, \ \ l = 1, ..., L, \ \ 1 \le i \ne j \le r && (4.72) \end{aligned}$$

where

$$\tilde{\boldsymbol{\Gamma}}_{ij}(\boldsymbol{\nu}, \boldsymbol{\omega}) = \begin{bmatrix} \tilde{\boldsymbol{\Lambda}}(\nu, \omega) & * \\ \tilde{\boldsymbol{D}}_i^T\boldsymbol{P}(\boldsymbol{\nu}) & -\rho^2\boldsymbol{I} \end{bmatrix},$$

$\tilde{\boldsymbol{\Pi}}_{lij} = (\tilde{\boldsymbol{A}}_{il} + \tilde{\boldsymbol{B}}_{il}\tilde{\boldsymbol{K}}_j)^T\boldsymbol{P}_l + (*)$ and $\tilde{\boldsymbol{\Lambda}}(\nu, \omega) = (\tilde{\boldsymbol{A}}_i(\boldsymbol{\nu}) + \tilde{\boldsymbol{B}}_i(\boldsymbol{\nu})\tilde{\boldsymbol{K}}_j)^T\boldsymbol{P}(\boldsymbol{\nu}) + (*) - (\boldsymbol{P}(\boldsymbol{\omega}) - \boldsymbol{P}_0) + \tilde{\boldsymbol{Q}}$, then the H_∞ performance given by (4.50) is guaranteed for the overall fuzzy system.

Proof: Let us consider the following parametric Lyapunov function,

$$V(t, \boldsymbol{\theta}) = \tilde{\boldsymbol{x}}^T(t)\boldsymbol{P}(\boldsymbol{\theta})\tilde{\boldsymbol{x}}(t). \tag{4.73}$$

Then from time derivative along the trajectory of the fuzzy system (4.67), it follows that

$$\dot{V}(t,\boldsymbol{\theta}) + \tilde{\boldsymbol{x}}^T(t)\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{x}}(t) - \gamma^2\tilde{\boldsymbol{w}}^T(t)\tilde{\boldsymbol{w}}(t)$$
$$= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j \begin{bmatrix} \tilde{\boldsymbol{x}}(t) \\ \tilde{\boldsymbol{w}}(t) \end{bmatrix}^T \begin{bmatrix} \tilde{\boldsymbol{\Lambda}}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) & * \\ \tilde{\boldsymbol{D}}_i^T\boldsymbol{P}(\boldsymbol{\theta}) & -\rho^2\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}}(t) \\ \tilde{\boldsymbol{w}}(t) \end{bmatrix} \tag{4.74}$$

where

$$\tilde{\boldsymbol{\Lambda}}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \Big((\boldsymbol{A}_i(\boldsymbol{\theta}) + \boldsymbol{B}_i(\boldsymbol{\theta})\boldsymbol{K}_j)^T\boldsymbol{P}(\boldsymbol{\theta}) + (*) + (\boldsymbol{P}(\dot{\boldsymbol{\theta}}) - \boldsymbol{P}_0) + \tilde{\boldsymbol{Q}}\Big)$$

Based on Lemma A.1 (Appendix) and the multi-convexity and the concepts explained in Theorems 4.2 and 4.3, if the conditions (4.68) to (4.72) are satisfied, then the following inequality holds,

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j \begin{bmatrix} \tilde{\boldsymbol{\Lambda}}(\theta,\dot{\theta}) & * \\ \tilde{\boldsymbol{D}}_i^T\boldsymbol{P}(\boldsymbol{\theta}) & -\rho^2\boldsymbol{I} \end{bmatrix} < 0 \tag{4.75}$$

From (4.74) and (4.75), the following condition is obtained

$$\dot{V}(t,\boldsymbol{\theta}) + \tilde{\boldsymbol{x}}^T(t)\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{x}}(t) - \rho^2\tilde{\boldsymbol{w}}^T(t)\tilde{\boldsymbol{w}}(t) < 0 \tag{4.76}$$

With zero initial condition, $V(0) = 0$. Integrating the above inequality from 0 to ∞, on both sides yields $\int_0^\infty \boldsymbol{e}^T(t)\boldsymbol{Q}\boldsymbol{e}(t) < \int_0^\infty \gamma^2\tilde{\boldsymbol{w}}^T(t)\tilde{\boldsymbol{w}}(t)$. Thus the proof is complete. □

Similar to the steps given in Algorithm 4.2, the robust H_∞ tracking controller with parametric Lyapunov function can be designed.

In the H_∞ stabilization design with parametric Lyapunov function, with the following minimization problem the disturbance attenuation level γ^2 in (4.32) can be minimized so that the H_∞ performance can be reduced:

$$\min_{P_0, P_l, K_j} \quad \gamma^2$$

subject to the inequalities (4.37) – (4.41).

The disturbance attenuation level in (4.50) for a tracking control problem can also be minimized in a similar way.

Remark 4.1. In some cases, the controller gains obtained with the fixed Lyapunov function based approach may be very high which may be unacceptable for practical applications. This problem can be overcome by including some additional LMI constraints. In robust H_∞ stabilization design, the controller gain is given by $\boldsymbol{K}_j = \boldsymbol{X}_j\boldsymbol{Y}^{-1}$. In this case, the controller gain can be limited by including the following constraints:

$$\boldsymbol{Y} > a_1\boldsymbol{I} \tag{4.77}$$

$$\begin{bmatrix} -a_2\boldsymbol{I} & * \\ \boldsymbol{X}_j^T & -\boldsymbol{I} \end{bmatrix} < 0, \quad j = 1, ..., r \tag{4.78}$$

where a_1 and a_2 are some positive constants. The matrix norm constraint $||\boldsymbol{Y}|| > a_1$ can be represented by (4.77). The second constraint (4.78) is included for $||\boldsymbol{X}_j^T|| < a_2$ [36]. Hence the choice of a_1 and a_2 limits the magnitude of the parameters of $\boldsymbol{Y}^{-1}$ and $\boldsymbol{X}_j$. Thereby the magnitude of $\boldsymbol{K}_j$ can be limited. Initially, a small positive value for a_1 and a large positive value for a_2 are to be chosen. If the obtained gains are very high and not acceptable, then the value of a_1 is to be increased while the value of a_2 is to be reduced till appropriate gains are obtained.

In a similar way, the gains in robust H_∞ tracking controller can be limited by the following additional LMI constraints:

$$\boldsymbol{Y} > a_1\boldsymbol{I} \tag{4.79}$$

$$\begin{bmatrix} -a_2\boldsymbol{I} & * & * \\ \boldsymbol{X}_{1j}^T & -\boldsymbol{I} & * \\ \boldsymbol{X}_{2j}^T & \boldsymbol{0} & -\boldsymbol{I} \end{bmatrix} < 0, \quad j = 1, ..., r \tag{4.80}$$

4.5 Simulation Results

4.5.1 Robust H_∞ Stabilization

Let us consider the example presented in Section 4.3.5 with $d_1 = \sin(t/2)/\sqrt{8}$ and $d_2 = \cos(t/2)/2$. The disturbance inputs $w_1(t)$ and $w_2(t)$ are assumed to be periodical square waves of period 1 sec with amplitude 1 for $t \leq 5$ and 0 otherwise. The matrices $\boldsymbol{D}_i$ $(i = 1, ..., 4)$ corresponding to the disturbance input are the same as in [2] and it is given by

$$\boldsymbol{D}_i = \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \\ 0 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

With $\boldsymbol{Q} = 1$, the H_∞ controller is designed for different values of γ^2 using the LMI toolbox in MATLAB. The trajectories of state variables with an initial condition of $\boldsymbol{x}(0) = [0 \ \ 0 \ \ 0 \ \ 0]^T$ using the controllers designed with $\gamma^2 = 0.08$, $\gamma^2 = 0.1$ and $\gamma^2 = 0.5$ are shown in Fig. 4.2. In Fig. 4.3, the

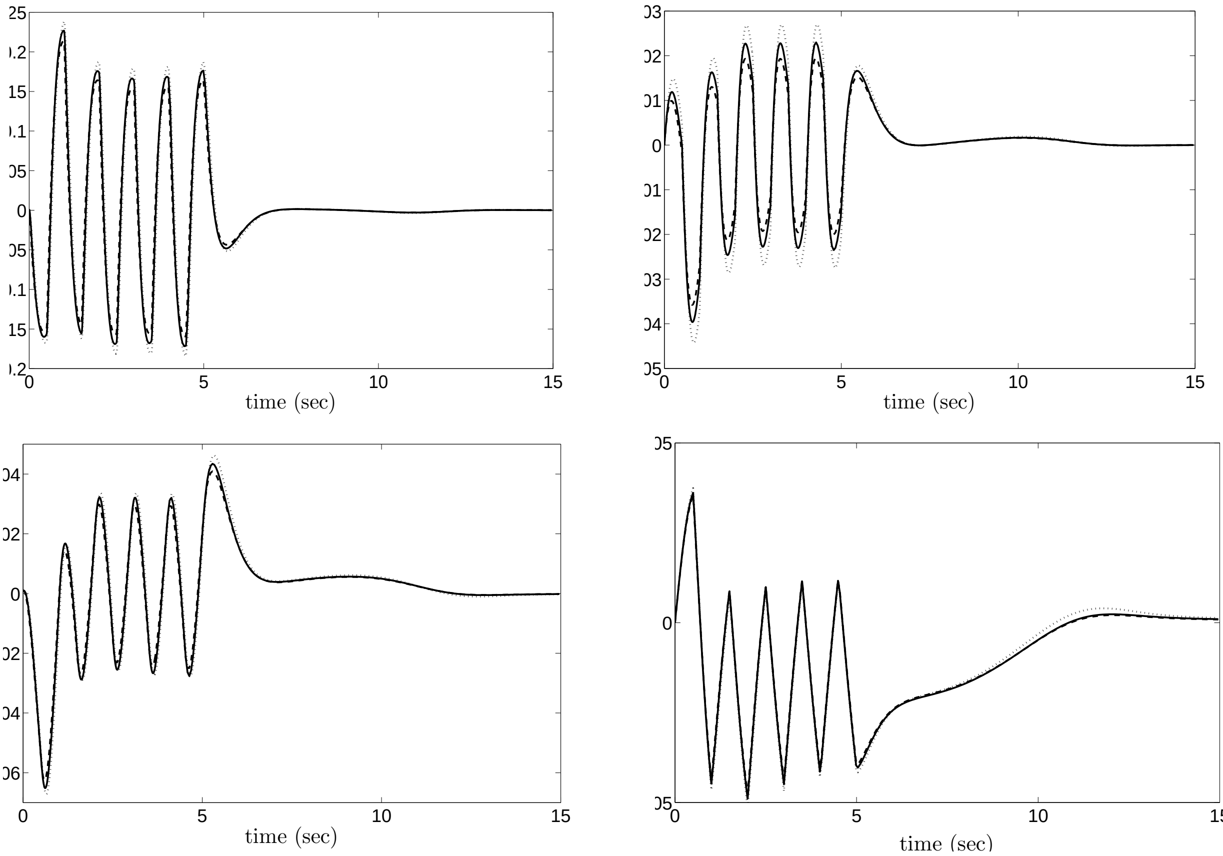


Fig. 4.2: Trajectories of state variables $\boldsymbol{x}(t)$ with parametric Lyapunov function based approach for $\gamma^2 = 0.08$ (dashed line), $\gamma^2 = 0.1$ (solid line)and $\gamma^2 = 0.5$ (dotted line).

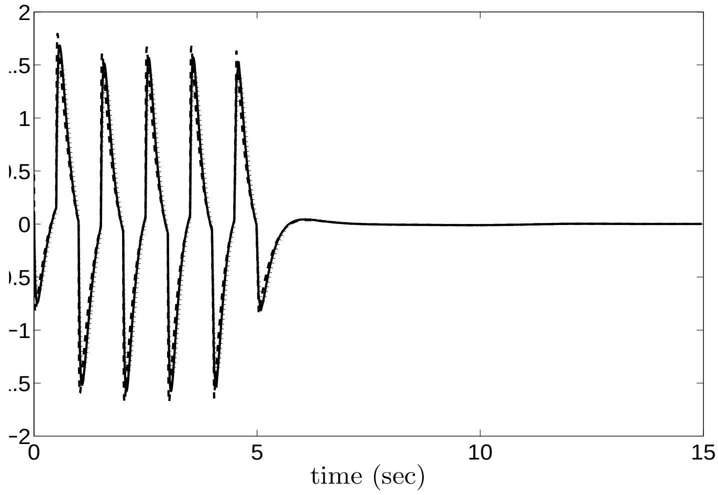


Fig. 4.3: Control input $\boldsymbol{u}(t)$ with parametric Lyapunov function based approach for $\gamma^2 = 0.08$ (dashed line), $\gamma^2 = 0.1$ (solid line)and $\gamma^2 = 0.5$ (dotted line).

Table. 4.1: Parameters of feedback gain matrices for H_∞ stabilization with $\gamma^2 = 0.1$ (for the example in Section 4.5.1)

	Parametric Lyapunov function based approach				Fixed Lyapunov function based approach			
	K_{1i}	K_{2i}	K_{3i}	K_{4i}	K_{1i}	K_{2i}	K_{3i}	K_{4i}
$i=1$	-41.87	-379.3	135.8	26.85	-124.7	-1472	604.2	99.60
$i=2$	-42.09	-381.6	136.7	27.02	-124.5	-1469	603.3	99.44
$i=3$	-34.73	-310.3	110.0	21.76	-105.4	-1238	506.7	83.60
$i=4$	-35.12	-314.1	111.4	22.06	-107.6	-1265	517.7	85.42

corresponding control input $\boldsymbol{u}(t)$ are plotted. Similar to the results shown in [2], the chattering and overshoot appearing in $\boldsymbol{x}(t)$ are due to disturbance and uncertainties present in the system.

Using one of the above values of γ^2, viz $\gamma^2 = 0.1$, let us consider the parametric Lyapunov function based approach and compare the results with those obtained by using the fixed Lyapunov function based approach. For the parametric Lyapunov function based approach, a feasible solution is obtained with $c\sigma = 0.9$. The values of feedback gain matrices $\boldsymbol{K}_i = [K_{1i} \;\; K_{2i} \;\; K_{3i} \;\; K_{4i}]$ for these cases are shown in Table 4.1. The parameters of the parametric Lyapunov function are given below:

$$\boldsymbol{P}_0 = \begin{bmatrix} 0.1814 & * & * & * \\ 1.6846 & 23.09 & * & * \\ -0.6131 & -9.4643 & 4.2464 & * \\ -0.1211 & -1.873 & 0.8597 & 0.263 \end{bmatrix}, \quad \boldsymbol{P}_1 = 10^{-6} \times \begin{bmatrix} -0.743 & * & * & * \\ -7.247 & 25.319 & * & * \\ 3.6347 & 20.19 & 93.15 & * \\ 0.5971 & -0.992 & 10.07 & 5004 \end{bmatrix}$$

$$\boldsymbol{P}_2 = \begin{bmatrix} 0.0202 & * & * & * \\ 0.2603 & 3.0825 & * & * \\ -0.0986 & -1.3846 & 0.6414 & * \\ -0.0196 & -0.2604 & 0.12 & 0.0191 \end{bmatrix}$$

The trajectories of the state variables $\boldsymbol{x}(t)$ for both the fixed and parametric Lyapunov function based controller with an initial condition $\boldsymbol{x}(0) = [0.8 \;\; 0.1 \;\; -0.1 - 0.5]^T$ are shown in Fig. 4.4. In Fig. 4.5, the corresponding control inputs $u(t)$ are plotted. In this example with the proposed design method based on parametric Lyapunov function, the controller design shows feasible solution with 10% of the uncertainties considered in Step 3 (Algorithm 4.2). The time taken in the final iteration with the feasible solution is 92.64 sec on a Pentium 4 processor with 2GB RAM.

It is observed in Fig. 4.5 that the control signal is very high in the case of the design with fixed Lyapunov function. This high gain is due to the conservatism existing in the fixed Lyapunov function based design. A large magnitude in the control effort can be seen in the transient region which is

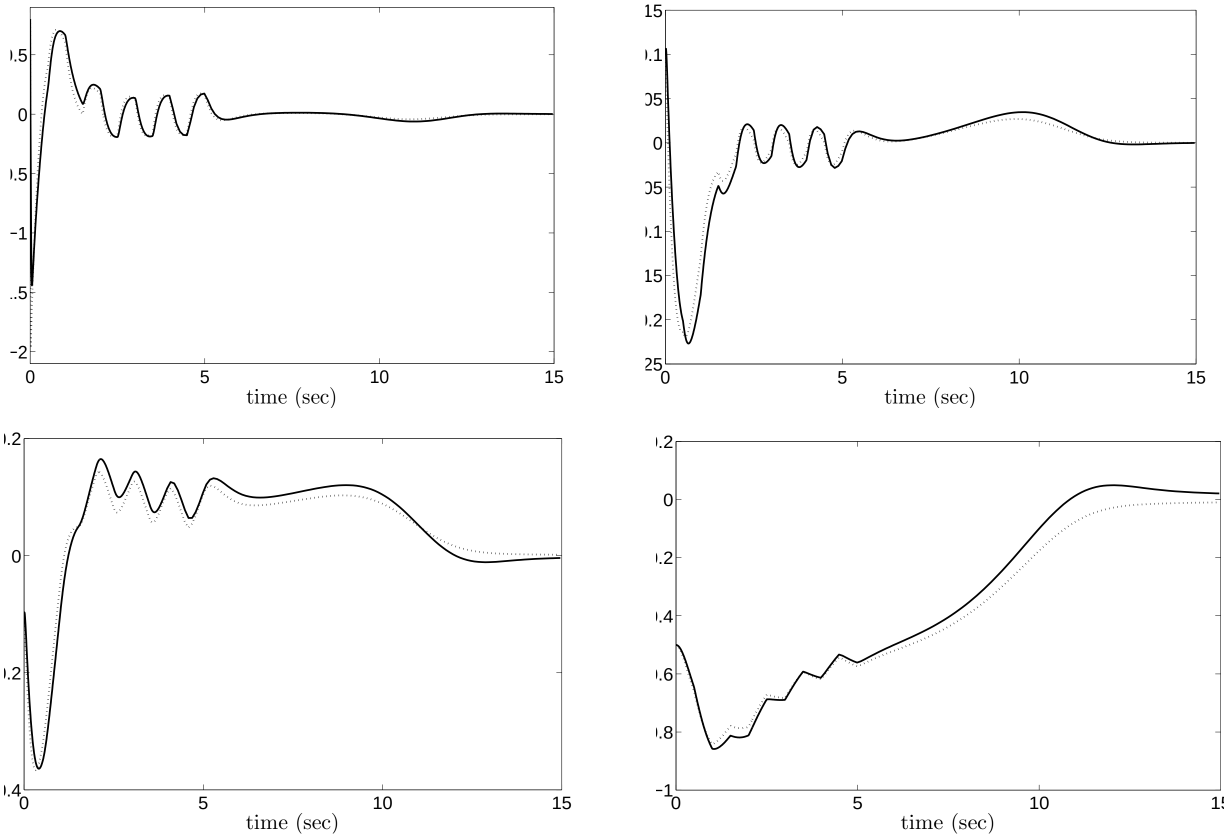


Fig. 4.4: Trajectories of state variables $\boldsymbol{x}(t)$ with parametric Lyapunov function based approach (solid line) and fixed Lyapunov function based approach (dotted line) [2] for $\gamma^2 = 0.1$.

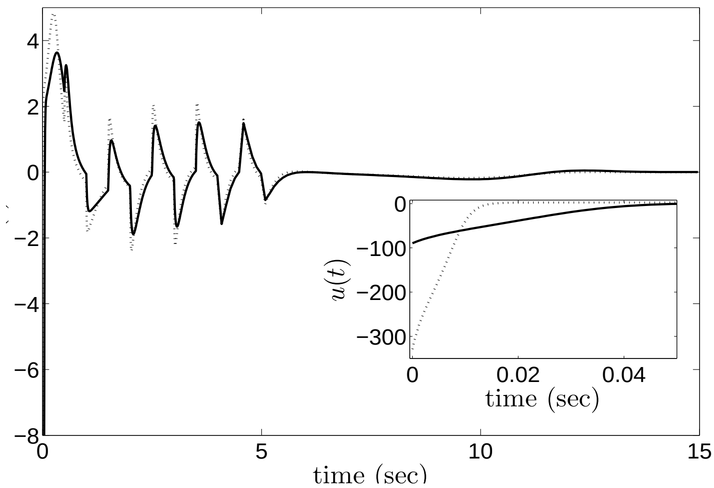


Fig. 4.5: Control input $u(t)$ with parametric Lyapunov function based approach (solid line) and fixed Lyapunov function based approach (dotted line) [2] for $\gamma^2 = 0.1$.

undesirable for the actuator and it can be seen in the initial portion of the simulation results. It is to be noted that the gain can be reduced by adjusting the desired performance measure (with a high value of γ^2). Hence the effect of conservatism will reduce the freedom in choosing the desired performance measure while designing the controller.

For both the above cases of fixed and parametric Lyapunov function based approaches, the constants a_1 and a_2 are chosen as 0.001 and 100 respectively and the controller is designed with the minimum possible disturbance attenuation level γ^2. In the parametric Lyapunov function based approach, the minimum value of γ^2 for which the H_∞ controller exists is found as $\gamma^2 = 0.076$, whereas, in the case of fixed Lyapunov function based approach, it is found to be $\gamma^2 = 0.087$. Hence, it is clear that the parametric Lyapunov function based approach is less conservative than the fixed Lyapunov function based approach.

4.5.2 Robust H_∞ Tracking Control

The problem of balancing an inverted pendulum is considered in this subsection to demonstrate the results obtained by using a robust H_∞ tracking controller. Let us consider the nonlinear equation representing the equation of motion of an inverted pendulum on a cart given in [66,93]. The equation of motion with an external disturbance $w(t)$ is given by

$$\begin{aligned} \dot{x_1}(t) &= x_2(t) \\ \dot{x_2}(t) &= \frac{g_r \sin(x_1(t)) - amlx_2^2(t)\sin(2x_1(t))/2 - \eta a \cos(x_1(t))u(t)}{4l/3 - aml\cos^2(x_1(t))} + w(t) \\ y(t) &= x_1(t). \end{aligned} \tag{4.81}$$

Here $x_1(t)$ and $x_2(t)$ represent the angular displacement about the vertical axis (in rad) and the angular velocity (in rad/sec) respectively, $g_r = 9.8$ m/s^2 is the acceleration due to gravity, $a = 1/(m+M)$, $m = 2$ kg is the mass of the pendulum, $M = 8$ kg is the mass of the cart, $2l = 1$ m is the length of the pendulum, $\eta = 1000$ is a constant and $u(t)$ is the force applied on the cart (in kN). Here the external disturbance is assumed to be $w(t) = (5\sin(3t)/(t+0.1))^2$.

The operating domain is considered as $x_1(t) \in [-75\pi/180 \;\; 75\pi/180]$, $x_2(t) \in [-6 \;\; 6]$ and the input $u(t) \in [-1000 \;\; 1000]$. The two rule fuzzy model is given by [93]:

Plant rule i:

IF x_1 is about N_i *THEN*

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i(\boldsymbol{\theta})\boldsymbol{x}(t) + \boldsymbol{B}_i(\boldsymbol{\theta})u(t) + \boldsymbol{D}_i w(t)$$

$$y(t) \;=\; \boldsymbol{C}_i \boldsymbol{x}(t),$$

where $N_1 = \cos(x_1)$, $N_2 = 1 - \cos(x_1)$. The parameters of matrices $\boldsymbol{A}_{0i}$ and $\boldsymbol{A}_{li}$ are

$$\begin{aligned}
\boldsymbol{A}_{01} &= \begin{bmatrix} 0 & 1 \\ \frac{g}{\frac{4l}{3}-aml} & 0 \end{bmatrix}, \quad \boldsymbol{A}_{02} = \begin{bmatrix} 0 & 1 \\ \frac{2g}{\pi(\frac{4l}{3}-aml\beta^2)} & 0 \end{bmatrix}, \\
\boldsymbol{B}_{01} &= \begin{bmatrix} 0 \\ \frac{-\eta a}{\frac{4l}{3}-aml} \end{bmatrix}, \quad \boldsymbol{B}_{02} = \begin{bmatrix} 0 \\ \frac{-\eta a\beta}{\frac{4l}{3}-aml\beta^2} \end{bmatrix}, \\
\boldsymbol{A}_{1i} &= \begin{bmatrix} 0 & 0 \\ 4.6757 & 0 \end{bmatrix}, \quad \boldsymbol{A}_{2i} = \begin{bmatrix} 0 & 0 \\ 0 & 0.813 \end{bmatrix}, \quad \boldsymbol{A}_{3i} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, i = 1,2 \\
\boldsymbol{B}_{1i} &= \boldsymbol{B}_{2i} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{B}_{3i} = \begin{bmatrix} 0 \\ 8.9697 \end{bmatrix}, \quad \boldsymbol{C}_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \boldsymbol{D}_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, i = 1,2.
\end{aligned}$$

For fixed Lyapunov function based design, the uncertain matrices are expressed in the following form:

$$\begin{aligned}
\Delta\boldsymbol{A}_i(t) &= \boldsymbol{M}_{i1}\Delta_{i1}(t)\boldsymbol{N}_{i11} + \boldsymbol{M}_{i2}\Delta_{i2}(t)\boldsymbol{N}_{i12} + \boldsymbol{M}_{i3}\Delta_{i3}(t)\boldsymbol{N}_{i13} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} d_1(t) \begin{bmatrix} 4.6757 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d_2(t) \begin{bmatrix} 0 & 0.813 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d_3(t) \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad i = 1,2 \\
\Delta\boldsymbol{B}_i(t) &= \boldsymbol{M}_{i1}\Delta_{i1}(t)\boldsymbol{N}_{i21} + \boldsymbol{M}_{i2}\Delta_{i2}(t)\boldsymbol{N}_{i22} + \boldsymbol{M}_{i3}\Delta_{i3}(t)\boldsymbol{N}_{i23} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} d_1(t) \begin{bmatrix} 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d_2(t) \begin{bmatrix} 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d_3(t) \begin{bmatrix} 8.9697 \end{bmatrix}, \quad i = 1,2.
\end{aligned}$$

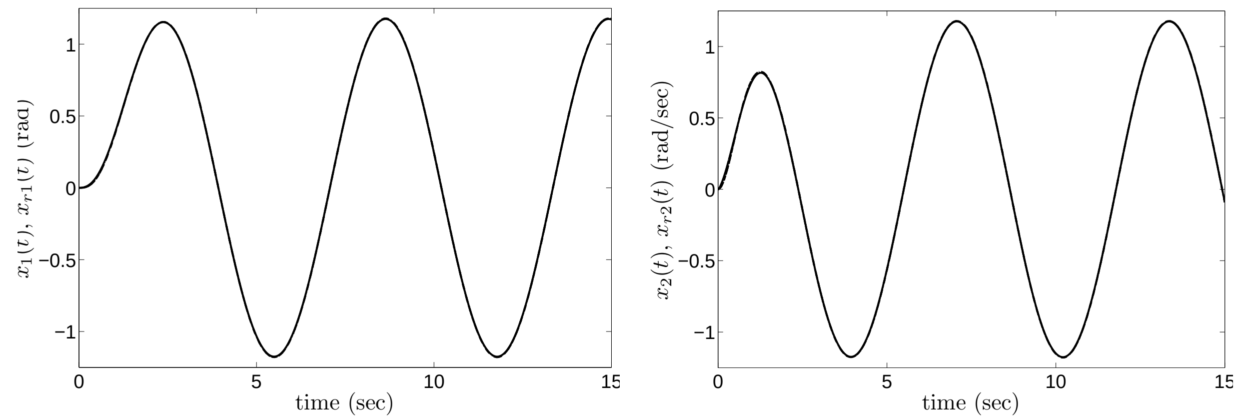


Fig. 4.6: Trajectories of state variables $\boldsymbol{x}(t)$, reference trajectories $\boldsymbol{x}_r(t)$ (solid line) for $\rho^2 = 0.01$ (dashed line), $\rho^2 = 0.05$ (dotted line) and $\rho^2 = 0.1$ (dash–dotted line).

The rate of variation of uncertain terms $\dot{\theta}_l$, $(l = 1, ..., L)$ are assumed to be within the extremal

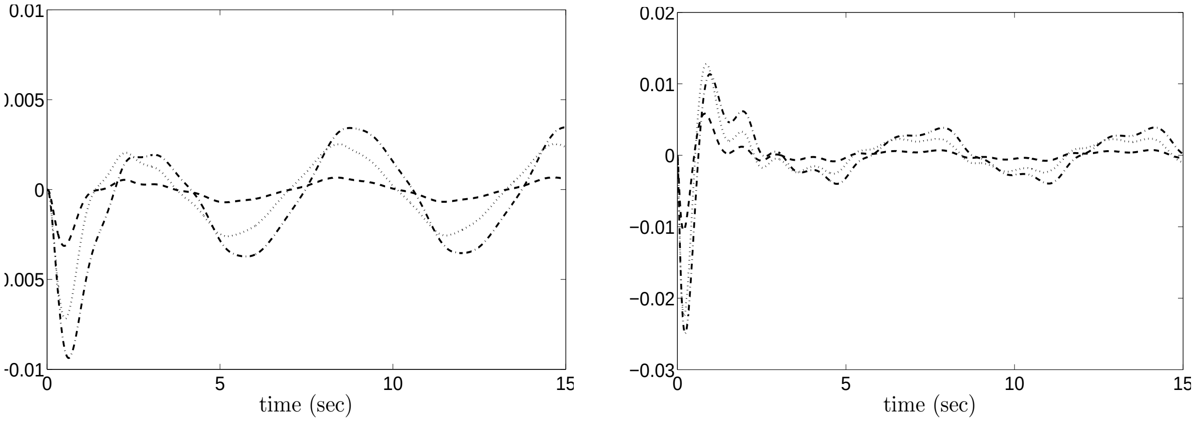


Fig. 4.7: Tracking error $\boldsymbol{e}(t)$ for $\rho^2 = 0.01$ (dashed line), $\rho^2 = 0.05$ (dotted line) and $\rho^2 = 0.1$ (dash–dotted line).

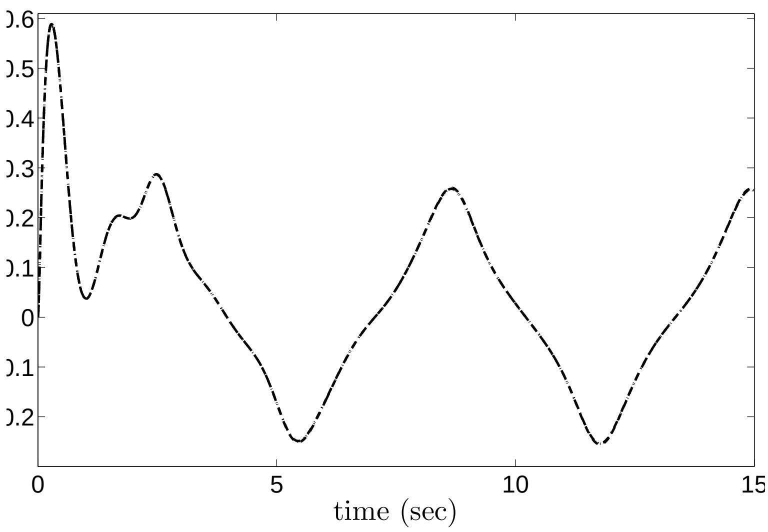


Fig. 4.8: Control input $\boldsymbol{u}(t)$ for $\rho^2 = 0.01$ (dashed line), $\rho^2 = 0.05$ (dotted line) and $\rho^2 = 0.1$ (dash–dotted line).

values of -4 and 4. Similar to [66], the following reference model and reference input are considered:

$$\begin{bmatrix} \dot{x}_{r1}(t) \\ \dot{x}_{r2}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} x_{r1}(t) \\ x_{r2}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 5\sin(t) \end{bmatrix}. \tag{4.82}$$

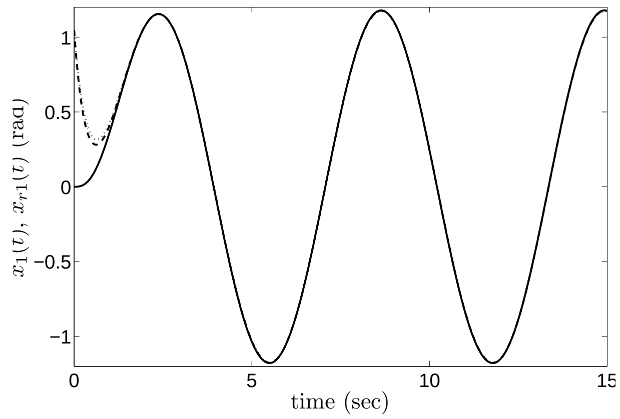


Fig. 4.9: Trajectories of state variables $\boldsymbol{x}(t)$, reference trajectories $\boldsymbol{x}_r(t)$ (solid line) for $\rho^2 = 0.05$ with parametric Lyapunov function based approach (dashed line) and fixed Lyapunov function based approach (dotted line).

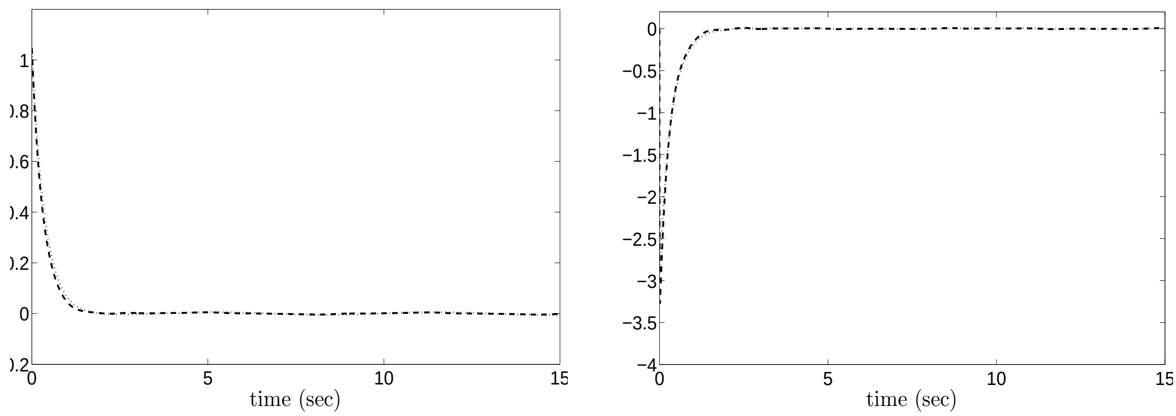


Fig. 4.10: Tracking error $\boldsymbol{e}(t)$ for $\rho^2 = 0.05$ with parametric Lyapunov function based approach (dashed line) and fixed Lyapunov function based approach (dotted line).

With $\boldsymbol{Q} = 100\boldsymbol{I}$, the H_∞ tracking controller is designed for different values of ρ^2 with the concept presented in Section 4.4.2. The trajectories of the state variable $\boldsymbol{x}(t)$ and the error $\boldsymbol{e}(t)$ are shown in Figs. 4.6 and 4.7, for different disturbance attenuation levels $\rho^2 = 0.01$, $\rho^2 = 0.05$ and $\rho^2 = 0.1$. The corresponding control inputs $\boldsymbol{u}(t)$ are plotted in Fig. 4.8.

Let us next consider the parametric Lyapunov function based approach and compare the results with those obtained using fixed Lyapunov function based approach. With $\boldsymbol{Q} = 100\boldsymbol{I}$ and $\rho^2 = 0.05$, a feasible solution is obtained with $ce = 1$, and hence the parametric Lyapunov function based approach is less conservative than the fixed Lyapunov function based approach. The parameters

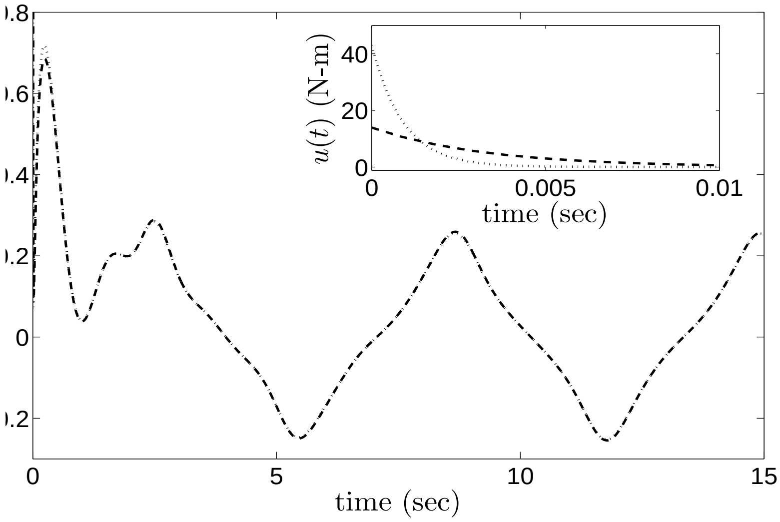


Fig. 4.11: Control input $\boldsymbol{u}(t)$ for $\rho^2 = 0.05$ with parametric Lyapunov function based approach (dashed line) and fixed Lyapunov function based approach (dotted line).

Table. 4.2: Parameters of feedback gain matrices for H_∞ tracking control with $\rho^2 = 0.05$ (for the example in Section 4.5.2)

	Parametric Lyapunov function based approach				Fixed Lyapunov function based approach			
	K_{11i}	K_{21i}	K_{12i}	K_{22i}	K_{11i}	K_{21i}	K_{12i}	K_{22i}
$i = 1$	10.139	2.4517	0.1307	0.0123	58.624	27.700	0.2215	0.2055
$i = 2$	9.3189	3.0388	0.2819	0.0627	63.365	29.703	0.4579	0.2542

of the feedback gain matrices $\boldsymbol{K}_{1i} = [K_{11i} \quad K_{21i}]$ and $\boldsymbol{K}_{2i} = [K_{12i} \quad K_{22i}]$ obtained for both fixed and parametric Lyapunov function based approaches are shown in Table 4.2. The parameters of parametric Lyapunov function are shown below:

$$\boldsymbol{P}_0 = \begin{bmatrix} 791.26 & * & * & * \\ 2.5845 & 0.9442 & * & * \\ -0.4872 & 0.0088 & 0.3973 & * \\ 1.7213 & 0.00528 & 0.0843 & 0.1047 \end{bmatrix}, \quad \boldsymbol{P}_1 = \begin{bmatrix} -19.747 & * & * & * \\ -0.00658 & 0 & * & * \\ -1.6611 & -0.00658 & 0.01407 & * \\ 0.03455 & 0 & -0.00262 & 0.00297 \end{bmatrix}$$

$$\boldsymbol{P}_2 = \begin{bmatrix} 4.9563 & * & * & * \\ 0 & -0.00216 & * & * \\ -0.3686 & 0 & -0.0002 & * \\ -0.4591 & -0.00216 & -0.00024 & -0.00103 \end{bmatrix}, \quad \boldsymbol{P}_3 = \begin{bmatrix} -119.09 & * & * & * \\ 0 & 0 & * & * \\ -0.8038 & 0 & 0.0155 & * \\ -0.3571 & 0 & -0.0014 & 0.001275 \end{bmatrix}$$

In the case of parametric Lyapunov function based approach, the time taken in the final iteration with feasible solution is 272.6 sec on a Pentium 4 processor with 2GB RAM. The state and reference trajectories with the initial condition $\boldsymbol{x}(0) = [\pi/3 \quad 0]^T$ and $\boldsymbol{x}_r(0) = [0 \quad 0]^T$ are shown in Fig. 4.9 for both fixed and parametric Lyapunov function based approaches. The corresponding tracking error plots are shown in Fig. 4.10. The control efforts $\boldsymbol{u}(t)$ for both fixed and parametric Lyapunov function based approach are shown in Fig. 4.11.

As in the previous example, the magnitude of the feedback gain is very high in the fixed Lyapunov function based approach and the effect of this high gain can be observed in the control input (transient region in Fig. 4.11). The high gain is due to the conservatism existing in the fixed Lyapunov function based approach.

For both the above cases of fixed and parametric Lyapunov function based approaches, the constants a_1 and a_2 are chosen as 0.001 and 10 respectively. In the parametric Lyapunov function based approach, the minimum possible value of disturbance attenuation level ρ^2 for which the H_∞ controller exists is $\rho^2 = 0.01$. In the case of fixed Lyapunov function based approach this condition is attained for $\rho^2 = 0.0326$. Hence, it is clear that the parametric Lyapunov function based approach shows less conservative results than the fixed Lyapunov function based approach.

The parametric Lyapunov function based approach proposed in this chapter produces less conservative results than the fixed Lyapunov function based approach for fuzzy systems with uncertain parameters. In the piecewise quadratic Lyapunov function based approach [67, 68, 73, 83] or fuzzy Lyapunov function based approach [74, 75, 79, 84], conservatism is reduced by varying the Lyapunov function across different regions. But in our proposed approach, conservatism is reduced by varying the Lyapunov function with respect to the uncertainties. Hence the proposed method is more suitable for fuzzy systems with large uncertainties.

4.6 Summary

This chapter has examined the problem of stabilizing an uncertain nonlinear system represented by its fuzzy model by using a fuzzy state feedback controller. An uncertain nonlinear system is first represented by a T-S fuzzy model. Then based on a parametric Lyapunov function based approach, a technique for designing a fuzzy state feedback control law is developed which guarantees stability over the entire range of uncertainties. In contrast to the result based on fixed Lyapunov function, the parametric Lyapunov function based approach shows less conservative result. Using the results of basic stabilization conditions, robust H_∞ controller design methods are presented for stabilization and tracking control problems. Finally, design examples of fuzzy controllers satisfying the H_∞ performance for stabilization and tracking control are presented. The advantage of the proposed parametric Lyapunov function based controller design method over the fixed Lyapunov function based design is that the proposed parametric Lyapunov function based method is less conservative, thereby admitting a wider range of uncertainties.

Chapter 5

Robust H_∞ Tracking Control for Uncertain Fuzzy Descriptor Systems

5.1 Introduction

In the robust control approaches discussed in previous chapters, a T-S fuzzy model is employed, where its consequent parts are represented by linear state-space systems. The descriptor system [8], which differs from a state-space representation describes a wider class of systems and it can be found in certain mechanical and electrical systems [8]. The ordinary T-S model is a special case of the descriptor fuzzy model. The advantage of choosing the descriptor representation over the state-space model is that the number of LMI conditions for designing the controller can be reduced for certain problems [86, 87]. Compared with the state-space based system representation, descriptor representation has more complicated structure and hence the controller design is also more complex.

In recent years, considerable work has been done involving stability analysis, stabilization control, H_∞ stabilization and model following control for fuzzy descriptor systems [86, 87, 89]. The need for such control techniques arises primarily from the increased practical interest for a more general system description which takes the intrinsic physical system model structure into account. Besides, the standard state-space system problem is a special case of descriptor systems formulations and therefore can be solved reliably by using descriptor system computational techniques.

A model following control is considered in [86] and observer based H_∞ tracking control problem is considered in [87]. For a state feedback H_∞ tracking control problem, the approach in [87] will yield the conditions in terms of bilinear matrix inequalities which are usually solved by a two step algorithm. With the approach in [87], the sufficient conditions for designing a state feedback controller cannot be framed as LMIs. Hence in this chapter, an LMI formulation of design conditions using fixed Lyapunov function is considered for a model reference trajectory tracking problem having H_∞ performance criteria. Next these results are combined with the concepts presented in the previous chapter (Chapter 4) and parametric Lyapunov function based design for controlling uncertain descriptor fuzzy systems is proposed here.

5.2 T-S Fuzzy Descriptor System

This section starts with introduction to T-S fuzzy model and then H_∞ tracking control problem is formulated.

The continuous fuzzy model proposed by Takagi and Sugeno [22] represents the dynamics of a nonlinear system using fuzzy IF-THEN rules. Let us consider the descriptor fuzzy model of a nonlinear system in the following form:

Plant rule:

IF $z_1^e(t)$ is N_{k1}^e,..., $z_{p^k}^e(t)$ is $N_{kp^k}^e$ and $z_1(t)$ is N_{i1},...,$z_p(t)$ is N_{ip} *THEN*

$$\begin{aligned} \boldsymbol{E}_k\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}_i\boldsymbol{x}(t), \qquad i = 1,2,...,r,\ k = 1,2,...,r^e \end{aligned} \tag{5.1}$$

where $z_1(t),...,z_p(t)$ are premise variables, p is the number of premise variables, N_{kj}^e $(j = 1...p^k)$, N_{ij} $(j = 1...p)$ are the fuzzy sets and r is the number of rules. Here, $\boldsymbol{x}(t) \in \mathbb{R}^{n\times 1}$ is the state vector, $\boldsymbol{y}(t) \in \mathbb{R}^{n_y\times 1}$ is the controlled output and $\boldsymbol{u}(t) \in \mathbb{R}^{m\times 1}$ is the input vector. $\boldsymbol{A}_i \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}_i \in \mathbb{R}^{n\times m}$, $\boldsymbol{C}_i \in \mathbb{R}^{n_y\times n}$ and $\boldsymbol{E}_k \in \mathbb{R}^{n\times n}$ are constant real matrices.

Given a pair of input and output $(\boldsymbol{x}(t), \boldsymbol{u}(t))$, the final output of the fuzzy system is inferred as follows:

$$\begin{aligned} \sum_{k=1}^{r_e}\mu_k^e(z(t))\boldsymbol{E}_k\dot{\boldsymbol{x}}(t) &= \sum_{i=1}^{r}\mu_i(z(t))\{\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t) + \boldsymbol{D}_i\boldsymbol{w}(t)\} \\ \boldsymbol{y}(t) &= \sum_{i=1}^{r}\mu_i(z(t))\boldsymbol{C}_i\boldsymbol{x}(t) \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} \mu_i(z(t)) &= \frac{\zeta_i(z(t))}{\sum_{j=1}^{r}\zeta_j(z(t))}, \quad \zeta_i(z(t)) = \prod_{j=1}^{p}N_{ij}(z_j(t)) \\ \mu_k^e(z(t)) &= \frac{\zeta_k^e(z^e(t))}{\sum_{j=1}^{r^e}\zeta_j^e(z^e(t))}, \quad \zeta_k^e(z^e(t)) = \prod_{j=1}^{p^e}N_{kj}^e(z_j^e(t)) \end{aligned}$$

and $N_{ij}(z_j(t))$, $N_{kj}^e(z_j^e(t))$ are the degrees of membership of $z_j(t)$ and $z_j^e(t)$ in the fuzzy set N_{ij} and

N_{kj}^e respectively. Here $\sum_{i=1}^{r}\mu_i(z(t))=1$ and $\sum_{k=1}^{r_e}\mu_k(z(t))=1$.

Let us consider a reference model as follows [92]:

$$\dot{\boldsymbol{x}}_r(t)=\boldsymbol{A}_r\boldsymbol{x}_r(t)+\boldsymbol{D}_r\boldsymbol{r}(t) \tag{5.3}$$

where $\boldsymbol{x}_r(t)$ is the reference state, $\boldsymbol{A}_r$ is a specific asymptotically stable matrix, $\boldsymbol{r}(t)$ is a bounded reference input.

The tracking error is defined as

$$\boldsymbol{e}(t)=\boldsymbol{x}(t)-\boldsymbol{x}_r(t) \tag{5.4}$$

Let us consider the H_∞ tracking performance related to the tracking error $\boldsymbol{e}(t)$ as follows [66]:

$$\int_0^{t_f}\boldsymbol{e}^T(t)\boldsymbol{Q}\boldsymbol{e}(t)dt\le\rho^2\int_0^{t_f}\boldsymbol{w}^T(t)\boldsymbol{w}(t)dt \tag{5.5}$$

where $\boldsymbol{Q}$ is a positive definite weight matrix, t_f is the terminal time of control and ρ is the prescribed disturbance attenuation level.

Let us consider the Parallel Distributed Compensation(PDC) fuzzy controller [8],

$$\boldsymbol{u}(t)=\sum_{i=1}^{r}\sum_{k=1}^{r_e}\mu_i\mu_k(\boldsymbol{K}_{1ik}\boldsymbol{e}(t)+\boldsymbol{K}_{2ik}\boldsymbol{x}_r(t)). \tag{5.6}$$

where $\boldsymbol{K}_{1ik}$ and $\boldsymbol{K}_{2ik}$ are the controller gains. A fuzzy controller is to be designed with the feedback gains $\boldsymbol{K}_{1ik}$ and $\boldsymbol{K}_{2ik}$ $(i=1,...,r,k=1,...,r^e)$ such that the resulting closed-loop system is asymptotically stable and also satisfies the H_∞ performance criterion given in (5.5).

Combining (5.2) and (5.3), the augmented system can be expressed as

$$\boldsymbol{E}^*\dot{\boldsymbol{x}}^*(t)=\sum_{i=1}^{r}\sum_{k=1}^{r^e}\mu_i\mu_k\{\boldsymbol{A}_{ik}^*\boldsymbol{x}^*(t)+\boldsymbol{B}_i^*\boldsymbol{u}(t)+\boldsymbol{D}_i^*\boldsymbol{w}^*(t)\} \tag{5.7}$$

where

$$\boldsymbol{x}^*(t)=\left[\begin{array}{c}\boldsymbol{e}(t)\\ \hdashline \boldsymbol{x}_r(t)\\ \hdashline \dot{\boldsymbol{e}}(t)\end{array}\right],\quad \boldsymbol{w}^*(t)=\left[\begin{array}{c}\boldsymbol{w}(t)\\ \hdashline \boldsymbol{r}(t)\end{array}\right],\quad \boldsymbol{E}^*=\left[\begin{array}{c:c:c}\boldsymbol{I}&\boldsymbol{0}&\boldsymbol{0}\\ \hdashline \boldsymbol{0}&\boldsymbol{I}&\boldsymbol{0}\\ \hdashline \boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\end{array}\right]$$

$$\boldsymbol{A}_{ik}^*=\left[\begin{array}{c:c:c}\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{I}\\ \hdashline \boldsymbol{0}&\boldsymbol{A}_r&\boldsymbol{0}\\ \hdashline \boldsymbol{A}_i&(\boldsymbol{A}_i-\boldsymbol{E}_k\boldsymbol{A}_r)&-\boldsymbol{E}_k\end{array}\right],\quad \boldsymbol{B}_i^*=\left[\begin{array}{c}\boldsymbol{0}\\ \hdashline \boldsymbol{0}\\ \hdashline \boldsymbol{B}_i\end{array}\right],\quad \boldsymbol{D}_i^*=\left[\begin{array}{c:c}\boldsymbol{0}&\boldsymbol{0}\\ \hdashline \boldsymbol{0}&\boldsymbol{D}_r\\ \hdashline \boldsymbol{D}_i&-\boldsymbol{E}_k\boldsymbol{D}_r\end{array}\right]$$

5.3 H_∞ Trajectory Tracking Control

For the augmented system represented by (5.7), the result of the H_∞ trajectory tracking control is summarized in the following theorem, followed by the proof.

Theorem 5.1. Let us consider the fuzzy descriptor system (5.2) with the control law (5.6). If there exist certain matrices $\boldsymbol{X}_{11}$, $\boldsymbol{X}_{21}$, $\boldsymbol{X}_{22}$, $\boldsymbol{X}_{31}$, $\boldsymbol{X}_{32}$, $\boldsymbol{X}_{33}$ and $\boldsymbol{W}_{1jk}$, $\boldsymbol{W}_{2jk}$ $(j = 1, ..., r, k = 1, ..., r^e)$ such that the following matrix inequalities are satisfied

$$\boldsymbol{S} = \boldsymbol{S}^T > 0 \tag{5.8}$$

$$\begin{aligned} \boldsymbol{\phi}_{iik} &< 0, \quad i = 1, 2, \ldots, r, \quad k = 1, 2, \ldots, r^e & (5.9) \\ \frac{1}{r-1}\phi_{ijk} + \frac{1}{2}(\phi_{ijk} + \phi_{jik}) &< 0, \quad 1 \leq i \neq j \leq r, \quad k = 1, 2, \ldots, r^e & (5.10) \end{aligned}$$

where

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}_{21}^T \\ \boldsymbol{X}_{21} & \boldsymbol{X}_{22} \end{bmatrix}$$

$$\boldsymbol{\phi}_{ijk} = \left[\begin{array}{ccc|cc|c} \boldsymbol{H}^{11} & * & * & * & * & * \\ \boldsymbol{H}^{21} & \boldsymbol{H}^{22} & * & * & * & * \\ \boldsymbol{H}^{31}_{ijk} & \boldsymbol{H}^{32}_{ijk} & \boldsymbol{H}^{33}_{k} & * & * & * \\ \hline \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{D}_i^T & -\rho^2\boldsymbol{I} & * & * \\ \boldsymbol{0} & \boldsymbol{D}_r^T & -\boldsymbol{D}_r^T\boldsymbol{E}_k^T & \boldsymbol{0} & -\rho^2\boldsymbol{I} & * \\ \hline \boldsymbol{X}_{11} & \boldsymbol{X}_{21}^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{Q}^{-1} \end{array}\right]$$

$$\begin{aligned} \boldsymbol{H}^{11} &= \boldsymbol{X}_{31}^T + (*) \\ \boldsymbol{H}^{21} &= \boldsymbol{X}_{32}^T + \boldsymbol{A}_r\boldsymbol{X}_{21} \\ \boldsymbol{H}^{22} &= \boldsymbol{A}_r\boldsymbol{X}_{22} + (*) \\ \boldsymbol{H}^{31}_{ijk} &= \boldsymbol{X}_{33}^T + \boldsymbol{A}_i\boldsymbol{X}_{11} + (\boldsymbol{A}_i - \boldsymbol{E}_k\boldsymbol{A}_r)\boldsymbol{X}_{21} - \boldsymbol{E}_k\boldsymbol{X}_{31} + \boldsymbol{B}_i\boldsymbol{W}_{1jk} \\ \boldsymbol{H}^{32}_{ijk} &= \boldsymbol{A}_i\boldsymbol{X}_{21}^T + (\boldsymbol{A}_i - \boldsymbol{E}_k\boldsymbol{A}_r)\boldsymbol{X}_{22} - \boldsymbol{E}_k\boldsymbol{X}_{32} + \boldsymbol{B}_i\boldsymbol{W}_{2jk} \\ \boldsymbol{H}^{33}_{k} &= -\boldsymbol{X}_{33}^T\boldsymbol{E}_k^T - (*) \end{aligned}$$

then the closed loop system with the controller gain matrices $[\boldsymbol{K}_{1jk} \quad \boldsymbol{K}_{2jk}] = [\boldsymbol{W}_{1jk} \quad \boldsymbol{W}_{2jk}] \times [\boldsymbol{X}_{11} \quad \boldsymbol{X}_{21}^T; \boldsymbol{X}_{21} \quad \boldsymbol{X}_{22}]^{-1}$ satisfy the given H_∞ performance criteria.

Proof: Let us consider a candidate of Lyapunov function

$$V(t) = \boldsymbol{x}^{*T}(t)\boldsymbol{E}^{*T}\boldsymbol{X}^{-1}\boldsymbol{x}^{*}(t) \tag{5.11}$$

where

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}_{21}^{T} & \mathbf{0} \\ \boldsymbol{X}_{21} & \boldsymbol{X}_{22} & \mathbf{0} \\ \boldsymbol{X}_{31} & \boldsymbol{X}_{32} & \boldsymbol{X}_{33} \end{bmatrix}$$

and

$$\boldsymbol{E}^{*T}\boldsymbol{X}^{-1} = \boldsymbol{X}^{-T}\boldsymbol{E}^{*} \geq 0$$

Applying Lemma A.1 (Appendix) to the conditions in Theorem 5.1, if the inequalities in (5.9) and (5.10) are satisfied then

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r^e} \mu_i\mu_j\mu_k\phi_{ijk} < 0 \tag{5.12}$$

The above inequality can be written as

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r^e} \mu_i\mu_j\mu_k \begin{bmatrix} \boldsymbol{X}^{T}\boldsymbol{\Omega}_{ijk}\boldsymbol{X} + \boldsymbol{X}^{T}\boldsymbol{Q}^{*}\boldsymbol{X} & * \\ \boldsymbol{D}_i^{*T} & -\rho^2\boldsymbol{I} \end{bmatrix} < 0 \tag{5.13}$$

where $\boldsymbol{\Omega}_{ijk} = (\boldsymbol{A}_{ik}^{*} + \boldsymbol{B}_i^{*}\boldsymbol{K}_{jk}^{*})^T\boldsymbol{X}^{-1} + (*)$ and $\boldsymbol{Q}^{*} = \text{block diag}\{\boldsymbol{Q}, \mathbf{0}, \mathbf{0}\}$.

Pre-multiplying and post multiplying the above inequality by block diag$[\boldsymbol{X}^{-T}, \mathbf{0}]$ and block diag$[\boldsymbol{X}^{-1}, \mathbf{0}]$, the following parameterized matrix inequality is obtained

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r^e} \mu_i\mu_j\mu_k \begin{bmatrix} \boldsymbol{\Omega}_{ijk} + \boldsymbol{Q}^{*} & * \\ \boldsymbol{D}_i^{*T}\boldsymbol{X}^{-1} & -\rho^2\boldsymbol{I} \end{bmatrix} < 0 \tag{5.14}$$

Let us consider the candidate of Lyapunov function (5.11)

$$V(t) = \boldsymbol{x}^{*T}(t)\boldsymbol{E}^{*T}\boldsymbol{X}^{-1}\boldsymbol{x}^{*}(t) \tag{5.15}$$

Let $\boldsymbol{K}_{ik}^{*} = [\boldsymbol{K}_{1ik} \;\; \boldsymbol{K}_{2ik} \;\; \mathbf{0}]$. Then from the derivative of the Lyapunov function, it follows that

$$\begin{aligned}
&\dot{V}(t) + \boldsymbol{x}^{*T}(t)\boldsymbol{Q}^{*}\boldsymbol{x}^{*}(t) - \rho^2\boldsymbol{w}^{*T}(t)\boldsymbol{w}^{*}(t) \\
&= \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r^e} \mu_i\mu_j\mu_k \Big\{\boldsymbol{x}^{*T}(t)\Big((\boldsymbol{A}_{ik}^{*} + \boldsymbol{B}_i^{*}\boldsymbol{K}_{jk}^{*})^T\boldsymbol{X}^{-1} + (*) + \boldsymbol{Q}^{*}\Big)\boldsymbol{x}^{*}(t)\Big\} + \boldsymbol{x}^{*T}(t)\boldsymbol{X}^{-T}\boldsymbol{D}_i^{*}\boldsymbol{w}^{*}(t) \\
&\qquad + \boldsymbol{w}^{*T}(t)\boldsymbol{D}_i^{*T}\boldsymbol{X}^{-1}\boldsymbol{x}^{*}(t) - \rho^2\boldsymbol{w}^{*T}(t)\boldsymbol{w}^{*}(t) \qquad (5.16) \\
&= \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r^e} \mu_i\mu_j\mu_k \begin{bmatrix} \boldsymbol{x}^{*T}(t) & \boldsymbol{w}^{*T}(t) \end{bmatrix} \begin{bmatrix} \boldsymbol{\Omega}_{ijk} + \boldsymbol{Q}^{*} & * \\ \boldsymbol{D}_i^{*T}\boldsymbol{X}^{-1} & -\rho^2\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^{*}(t) \\ \boldsymbol{w}^{*}(t) \end{bmatrix} \qquad (5.17)
\end{aligned}$$

From (5.17) and (5.14), the following inequality is obtained

$$\dot{V}(t) + \boldsymbol{x}^{*T}(t)\boldsymbol{Q}^{*}\boldsymbol{x}^{*}(t) - \rho^2\boldsymbol{w}^{*T}(t)\boldsymbol{w}^{*}(t) < 0 \tag{5.18}$$

Integrating the above inequality from 0 to ∞ on both sides, yields

$$V(\infty) - V(0) + \int_0^\infty (\boldsymbol{x}^{*T}(t)\boldsymbol{Q}^{*}\boldsymbol{x}^{*}(t) - \rho^2\boldsymbol{w}^{*T}(t)\boldsymbol{w}^{*}(t))dt < 0 \tag{5.19}$$

With zero initial condition, $V(0) = 0$ and hence

$$\int_0^\infty \boldsymbol{x}^{*T}(t)\boldsymbol{Q}^{*}\boldsymbol{x}^{*}(t)dt \;<\; \int_0^\infty \rho^2\boldsymbol{w}^{*T}(t)\boldsymbol{w}^{*}(t)dt \tag{5.20}$$

$$\int_0^\infty \boldsymbol{e}^{T}(t)\boldsymbol{Q}\boldsymbol{e}(t)dt \;<\; \int_0^\infty \rho^2\boldsymbol{w}^{*T}(t)\boldsymbol{w}^{*}(t)dt \tag{5.21}$$

Thus the proof is complete. □

5.3.1 Stability Analysis

Let us consider (5.18). If $\boldsymbol{w}^{*}(t) = 0$, then $\dot{V}(t) < 0$, which implies that the closed loop system is asymptotically stable.

5.3.2 Common $\boldsymbol{B}$ Matrix case

In this subsection, the common $\boldsymbol{B}$ matrix case is considered, where $\boldsymbol{B}_i = \boldsymbol{B}$ $(i = 1, 2, ..., r)$. The LMI conditions for designing the controller are given by the following Theorem.

Theorem 5.2. Let us consider the fuzzy descriptor system (5.2) with the control law (5.6). If there exist some matrices $\boldsymbol{X}_{11}$, $\boldsymbol{X}_{21}$, $\boldsymbol{X}_{22}$, $\boldsymbol{X}_{31}$, $\boldsymbol{X}_{32}$, $\boldsymbol{X}_{33}$ and $\boldsymbol{W}_{1ik}$, $\boldsymbol{W}_{2ik}$ $(i = 1, ..., r, k = 1, ..., r^e)$ such that the following matrix inequalities are satisfied

$$\boldsymbol{S} = \boldsymbol{S}^T > 0 \tag{5.22}$$

$$\left[\begin{array}{ccc|cc|c}
\boldsymbol{M}_{11} & * & * & * & * & * \\
\boldsymbol{M}_{21} & \boldsymbol{M}_{22} & * & * & * & * \\
\boldsymbol{M}_{31} & \boldsymbol{M}_{32} & \boldsymbol{M}_{33} & * & * & * \\
\hline
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{D}_i^T & -\rho^2\boldsymbol{I} & * & * \\
\boldsymbol{0} & \boldsymbol{D}_r^T & -\boldsymbol{D}_r^T\boldsymbol{E}_k^T & \boldsymbol{0} & -\rho^2\boldsymbol{I} & * \\
\hline
\boldsymbol{X}_{11} & \boldsymbol{X}_{21}^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{Q}^{-1}
\end{array}\right] < 0, \quad i = i, ..., r, \quad k = 1, ..., r^e$$

where

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}_{21}^T \\ \boldsymbol{X}_{21} & \boldsymbol{X}_{22} \end{bmatrix}$$

$$\begin{aligned}
\boldsymbol{M}_{11} &= \boldsymbol{X}_{31}^T + (*) \\
\boldsymbol{M}_{21} &= \boldsymbol{X}_{32}^T + \boldsymbol{A}_r\boldsymbol{X}_{21} \\
\boldsymbol{M}_{22} &= \boldsymbol{A}_r\boldsymbol{X}_{22} + (*) \\
\boldsymbol{M}_{31} &= \boldsymbol{X}_{33}^T + \boldsymbol{A}_i\boldsymbol{X}_{11} + (\boldsymbol{A}_i - \boldsymbol{E}_k\boldsymbol{A}_r)\boldsymbol{X}_{21} - \boldsymbol{E}_k\boldsymbol{X}_{31} + \boldsymbol{B}\boldsymbol{W}_{1ik} \\
\boldsymbol{M}_{32} &= \boldsymbol{A}_i\boldsymbol{X}_{21}^T + (\boldsymbol{A}_i - \boldsymbol{E}_k\boldsymbol{A}_r)\boldsymbol{X}_{22} - \boldsymbol{E}_k\boldsymbol{X}_{32} + \boldsymbol{B}\boldsymbol{W}_{2ik} \\
\boldsymbol{M}_{33} &= -\boldsymbol{X}_{33}^T\boldsymbol{E}_k^T - (*)
\end{aligned}$$

then the closed loop system with the controller gain matrices $[\boldsymbol{K}_{1ik} \quad \boldsymbol{K}_{2ik}] = [\boldsymbol{W}_{1ik} \quad \boldsymbol{W}_{2ik}] \times [\boldsymbol{X}_{11} \quad \boldsymbol{X}_{21}^T; \boldsymbol{X}_{21} \quad \boldsymbol{X}_{22}]^{-1}$ satisfy the given H_∞ performance criteria.

In this case, the LMI conditions for controller design are simpler and number of LMI conditions is also less than that of the general case.

5.3.3 Simulation Results

Let us consider the simple nonlinear system presented in [86] with some external disturbance. The system is represented by

$$(1 + a\cos(\theta(t)))\ddot{\theta}(t) = -b\dot{\theta}^3(t) + c\theta(t) + du(t) + 0.1w(t) \tag{5.23}$$

where $a = 0.2$, $b = 1$, $c = -1$, $d = 10$, $w(t) = \sin(5t)$ and the range of $\dot{\theta}(t)$ is $|\dot{\theta}(t)| < \phi$, $\phi = 4$. This descriptor fuzzy model is given by [86]

$$\begin{aligned}
\sum_{k=1}^{2} \mu_k^e(z(t))\boldsymbol{E}_k\dot{\boldsymbol{x}}(t) &= \sum_{i=1}^{2} \mu_i(z(t))\{\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t) + \boldsymbol{D}_i\boldsymbol{w}(t)\} \\
\boldsymbol{y}(t) &= \sum_{i=1}^{2} \mu_i(z(t))\boldsymbol{C}_i\boldsymbol{x}(t)
\end{aligned} \tag{5.24}$$

where $\boldsymbol{x}(t) = [x_1(t) \quad x_2(t)]^T = [\theta(t) \quad \dot{\theta}(t)]^T$. The parameters of the constant matrices are

$$\begin{aligned}
\boldsymbol{E}_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1+a \end{bmatrix}, \quad \boldsymbol{E}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1-a \end{bmatrix}, \\
\boldsymbol{A}_1 &= \begin{bmatrix} 0 & 1 \\ c & -b\cdot\phi^2 \end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 \\ c & 0 \end{bmatrix}, \quad \boldsymbol{B}_1 = \boldsymbol{B}_2 = \begin{bmatrix} 0 \\ d \end{bmatrix}, \quad \boldsymbol{D}_i = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, i = 1, 2
\end{aligned}$$

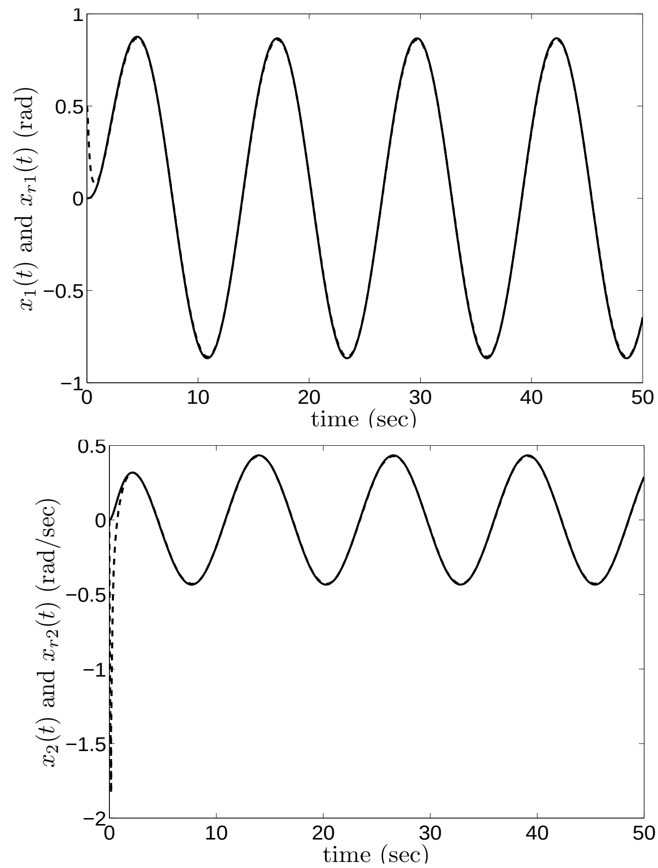


Fig. 5.1: Trajectories of state variables $\boldsymbol{x}(t)$ (dashed line) and the reference trajectories $\boldsymbol{x}_r(t)$ (solid line).

$$\begin{aligned}\mu_1(x_2(t)) &= \frac{x_2^2(t)}{2}, \quad \mu_2(x_2(t)) = 1 - \frac{x_2^2(t)}{2},\\ \mu_1^e(x_1(t)) &= \frac{1+\cos(x_1(t))}{2}, \quad \mu_2^e(x_1(t)) = \frac{1-\cos(x_1(t))}{2}.\end{aligned}$$

The following reference model and reference input are considered:

$$\begin{bmatrix}\dot{x}_{r1}\\ \dot{x}_{r2}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_{r1}\\ x_{r2}\end{bmatrix} + \begin{bmatrix}0\\ 2\sin(t/2)\end{bmatrix}.$$

The H_∞ tracking controller is designed by solving the LMI conditions in Theorem 5.2. With $\boldsymbol{Q} = 0.1\boldsymbol{I}$ and $\rho^2 = 0.01$, the parameters of Lyapunov function and the feedback gain matrices $\boldsymbol{K}_{1ik}$, $\boldsymbol{K}_{2ik}$ obtained are given below:

$$\boldsymbol{X}_{11} = \begin{bmatrix}3.6783 & -9.2633\\ -9.2633 & 109.4131\end{bmatrix}, \quad \boldsymbol{X}_{21} = \begin{bmatrix}0.5044 & 0.9124\\ -0.5114 & -2.8609\end{bmatrix}, \quad \boldsymbol{X}_{22} = \begin{bmatrix}362.01 & -71.42\\ -71.42 & 227.28\end{bmatrix},$$

$$\boldsymbol{X}_{31} = \begin{bmatrix}-4.92\times 10^8 & 112.29\\ -0.6444 & -81775\end{bmatrix}, \quad \boldsymbol{X}_{32} = \begin{bmatrix}1.0239 & -2.8364\\ 79.828 & -352.48\end{bmatrix}, \quad \boldsymbol{X}_{33} = \begin{bmatrix}4.92\times 10^8 & -3.107\\ -3.1071 & 81291\end{bmatrix},$$

$$\boldsymbol{K}_{111} = \begin{bmatrix} -8.5519 & -1.7772 \end{bmatrix}, \qquad \boldsymbol{K}_{112} = \begin{bmatrix} -8.7643 & -1.8713 \end{bmatrix},$$
$$\boldsymbol{K}_{121} = \begin{bmatrix} -6.9749 & -2.7477 \end{bmatrix}, \qquad \boldsymbol{K}_{122} = \begin{bmatrix} -6.8766 & -2.7285 \end{bmatrix},$$
$$\boldsymbol{K}_{211} = \begin{bmatrix} -0.0271 & 1.0333 \end{bmatrix}, \qquad \boldsymbol{K}_{212} = \begin{bmatrix} -0.0201 & 1.0119 \end{bmatrix},$$
$$\boldsymbol{K}_{221} = \begin{bmatrix} -0.0111 & -0.2714 \end{bmatrix}, \qquad \boldsymbol{K}_{222} = \begin{bmatrix} 0.0302 & -0.1605 \end{bmatrix}.$$

State and reference trajectories $\boldsymbol{x}(t)$ and $\boldsymbol{x}_r(t)$ with the initial condition $\boldsymbol{x}(0) = [0.5 \;\; 0]^T$ and $\boldsymbol{x}_r(0) = [0 \;\; 0]^T$ are shown in Fig. 5.1.

5.4 Uncertain T-S Fuzzy Descriptor System

This section starts with introduction to uncertain T-S descriptor fuzzy model and then the robust H_∞ tracking control problem is formulated.

The continuous fuzzy model proposed by Takagi and Sugeno [22] represents the dynamics of a nonlinear system using fuzzy IF-THEN rules. Let us consider the descriptor fuzzy model of an uncertain nonlinear system in the following form:

Plant rule:

IF $z_1^e(t)$ is $N_{k1}^e, \cdots$, $z_{p^k}^e(t)$ is $N_{kp^k}^e$ and $z_1(t)$ is $N_{i1}, \cdots$,$z_p(t)$ is N_{ip} *THEN*

$$\begin{aligned}(\boldsymbol{E}_k(\boldsymbol{\theta}) + \Delta\boldsymbol{E}_k(t))\dot{\boldsymbol{x}}(t) &= (\boldsymbol{A}_i(\boldsymbol{\theta}) + \Delta\boldsymbol{A}_i(t))\boldsymbol{x}(t) + (\boldsymbol{B}_i(\boldsymbol{\theta}) + \Delta\boldsymbol{B}_i(t))\boldsymbol{u}(t) + \boldsymbol{D}_i\boldsymbol{w}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}_i\boldsymbol{x}(t), \qquad i = 1, 2, \cdots, r,\; k = 1, 2, \cdots, r^e \end{aligned} \tag{5.25}$$

where $\boldsymbol{A}_i(\boldsymbol{\theta}) = \boldsymbol{A}_{i0} + \sum_{l=1}^{L} \theta_l(t)\boldsymbol{A}_{il}$, $\boldsymbol{B}_i(\boldsymbol{\theta}) = \boldsymbol{B}_{i0} + \sum_{l=1}^{L} \theta_l(t)\boldsymbol{B}_{il}$, $\boldsymbol{E}_k(\boldsymbol{\theta}) = \boldsymbol{E}_{k0} + \sum_{l=1}^{L} \theta_l(t)\boldsymbol{E}_{kl}$, $z_1(t), \cdots, z_p(t)$ are premise variables, p is the number of premise variables, N_{kj}^e ($j = 1...p^k$), N_{ij} ($j = 1...p$) are the fuzzy sets and r is the number of rules. For simplicity $\boldsymbol{\theta}(t)$ is denoted as $\boldsymbol{\theta}$. Here, $\boldsymbol{x}(t) \in \mathbb{R}^{n\times 1}$ is the state vector, $\boldsymbol{y}(t) \in \mathbb{R}^{n_y\times 1}$ is the controlled output and $\boldsymbol{u}(t) \in \mathbb{R}^{m\times 1}$ is the input vector. $\boldsymbol{A}_{i0} \in \mathbb{R}^{n\times n}$, $\boldsymbol{A}_{il} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}_{i0} \in \mathbb{R}^{n\times m}$, $\boldsymbol{B}_{il} \in \mathbb{R}^{n\times m}$, $\boldsymbol{E}_{k0} \in \mathbb{R}^{n\times n}$, $\boldsymbol{E}_{kl} \in \mathbb{R}^{n\times n}$, $\boldsymbol{C}_i \in \mathbb{R}^{n_y\times n}$ are constant real matrices, $\theta_l(t)$ represents time varying parametric uncertainties; $\Delta\boldsymbol{A}_i(t)$, $\Delta\boldsymbol{B}_i(t)$ and $\Delta\boldsymbol{E}_k(t)$ are time varying matrices of appropriate dimensions, which represent modeling errors.

Given a pair of input and output $(\boldsymbol{x}(t), \boldsymbol{u}(t))$, the final output of the fuzzy system is inferred as follows:

$$\sum_{k=1}^{r_e} \mu_k^e(\boldsymbol{E}_k(\boldsymbol{\theta}) + \Delta\boldsymbol{E}_k(t))\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} \mu_i\{(\boldsymbol{A}_i(\boldsymbol{\theta}) + \Delta\boldsymbol{A}_i(t))\boldsymbol{x}(t) + (\boldsymbol{B}_i(\boldsymbol{\theta}) + \Delta\boldsymbol{B}_i(t))\boldsymbol{u}(t) + \boldsymbol{D}_i\boldsymbol{w}(t)\}$$

$$\boldsymbol{y}(t) \;=\; \sum_{i=1}^{r} \mu_i \boldsymbol{C}_i \boldsymbol{x}(t) \tag{5.26}$$

where

$$\begin{aligned}
\mu_i &= \frac{\zeta_i(z(t))}{\sum_{j=1}^{r}\zeta_j(z(t))}, \quad \zeta_i(z(t)) = \prod_{j=1}^{p} N_{ij}(z_j(t)) \\
\mu_k^e &= \frac{\zeta_k^e(z^e(t))}{\sum_{j=1}^{r^e}\zeta_j^e(z^e(t))}, \quad \zeta_k^e(z^e(t)) = \prod_{j=1}^{p^e} N_{kj}^e(z_j^e(t))
\end{aligned}$$

$N_{ij}(z_j(t))$ and $N_{kj}^e(z_j^e(t))$ are the degrees of membership of $z_j(t)$ and $z_j^e(t)$ in the fuzzy set N_{ij} and N_{kj}^e respectively. Here $\sum_{i=1}^{r}\mu_i(z(t)) = 1$ and $\sum_{k=1}^{r_e}\mu_k(z(t)) = 1$. For simplicity, $\mu_k^e(z(t))$ and $\mu_i(z(t))$ are represented as μ_k^e and μ_i respectively.

The uncertain matrices $\Delta\boldsymbol{A}_i(t)$, $\Delta\boldsymbol{B}_i(t)$ and $\Delta\boldsymbol{E}_k(t)$ are assumed to be norm bounded and are described by [2]:

$$\begin{aligned}
\begin{bmatrix} \Delta\boldsymbol{A}_i(t) & \Delta\boldsymbol{B}_i(t) \end{bmatrix} &= \sum_{l=1}^{L_a} \boldsymbol{M}_{il}^a \Delta_{il}^a(t) \begin{bmatrix} \boldsymbol{N}_{i1l}^a & \boldsymbol{N}_{i2l}^a \end{bmatrix}, \\
\Delta\boldsymbol{E}_k(t) &= \sum_{l=1}^{L_e} \boldsymbol{M}_{kl}^e \Delta_{kl}^e(t) \boldsymbol{N}_{kl}^e
\end{aligned} \tag{5.27}$$

where $\boldsymbol{M}_{il}^a$, $\boldsymbol{M}_{kl}^e$, $\boldsymbol{N}_{i1l}^a$, $\boldsymbol{N}_{i2l}^a$ and $\boldsymbol{N}_{k1l}^e$ are known real constant matrices of appropriate dimension and $\Delta_{il}^a(t)$, $\Delta_{il}^e(t)$ are time varying functions, satisfying $|\Delta_{il}^a(t)| < 1$, $|\Delta_{kl}^e(t)| < 1$, $\forall\ t > 0$.

Let us consider a reference model and the H_∞ performance measure as given in Section 5.2 with the Parallel Distributed Compensation(PDC) fuzzy controller [8],

$$\boldsymbol{u}(t) = \sum_{i=1}^{r}\sum_{k=1}^{r_e} \mu_i \mu_k^e (\boldsymbol{K}_{1ik}\boldsymbol{e}(t) + \boldsymbol{K}_{2ik}\boldsymbol{x}_r(t)). \tag{5.28}$$

where $\boldsymbol{K}_{1ik}$ and $\boldsymbol{K}_{2ik}$ are the controller gains. A fuzzy controller is to be designed with the feedback gains $\boldsymbol{K}_{1ik}$ and $\boldsymbol{K}_{2ik}$ $(i = 1, ..., r, k = 1, ..., r^e)$ such that the resulting closed-loop system is asymptotically stable and also satisfies the H_∞ performance given in (5.5).

Combining (5.26) and (5.3) with the control law (5.28), the augmented fuzzy descriptor system can be expressed as

$$\boldsymbol{E}^*\dot{\boldsymbol{x}}^*(t) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r^e} \mu_i\mu_j\mu_k^e\{(\boldsymbol{A}_{ik}^*(\boldsymbol{\theta}) + \Delta\boldsymbol{A}_{ik}^*(t) + (\boldsymbol{B}_i^*(\boldsymbol{\theta}) + \Delta\boldsymbol{B}_i^*(t))\boldsymbol{K}_{jk}^*)\boldsymbol{x}^*(t) + \boldsymbol{D}_i^*\boldsymbol{w}^*(t)\} \tag{5.29}$$

where

$$\boldsymbol{x}^*(t) = \left[\begin{array}{c} \boldsymbol{e}(t) \\ \hdashline \boldsymbol{x}_r(t) \\ \hdashline \dot{\boldsymbol{e}}(t) \end{array}\right], \boldsymbol{w}^*(t) = \left[\begin{array}{c} \boldsymbol{w}(t) \\ \hdashline \boldsymbol{r}(t) \end{array}\right], \boldsymbol{A}_{ik}^*(\boldsymbol{\theta}) = \left[\begin{array}{c:c:c} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \\ \hdashline \boldsymbol{0} & \boldsymbol{A}_r & \boldsymbol{0} \\ \hdashline \boldsymbol{A}_i(\boldsymbol{\theta}) & \boldsymbol{A}_i(\boldsymbol{\theta}) - \boldsymbol{E}_k(\boldsymbol{\theta})\boldsymbol{A}_r & -\boldsymbol{E}_k(\boldsymbol{\theta}) \end{array}\right]$$

$$\Delta \boldsymbol{A}_{ik}^*(t) = \left[\begin{array}{c|c|c} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \Delta \boldsymbol{A}_i(t) & \Delta \boldsymbol{A}_i(t) - \Delta \boldsymbol{E}_k(t)\boldsymbol{A}_r & -\Delta \boldsymbol{E}_k(t) \end{array}\right], \quad \boldsymbol{B}_i^*(\boldsymbol{\theta}) = \left[\begin{array}{c} \mathbf{0} \\ \hline \mathbf{0} \\ \hline \boldsymbol{B}_i(\boldsymbol{\theta}) \end{array}\right], \quad \Delta \boldsymbol{B}_i^* = \left[\begin{array}{c} \mathbf{0} \\ \hline \mathbf{0} \\ \hline \Delta \boldsymbol{B}_i(t) \end{array}\right]$$

$$\boldsymbol{K}_{jk}^* = \left[\begin{array}{c|c|c} \boldsymbol{K}_{1jk} & \boldsymbol{K}_{2jk} & \mathbf{0} \end{array}\right], \quad \boldsymbol{D}_{ik}^* = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{D}_r \\ \hline \boldsymbol{D}_i & -\boldsymbol{E}_k(\boldsymbol{\theta})\boldsymbol{D}_r \end{array}\right], \quad \boldsymbol{E}^* = \left[\begin{array}{c|c|c} \boldsymbol{I} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{I} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$$

$$\left[\begin{array}{cc} \Delta \boldsymbol{A}_{ik}^*(t) & \Delta \boldsymbol{B}_i^*(t) \end{array}\right] = \sum_{l=1}^{L_a} \boldsymbol{M}_{il}^{a*} \Delta_{il}^a(t) \left[\begin{array}{cc} \boldsymbol{N}_{i1l}^{a*} & \boldsymbol{N}_{i2l}^{a*} \end{array}\right] + \sum_{l=1}^{L_e} \boldsymbol{M}_{kl}^{e*} \Delta_{kl}^e(t) \boldsymbol{N}_{kl}^{e*} \tag{5.30}$$

$$\begin{aligned} \boldsymbol{M}_{il}^{a*} &= \left[\begin{array}{c} \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{M}_{il}^a \end{array}\right], \quad \boldsymbol{M}_{kl}^{e*} = \left[\begin{array}{c} \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{M}_{kl}^e \end{array}\right], \\ \boldsymbol{N}_{i1l}^{a*} &= \left[\begin{array}{ccc} \boldsymbol{N}_{i1l}^a & \boldsymbol{N}_{i1l}^a & \mathbf{0} \end{array}\right], \quad \boldsymbol{N}_{i2l}^{a*} = \boldsymbol{N}_{i2l}^a, \quad \boldsymbol{N}_{kl}^{e*} = \left[\begin{array}{ccc} \mathbf{0} & \boldsymbol{N}_{kl}^e & \mathbf{0} \end{array}\right] \end{aligned}$$

The fuzzy descriptor system (5.29) affinely depends on the parameter vector $\boldsymbol{\theta}$. As in [124] and [125], the lower and upper bounds of the uncertain parameter and their rates of variation are assumed to be known. Specifically:

1. Each parameter θ_l ranges between the known lower bound $\underline{\theta}_l$ and upper bound $\overline{\theta}_l$, i.e.,

$$\theta_l \in [\underline{\theta}_l \ \ \overline{\theta}_l], \tag{5.31}$$

2. The rate of variation $\dot{\theta}_l$ is well defined at all times and satisfies

$$\dot{\theta}_l \in [\underline{v}_l \ \ \overline{v}_l], \tag{5.32}$$

   where $\underline{v}_l$ and $\overline{v}_l$ are known lower and upper bounds of $\dot{\theta}_l$

With these assumptions, the parameter vector θ_l takes values within the hyper-rectangle called parameter box and the rate vector $\dot{\theta}_l$ takes values in another hyper-rectangle called rate box. It is denoted as,

$$\begin{aligned} \mathcal{V} &:= \{(\nu_1, \nu_2, ..., \nu_L)^T : \nu_l \in \{\underline{\theta}_l \ \ \overline{\theta}_l\}\}, && (5.33) \\ \mathcal{W} &:= \{(\omega_1, \omega_2, ..., \omega_L)^T : \omega_l \in \{\underline{v}_l \ \ \overline{v}_l\}\}, && (5.34) \end{aligned}$$

which are the set of 2^L vertices of the parameter box and the rate box respectively.

5.5 Robust H_∞ Tracking Control with Fixed Lyapunov Function

In this section, fixed Lyapunov function based robust H_∞ tracking controller design for fuzzy descriptor systems is presented. Let us consider the fixed Lyapunov function in the following form

$$V(t) = \boldsymbol{x}^{*T}(t)\boldsymbol{E}^{*T}\boldsymbol{X}^{-1}\boldsymbol{x}^*(t) \tag{5.35}$$

where

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}_{21}^T & \mathbf{0} \\ \boldsymbol{X}_{21} & \boldsymbol{X}_{22} & \mathbf{0} \\ \boldsymbol{X}_{31} & \boldsymbol{X}_{32} & \boldsymbol{X}_{33} \end{bmatrix}$$

and

$$\boldsymbol{E}^{*T}\boldsymbol{X}^{-1} = \boldsymbol{X}^{-T}\boldsymbol{E}^* \geq 0$$

Theorem 5.3. Let us consider the fuzzy descriptor system (5.29) with the control law (5.28). If there exist certain matrices $\boldsymbol{X}$ as defined in (5.35) and $\boldsymbol{W}_{jk}$ $(j = 1, ..., r, k = 1, ..., r^e)$ such that the following matrix inequalities are satisfied,

$$\begin{aligned} \boldsymbol{S}^T = \boldsymbol{S} &> 0 && (5.36)\\ \boldsymbol{\phi}^*_{iik}(\boldsymbol{\nu}) &< 0, \ \forall \ \boldsymbol{\nu} \in \mathcal{V}, \ i = 1, 2, \ldots, r, \ k = 1, 2, \ldots, r^e && (5.37)\\ \frac{1}{r-1}\boldsymbol{\phi}^*_{iik}(\boldsymbol{\nu}) + \frac{1}{2}(\boldsymbol{\phi}^*_{ijk}(\boldsymbol{\nu}) + \phi^*_{jik}(\boldsymbol{\nu})) &< 0, \ \forall \ \boldsymbol{\nu} \in \mathcal{V}, \ 1 \leq i \neq j \leq r, \ k = 1, 2, \ldots, r^e && (5.38) \end{aligned}$$

where

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}_{21}^T \\ \boldsymbol{X}_{21} & \boldsymbol{X}_{22} \end{bmatrix}$$

$$\boldsymbol{\phi}^*_{ijk}(\boldsymbol{\nu}) = \left[\begin{array}{ccc:cc:cc} \boldsymbol{\mathcal{A}}^{11}_{ijk}(\boldsymbol{\nu}) & * & * & * & * & * & * \\ \boldsymbol{D}^{*T}_{ik} & -\rho^2\boldsymbol{I} & * & * & * & * & * \\ \boldsymbol{\mathcal{Y}} & \mathbf{0} & -\boldsymbol{Q}^{-1} & * & * & * & * \\ \hdashline \boldsymbol{\mathcal{A}}^{31}_{i} & \mathbf{0} & \mathbf{0} & -\boldsymbol{\epsilon}^a_i & * & * & * \\ \boldsymbol{\mathcal{A}}^{41}_{ijk} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\boldsymbol{\epsilon}^a_i & * & * \\ \hdashline \boldsymbol{\mathcal{A}}^{51}_{k} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\boldsymbol{\epsilon}^e_k & * \\ \boldsymbol{\mathcal{A}}^{61}_{k} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\boldsymbol{\epsilon}^e_k \end{array}\right]$$

$$\boldsymbol{\mathcal{A}}_i^{31} = \begin{bmatrix} \epsilon_{i1}^a \boldsymbol{M}_{i1}^{a*T} \\ \vdots \\ \epsilon_{iL_a}^a \boldsymbol{M}_{iL_a}^{a*\ T} \end{bmatrix}, \boldsymbol{\mathcal{A}}_{ijk}^{41} = \begin{bmatrix} (\boldsymbol{N}_{i11}^{a*}\boldsymbol{X} + \boldsymbol{N}_{i21}^{a*}\boldsymbol{W}_{jk}) \\ \vdots \\ (\boldsymbol{N}_{i1L_a}^{a*}\boldsymbol{X} + \boldsymbol{N}_{i2L_a}^{a*}\boldsymbol{W}_{jk}) \end{bmatrix},$$

$$\boldsymbol{\mathcal{A}}_k^{51} = \begin{bmatrix} \epsilon_{k1}^e \boldsymbol{M}_{k1}^{e*T} \\ \vdots \\ \epsilon_{kL_e}^e \boldsymbol{M}_{kL_e}^{e*\ T} \end{bmatrix}, \boldsymbol{\mathcal{A}}_k^{61} = \begin{bmatrix} \boldsymbol{N}_{k1}^{e*}\boldsymbol{X} \\ \vdots \\ \boldsymbol{N}_{kL_e}^{e*}\boldsymbol{X} \end{bmatrix}, \quad \boldsymbol{\mathcal{Y}} = \boldsymbol{X}\left[\begin{array}{c:c:c} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \end{array}\right]$$

$\boldsymbol{\mathcal{A}}_{ijk}^{11}(\boldsymbol{\nu}) = \boldsymbol{X}^T\boldsymbol{A}_{ik}^{*T}(\boldsymbol{\nu}) + \boldsymbol{W}_{jk}^{*T}\boldsymbol{B}_i^{*T}(\boldsymbol{\nu}) + (*)$, $\boldsymbol{\epsilon}_i^a = \text{diag}(\epsilon_{i1}^a, \cdots, \epsilon_{iL_a}^a)$, $\boldsymbol{\epsilon}_k^e = \text{diag}(\epsilon_{k1}^e, \cdots, \epsilon_{kL_e}^e)$ and $\boldsymbol{W}_{jk}^* = \boldsymbol{K}_{jk}^*\boldsymbol{X}$, then the closed loop system is asymptotically stable and satisfies the given H_∞ performance criteria.

Proof: Applying Lemma A.1 (Appendix) to the inequalities in Theorem 5.3, the following parameterized inequality is obtained

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r^e} \mu_i\mu_j\mu_k^e \boldsymbol{\phi}_{ijk}^*(\boldsymbol{\nu}) < 0, \quad \forall \ \boldsymbol{\nu} \in \mathcal{V} \tag{5.39}$$

If the above inequality is satisfied in the vertices $\boldsymbol{\nu}$ of the parameter box $\mathcal{V}$, then the inequality holds for the range of $\boldsymbol{\theta}$ defined in the parameter box [128]. Hence,

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r^e} \mu_i\mu_j\mu_k^e \boldsymbol{\phi}_{ijk}^*(\boldsymbol{\theta}) < 0 \tag{5.40}$$

With (5.27), using the Schur complement Lemma and the inequality $\boldsymbol{Y}^T\boldsymbol{Z}+\boldsymbol{Z}^T\boldsymbol{Y} \leq \boldsymbol{Y}^T\boldsymbol{Y}+\boldsymbol{Z}^T\boldsymbol{Z}$ given in [66], the matrices related to $\Delta\boldsymbol{A}_i$, $\Delta\boldsymbol{B}_i$ and $\Delta\boldsymbol{E}_k$ can be rewritten as follows

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r^e} \mu_i\mu_j\mu_k^e \boldsymbol{\Upsilon}_{ijk}^*(t,\boldsymbol{\theta}) < 0 \tag{5.41}$$

where

$$\boldsymbol{\Upsilon}_{ijk}^*(t,\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{\Omega}_{ijk}^*(t,\boldsymbol{\theta}) & * & * \\ \boldsymbol{D}_i^{*T} & -\rho^2\boldsymbol{I} & * \\ \boldsymbol{\mathcal{Y}} & \boldsymbol{0} & -\boldsymbol{Q}^{-1} \end{bmatrix}$$

$$\boldsymbol{\Omega}_{ijk}^*(t,\boldsymbol{\theta}) = \boldsymbol{X}^T\boldsymbol{A}_{ik}^{*T}(\boldsymbol{\theta}) + \boldsymbol{W}_{jk}^{*T}\boldsymbol{B}_i^{*T}(\boldsymbol{\theta}) + \boldsymbol{X}^T\Delta\boldsymbol{A}_{ik}^{*T}(t) + \boldsymbol{W}_{jk}^{*T}\Delta\boldsymbol{B}_i^{*T}(t) + (*)$$

Again by Schur complement, the above inequality can be expressed as

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r^e} \mu_i\mu_j\mu_k^e \begin{bmatrix} \boldsymbol{\Omega}_{ijk}^*(t,\boldsymbol{\theta}) + \boldsymbol{X}^T\boldsymbol{Q}^*\boldsymbol{X} & * \\ \boldsymbol{D}_i^{*T} & -\rho^2\boldsymbol{I} \end{bmatrix} < 0 \tag{5.42}$$

where $\boldsymbol{Q}^* = \text{diag}(\boldsymbol{Q}, \boldsymbol{0}, \boldsymbol{0})$.

Pre-multiplying (5.42) by diag($\boldsymbol{X}^{-T}, \boldsymbol{I}$) and post-multiplying by diag($\boldsymbol{X}^{-1}, \boldsymbol{I}$) yields,

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r^e}\mu_i\mu_j\mu_k^e\begin{bmatrix}\boldsymbol{X}^{-T}\boldsymbol{\Omega}_{ijk}^*(t,\boldsymbol{\theta})\boldsymbol{X}^{-1}+\boldsymbol{Q}^* & * \\ \boldsymbol{D}_i^{*T}\boldsymbol{X}^{-1} & -\rho^2\boldsymbol{I}\end{bmatrix}<0 \tag{5.43}$$

Let us consider a candidate of Lyapunov function $V(t) = \boldsymbol{x}^*(t)\boldsymbol{E}^{*T}\boldsymbol{X}^{-1}\boldsymbol{x}^*(t)$. Then from the derivative of $V(t)$, it follows,

$$\begin{aligned}&\dot{V}(t)+\boldsymbol{x}^{*T}(t)\boldsymbol{Q}^*\boldsymbol{x}^*(t)-\rho^2\boldsymbol{w}^{*T}(t)\boldsymbol{w}^*(t)\\ &=\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r^e}\mu_i\mu_j\mu_k^e\Big\{\boldsymbol{x}^{*T}(t)\Big(\boldsymbol{X}^{-T}\boldsymbol{\Omega}_{ijk}^*(t,\boldsymbol{\theta})\boldsymbol{X}^{-1}+\boldsymbol{Q}^*\Big)\boldsymbol{x}^*(t)\Big\}+\boldsymbol{x}^{*T}(t)\boldsymbol{X}^{-T}\boldsymbol{D}_i^*\boldsymbol{w}^*(t)\\ &\qquad +\boldsymbol{w}^{*T}(t)\boldsymbol{D}_i^{*T}\boldsymbol{X}^{-1}\boldsymbol{x}^*(t)-\rho^2\boldsymbol{w}^{*T}(t)\boldsymbol{w}^*(t) \qquad (5.44)\\ &=\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r^e}\mu_i\mu_j\mu_k\begin{bmatrix}\boldsymbol{x}^{*T}(t) & \boldsymbol{w}^{*T}(t)\end{bmatrix}\begin{bmatrix}\boldsymbol{X}^{-T}\boldsymbol{\Omega}_{ijk}^*(t,\boldsymbol{\theta})\boldsymbol{X}^{-1}+\boldsymbol{Q}^* & * \\ \boldsymbol{D}_i^{*T}\boldsymbol{X}^{-1} & -\rho^2\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}^*(t)\\ \boldsymbol{w}^*(t)\end{bmatrix} \qquad (5.45)\end{aligned}$$

From (5.43) and (5.45), the following inequality can be obtained,

$$\dot{V}(t)+\boldsymbol{x}^{*T}(t)\boldsymbol{Q}^*\boldsymbol{x}^*(t)-\rho^2\boldsymbol{w}^{*T}(t)\boldsymbol{w}^*(t)<0 \tag{5.46}$$

Integrating the above inequality from 0 to ∞, yields

$$V(\infty)-V(0)+\int_0^\infty(\boldsymbol{x}^{*T}(t)\boldsymbol{Q}^*\boldsymbol{x}^*(t)-\rho^2\boldsymbol{w}^{*T}(t)\boldsymbol{w}^*(t))dt<0 \tag{5.47}$$

With zero initial condition, $V(0) = 0$ and hence

$$\int_0^\infty \boldsymbol{x}^{*T}(t)\boldsymbol{Q}^*\boldsymbol{x}^*(t)dt \;<\; \int_0^\infty \rho^2\boldsymbol{w}^{*T}(t)\boldsymbol{w}^*(t)dt \tag{5.48}$$

$$\int_0^\infty \boldsymbol{e}^{T}(t)\boldsymbol{Q}\boldsymbol{e}(t)dt \;<\; \int_0^\infty \rho^2\boldsymbol{w}^{*T}(t)\boldsymbol{w}^*(t)dt \tag{5.49}$$

Thus the proof is complete. □

5.5.1 Special cases

In this subsection, some special cases are considered and the condition for designing the controller for these cases are presented. When $\boldsymbol{B}_1(\boldsymbol{\theta}) = \boldsymbol{B}_2(\boldsymbol{\theta}) = \cdots = \boldsymbol{B}_r(\boldsymbol{\theta})$ and $\Delta\boldsymbol{B} = \boldsymbol{0}$, the conditions for controller design can be simplified as given below in Theorem 5.4.

Theorem 5.4. Let us consider the fuzzy descriptor system (5.29) with $\boldsymbol{B}_i(\boldsymbol{\theta}) = \boldsymbol{B}(\boldsymbol{\theta})$ $(i = 1,\ldots,r)$ and $\Delta\boldsymbol{B} = \boldsymbol{0}$ and the control law given by (5.28). If there exist some matrices $\boldsymbol{X}$ as defined in (5.35) and $\boldsymbol{W}_{ik}^*$ $(i = 1,\cdots,r, k = 1,\cdots,r^e)$ such that the following matrix inequalities are satisfied

$$\boldsymbol{S}^T = \boldsymbol{S} \;>\; 0 \tag{5.50}$$

$$\boldsymbol{\phi}_{ik}^*(\boldsymbol{\nu}) \;<\; 0,\ \forall\ \boldsymbol{\nu}\in\mathcal{V},\quad i=1,2,\cdots,r,\quad k=1,2,\ldots,r^e \tag{5.51}$$

where

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}_{21}^T \\ \boldsymbol{X}_{21} & \boldsymbol{X}_{22} \end{bmatrix}$$

$$\boldsymbol{\phi}_{ik}^*(\boldsymbol{\nu}) = \left[\begin{array}{ccc:cc:cc} \boldsymbol{\mathcal{A}}_{ik}^{11}(\boldsymbol{\nu}) & * & * & * & * & * & * \\ \boldsymbol{D}_{ik}^{*T} & -\rho^2\boldsymbol{I} & * & * & * & * & * \\ \boldsymbol{\mathcal{Y}} & \boldsymbol{0} & -\boldsymbol{Q}^{-1} & * & * & * & * \\ \hdashline \boldsymbol{\mathcal{A}}_{i}^{31} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{\epsilon}_i^a & * & * & * \\ \boldsymbol{\mathcal{A}}_{ik}^{41} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{\epsilon}_i^a & * & * \\ \hdashline \boldsymbol{\mathcal{A}}_{k}^{51} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{\epsilon}_k^e & * \\ \boldsymbol{\mathcal{A}}_{k}^{61} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{\epsilon}_k^e \end{array}\right]$$

$$\begin{aligned} \boldsymbol{\mathcal{A}}_i^{31} &= \begin{bmatrix} \epsilon_{i1}^a \boldsymbol{M}_{i1}^{a*T} \\ \vdots \\ \epsilon_{iL_a}^a \boldsymbol{M}_{iL_a}^{a*\;T} \end{bmatrix}, \boldsymbol{\mathcal{A}}_{ik}^{41} = \begin{bmatrix} \boldsymbol{N}_{i11}^{a*}\boldsymbol{X} \\ \vdots \\ \boldsymbol{N}_{i1L_a}^{a*}\boldsymbol{X} \end{bmatrix}, \boldsymbol{\mathcal{A}}_k^{51} = \begin{bmatrix} \epsilon_{k1}^e \boldsymbol{M}_{k1}^{e*T} \\ \vdots \\ \epsilon_{kL_e}^e \boldsymbol{M}_{kL_e}^{e*\;T} \end{bmatrix}, \boldsymbol{\mathcal{A}}_k^{61} = \begin{bmatrix} \boldsymbol{N}_{k1}^{e*}\boldsymbol{X} \\ \vdots \\ \boldsymbol{N}_{kL_e}^{e*}\boldsymbol{X} \end{bmatrix}, \\ \boldsymbol{\mathcal{Y}} &= \boldsymbol{X}\left[\begin{array}{c:c:c} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \end{array}\right] \end{aligned}$$

$\boldsymbol{\mathcal{A}}_{ik}^{11}(\boldsymbol{\nu}) = \boldsymbol{X}^T\boldsymbol{A}_{ik}^{*T}(\boldsymbol{\nu}) + \boldsymbol{W}_{ik}^{*T}\boldsymbol{B}^{*T}(\boldsymbol{\nu}) + (*)$, $\boldsymbol{\epsilon}_i^a = \mathrm{diag}(\epsilon_{i1}^a, \cdots, \epsilon_{iL_a}^a)$, $\boldsymbol{\epsilon}_k^e = \mathrm{diag}(\epsilon_{k1}^e, \cdots, \epsilon_{kL_e}^e)$ and $\boldsymbol{W}_{jk}^* = \boldsymbol{K}_{jk}^*\boldsymbol{X}$, then the closed loop system is asymptotically stable and satisfies the given H_∞ performance criteria.

Next the special case, where $\mu_i(z(t)) = \mu_k^e(z(t))$ and $r = r^e$ is considered. In this case, the H_∞ controller design can be simplified as given in Theorem 5.5.

Theorem 5.5. Let us consider the fuzzy descriptor system (5.29) with the special case $\mu_i(z(t)) = \mu_k^e(z(t))$, $r = r^e$ and the control law given by (5.28). If there exist certain matrices $\boldsymbol{X}$ as defined in (5.35) and $\boldsymbol{W}_j^*$ $(j = 1, ..., r)$ such that the following matrix inequalities are satisfied

$$\boldsymbol{S}^T = \boldsymbol{S} \;>\; 0 \tag{5.52}$$

$$\boldsymbol{\phi}_{ii}^*(\boldsymbol{\nu}) \;<\; 0, \;\; \forall \;\; \boldsymbol{\nu} \in \mathcal{V}, \;\; i = 1, 2, \ldots, r \tag{5.53}$$

$$\frac{1}{r-1}\boldsymbol{\phi}_{ii}^*(\boldsymbol{\nu}) + \frac{1}{2}(\boldsymbol{\phi}_{ij}^*(\boldsymbol{\nu}) + \phi_{ji}^*(\boldsymbol{\nu})) \;<\; 0, \;\; \forall \;\; \boldsymbol{\nu} \in \mathcal{V}, \;\; 1 \le i \ne j \le r \tag{5.54}$$

where

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}_{21}^T \\ \boldsymbol{X}_{21} & \boldsymbol{X}_{22} \end{bmatrix}$$

$$\boldsymbol{\phi}^*_{ij}(\boldsymbol{\nu}) = \left[\begin{array}{ccc|cc|cc} \boldsymbol{\mathcal{A}}^{11}_{ij}(\boldsymbol{\nu}) & * & * & * & * & * & * \\ \boldsymbol{D}^{*T}_i & -\rho^2\boldsymbol{I} & * & * & * & * & * \\ \boldsymbol{\mathcal{Y}} & \mathbf{0} & -\boldsymbol{Q}^{-1} & * & * & * & * \\ \hline \boldsymbol{\mathcal{A}}^{31}_i & \mathbf{0} & \mathbf{0} & -\boldsymbol{\epsilon}^a_i & * & * & * \\ \boldsymbol{\mathcal{A}}^{41}_{ij} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\boldsymbol{\epsilon}^a_i & * & * \\ \hline \boldsymbol{\mathcal{A}}^{51}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\boldsymbol{\epsilon}^e_i & * \\ \boldsymbol{\mathcal{A}}^{61}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\boldsymbol{\epsilon}^e_i \end{array}\right]$$

$$\boldsymbol{\mathcal{A}}^{31}_i = \begin{bmatrix} \epsilon^a_{i1}\boldsymbol{M}^{a*T}_{i1} \\ \vdots \\ \epsilon^a_{iL_a}\boldsymbol{M}^{a*\ T}_{iL_a} \end{bmatrix}, \boldsymbol{\mathcal{A}}^{41}_{ij} = \begin{bmatrix} \boldsymbol{N}^{a*}_{i11}\boldsymbol{X} + \boldsymbol{N}^{a*}_{i21}\boldsymbol{W}_j \\ \vdots \\ \boldsymbol{N}^{a*}_{i1L_a}\boldsymbol{X} + \boldsymbol{N}^{a*}_{i2L_a}\boldsymbol{W}_j) \end{bmatrix},$$

$$\boldsymbol{\mathcal{A}}^{51}_i = \begin{bmatrix} \epsilon^e_{i1}\boldsymbol{M}^{e*T}_{i1} \\ \vdots \\ \epsilon^e_{iL_e}\boldsymbol{M}^{e*\ T}_{iL_e} \end{bmatrix}, \boldsymbol{\mathcal{A}}^{61}_i = \begin{bmatrix} \boldsymbol{N}^{e*}_{i1}\boldsymbol{X} \\ \vdots \\ \boldsymbol{N}^{e*}_{iL_e}\boldsymbol{X} \end{bmatrix}, \quad \boldsymbol{\mathcal{Y}} = \boldsymbol{X}\left[\begin{array}{c|c|c} \boldsymbol{I} & \mathbf{0} & \mathbf{0} \end{array}\right]$$

$\boldsymbol{\mathcal{A}}^{11}_{ij}(\boldsymbol{\nu}) = \boldsymbol{X}^T\boldsymbol{A}^{*T}_i(\boldsymbol{\nu}) + \boldsymbol{W}^{*T}_j\boldsymbol{B}^{*T}_i(\boldsymbol{\nu}) + (*)$, $\boldsymbol{\epsilon}^a_i = \mathrm{diag}(\epsilon^a_{i1}, \cdots, \epsilon^a_{iL_a})$, $\boldsymbol{\epsilon}^e_i = \mathrm{diag}(\epsilon^e_{i1}, \cdots, \epsilon^e_{iL_e})$ and $\boldsymbol{W}^*_j = \boldsymbol{K}^*_j\boldsymbol{X}$, then the closed loop system is asymptotically stable and satisfies the given H_∞ performance criteria.

When $\boldsymbol{B}_1(\boldsymbol{\theta}) = \boldsymbol{B}_2(\boldsymbol{\theta}) = \cdots = \boldsymbol{B}_r(\boldsymbol{\theta})$, $\Delta\boldsymbol{B} = 0$, $\mu_i(z(t)) = \mu^e_k(z(t))$ and $r = r^e$, the conditions in Theorem 5.3 can be simplified as in Theorem 5.6 given below.

Theorem 5.6. Let us consider the fuzzy descriptor system (5.29) (special case with $\boldsymbol{B}_i(\boldsymbol{\theta}) = \boldsymbol{B}(\boldsymbol{\theta})$ $(i = 1, \ldots, r)$, $\Delta\boldsymbol{B} = 0$, $\mu_i(z(t)) = \mu^e_k(z(t))$ and $r = r^e$) with the control law given by (5.28). If there exist certain matrices $\boldsymbol{X}$ as defined in (5.35) and $\boldsymbol{W}^*_i$ $(i = 1, ..., r)$ such that the following matrix inequalities are satisfied

$$\boldsymbol{S}^T = \boldsymbol{S} \quad > \quad 0 \tag{5.55}$$

$$\boldsymbol{\phi}^*_i(\boldsymbol{\nu}) \quad < \quad 0, \quad \forall \ \boldsymbol{\nu} \in \mathcal{V}, \quad i = 1, 2, \ldots, r \tag{5.56}$$

where

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}^T_{21} \\ \boldsymbol{X}_{21} & \boldsymbol{X}_{22} \end{bmatrix}$$

$$\phi_i^*(\boldsymbol{\nu}) = \left[\begin{array}{ccc|cc|cc} \mathcal{A}_i^{11}(\boldsymbol{\nu}) & * & * & * & * & * & * \\ \boldsymbol{D}_i^{*T} & -\rho^2\boldsymbol{I} & * & * & * & * & * \\ \mathcal{Y} & \mathbf{0} & -\boldsymbol{Q}^{-1} & * & * & * & * \\ \hline \mathcal{A}_i^{31} & \mathbf{0} & \mathbf{0} & -\boldsymbol{\epsilon}_i^a & * & * & * \\ \mathcal{A}_i^{41} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\boldsymbol{\epsilon}_i^a & * & * \\ \hline \mathcal{A}_i^{51} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\boldsymbol{\epsilon}_i^e & * \\ \mathcal{A}_i^{61} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\boldsymbol{\epsilon}_i^e \end{array}\right]$$

$$\begin{aligned} \mathcal{A}_i^{31} &= \begin{bmatrix} \epsilon_{i1}^a \boldsymbol{M}_{i1}^{a*T} \\ \vdots \\ \epsilon_{iL_a}^a \boldsymbol{M}_{iL_a}^{a*\ T} \end{bmatrix}, \mathcal{A}_i^{41} = \begin{bmatrix} \boldsymbol{N}_{i11}^{a*}\boldsymbol{X} \\ \vdots \\ \boldsymbol{N}_{i1L_a}^{a*}\boldsymbol{X} \end{bmatrix}, \mathcal{A}_i^{51} = \begin{bmatrix} \epsilon_{i1}^e \boldsymbol{M}_{i1}^{e*T} \\ \vdots \\ \epsilon_{iL_e}^e \boldsymbol{M}_{iL_e}^{e*\ T} \end{bmatrix}, \mathcal{A}_i^{61} = \begin{bmatrix} \boldsymbol{N}_{i1}^{e*}\boldsymbol{Y} \\ \vdots \\ \boldsymbol{N}_{iL_e}^{e*}\boldsymbol{Y} \end{bmatrix}, \\ \mathcal{Y} &= \boldsymbol{X}\left[\begin{array}{c|c|c} \boldsymbol{I} & \mathbf{0} & \mathbf{0} \end{array}\right] \end{aligned}$$

$\mathcal{A}_i^{11}(\boldsymbol{\nu}) = \boldsymbol{X}^T\boldsymbol{A}_i^{*T}(\boldsymbol{\nu}) + \boldsymbol{W}_i^{*T}\boldsymbol{B}^{*T}(\boldsymbol{\nu}) + (*)$, $\boldsymbol{\epsilon}_i^a = \mathrm{diag}(\epsilon_{i1}^a, \cdots, \epsilon_{iL_a}^a)$, $\boldsymbol{\epsilon}_i^e = \mathrm{diag}(\epsilon_{i1}^e, \cdots, \epsilon_{iL_e}^e)$ and $\boldsymbol{W}_i^* = \boldsymbol{K}_i^*\boldsymbol{X}$, then the closed loop system is asymptotically stable and satisfies the given H_∞ performance criteria.

5.6 Robust H_∞ Tracking Control with Parametric Lyapunov Function

In this section, the robust H_∞ tracking controller design for uncertain descriptor fuzzy systems is discussed.

Let us consider a parametric Lyapunov function in the following form

$$V(t,\boldsymbol{\theta}) = \boldsymbol{x}^{*T}(t)\boldsymbol{E}^{*T}\boldsymbol{X}^{-1}(\boldsymbol{\theta})\boldsymbol{x}^*(t) \tag{5.57}$$

where

$$\begin{aligned} \boldsymbol{E}^{*T}\boldsymbol{X}^{-1}(\boldsymbol{\theta}) &= \boldsymbol{X}^{-T}(\boldsymbol{\theta})\boldsymbol{E}^* \geq 0 \\ \boldsymbol{X}(\boldsymbol{\theta}) &= \boldsymbol{X}_0 + \sum_{i=1}^{L}\theta_l\boldsymbol{X}_l \end{aligned}$$

and

$$\boldsymbol{X}_0 = \begin{bmatrix} \boldsymbol{X}_{011} & \boldsymbol{X}_{021}^T & \mathbf{0} \\ \boldsymbol{X}_{021} & \boldsymbol{X}_{022} & \mathbf{0} \\ \boldsymbol{X}_{031} & \boldsymbol{X}_{032} & \boldsymbol{X}_{033} \end{bmatrix}, \boldsymbol{X}_l = \begin{bmatrix} \boldsymbol{X}_{l11} & \boldsymbol{X}_{l21}^T & \mathbf{0} \\ \boldsymbol{X}_{l21} & \boldsymbol{X}_{l22} & \mathbf{0} \\ \boldsymbol{X}_{l31} & \boldsymbol{X}_{l32} & \boldsymbol{X}_{l33} \end{bmatrix}$$

For the uncertain descriptor system represented by (5.29), the result for H_∞ tracking control problem with parametric Lyapunov function (5.57) is summarized in the following theorem, followed by the proof.

Theorem 5.7. Let us consider the uncertain fuzzy descriptor system represented by (5.29) with the PDC control law (5.28). For a given disturbance attenuation level ρ^2, if there exist certain matrices $\boldsymbol{X}_0$, $\boldsymbol{X}_l$, $(l = 1, ..., L)$ as defined in (5.57) and certain matrices $\boldsymbol{K}^*_{jk}$, $(j = 1, ..., r,\ k = 1, ..., r^e)$ such that

$$\begin{aligned}
\boldsymbol{S}^T(\boldsymbol{\nu}) = \boldsymbol{S}(\boldsymbol{\nu}) &> 0, \ \forall \ \nu \in \mathcal{V} && (5.58)\\
\boldsymbol{\Gamma}^*_{iik}(\boldsymbol{\nu},\boldsymbol{\omega}) &< 0, \ \forall \ (\nu,\omega) \in \mathcal{V}\times\mathcal{W}, \\
& \qquad i = 1, 2, \cdots, r, \quad k = 1, 2, \cdots, r^e && (5.59)\\
\frac{1}{r-1}\boldsymbol{\Gamma}^*_{iik}(\boldsymbol{\nu},\boldsymbol{\omega}) + \frac{1}{2}(\boldsymbol{\Gamma}^*_{ijk}(\boldsymbol{\nu},\boldsymbol{\omega}) + \boldsymbol{\Gamma}^*_{jik}(\boldsymbol{\nu},\boldsymbol{\omega})) &< 0, \ \forall \ (\nu,\omega) \in \mathcal{V}\times\mathcal{W}, \\
& \qquad 1 \le i \ne j \le r, \quad k = 1, 2, \cdots, r^e && (5.60)\\
\boldsymbol{\Pi}^*_{liik} &\ge 0, \ l = 1, ..., L, \\
& \qquad i = 1, 2, \ldots, r, \quad k = 1, 2, \cdots, r^e && (5.61)\\
\frac{1}{r-1}\boldsymbol{\Pi}^*_{liik} + \frac{1}{2}(\boldsymbol{\Pi}^*_{lijk} + \boldsymbol{\Pi}^*_{ljik}) &\ge 0, \ l = 1, ..., L, \\
& \qquad 1 \le i \ne j \le r, \quad k = 1, 2, \cdots, r^e && (5.62)
\end{aligned}$$

where

$$\boldsymbol{S}(\boldsymbol{\nu}) = \begin{bmatrix} \boldsymbol{X}_{11}(\boldsymbol{\nu}) & \boldsymbol{X}^T_{21}(\boldsymbol{\nu}) \\ \boldsymbol{X}_{21}(\boldsymbol{\nu}) & \boldsymbol{X}_{22}(\boldsymbol{\nu}) \end{bmatrix}$$

$$\boldsymbol{\Gamma}^*_{ijk}(\boldsymbol{\nu},\boldsymbol{\omega}) = \left[\begin{array}{ccc:cc:cc}
\boldsymbol{\Lambda}^*_{ijk}(\nu,\omega) & * & * & * & * & * & * \\
\boldsymbol{D}^{*T}_{ik} & -\rho^2\boldsymbol{I} & * & * & * & * & * \\
\boldsymbol{\mathcal{Y}}(\boldsymbol{\nu}) & \boldsymbol{0} & -\boldsymbol{Q}^{-1} & * & * & * & * \\
\hdashline
\boldsymbol{\mathcal{A}}^{31}_i & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{\epsilon}^a_i & * & * & * \\
\boldsymbol{\mathcal{A}}^{41}_{ijk}(\boldsymbol{\nu}) & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{\epsilon}^a_i & * & * \\
\hdashline
\boldsymbol{\mathcal{A}}^{51}_k & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{\epsilon}^e_k & * \\
\boldsymbol{\mathcal{A}}^{61}_k(\boldsymbol{\nu}) & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{\epsilon}^e_k
\end{array}\right],$$

$$\boldsymbol{\mathcal{A}}_i^{31} = \begin{bmatrix} \epsilon_{i1}^a \boldsymbol{M}_{i1}^{a*T} \\ \vdots \\ \epsilon_{iL_a}^a \boldsymbol{M}_{iL_a}^{a*\,T} \end{bmatrix}, \boldsymbol{\mathcal{A}}_{ijk}^{41} = \begin{bmatrix} \boldsymbol{N}_{i11}^{a*}\boldsymbol{X}(\boldsymbol{\nu}) + \boldsymbol{N}_{i21}^{a*}\boldsymbol{W}_{jk} \\ \vdots \\ \boldsymbol{N}_{i1L_a}^{a*}\boldsymbol{X}(\boldsymbol{\nu}) + \boldsymbol{N}_{i2L_a}^{a*}\boldsymbol{W}_{jk} \end{bmatrix},$$

$$\boldsymbol{\mathcal{A}}_k^{51} = \begin{bmatrix} \epsilon_{k1}^e \boldsymbol{M}_{k1}^{e*T} \\ \vdots \\ \epsilon_{kL_e}^e \boldsymbol{M}_{kL_e}^{e*\,T} \end{bmatrix}, \boldsymbol{\mathcal{A}}_k^{61} = \begin{bmatrix} \boldsymbol{N}_{k1}^{e*}\boldsymbol{X}(\boldsymbol{\nu}) \\ \vdots \\ \boldsymbol{N}_{kL_e}^{e*}\boldsymbol{X}(\boldsymbol{\nu}) \end{bmatrix}, \quad \boldsymbol{\mathcal{Y}}(\boldsymbol{\nu}) = \boldsymbol{X}(\boldsymbol{\nu}) \left[\begin{array}{c:c:c} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \end{array} \right]$$

$\boldsymbol{\Lambda}_{ijk}^*(\boldsymbol{\nu},\boldsymbol{\omega}) = \boldsymbol{X}^T(\boldsymbol{\nu})(\boldsymbol{A}_i^*(\boldsymbol{\nu})+\boldsymbol{B}_i^*(\boldsymbol{\nu})\boldsymbol{K}_{jk}^*)^T+(*)-\boldsymbol{E}^*(\boldsymbol{X}(\boldsymbol{\omega})-\boldsymbol{X}_0)$, $\boldsymbol{\Pi}_{lijk}^* = \boldsymbol{X}_l^T(\boldsymbol{A}_{il}^*+\boldsymbol{B}_{il}^*\boldsymbol{K}_{jk}^*)^T+(*)$, $\boldsymbol{\epsilon}_i^a = \mathrm{diag}(\epsilon_{i1}^a,\cdots,\epsilon_{iL_a}^a)$ and $\boldsymbol{\epsilon}_k^e = \mathrm{diag}(\epsilon_{k1}^e,\cdots,\epsilon_{kL_e}^e)$ then the H_∞ performance given by (5.5) is guaranteed for the overall fuzzy system.

Proof: Let us consider the inequalities (5.59) to (5.62) given in Theorem 5.7. By the multi-convexity concept explained in [124], if these inequalities are satisfied in the corners defined by the rate box $\mathcal{V}$ and the parameter box $\mathcal{W}$ then it will hold for all values of $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$ inside the parameter box $\mathcal{V}$ and rate box $\mathcal{W}$. Hence,

$$\boldsymbol{\Gamma}_{iik}^*(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) < 0, \quad i=1,2,\cdots,r, \quad k=1,2,\cdots,r^e \tag{5.63}$$

$$\frac{1}{r-1}\boldsymbol{\Gamma}_{iik}^*(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) + \frac{1}{2}(\boldsymbol{\Gamma}_{ijk}^*(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) + \boldsymbol{\Gamma}_{jik}^*(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) < 0, \quad 1 \le i \ne j \le r, \quad k=1,2,\cdots,r^e \tag{5.64}$$

By applying Lemma A.1 (Appendix) to the above inequality, the following parameterized inequality is obtained

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r^e}\mu_i\mu_j\mu_k^e\boldsymbol{\Gamma}_{ijk}^*(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) < 0 \tag{5.65}$$

With (5.27), applying Schur complement and using the inequality $\boldsymbol{Y}^T\boldsymbol{Z}+\boldsymbol{Z}^T\boldsymbol{Y} \le \boldsymbol{Y}^T\boldsymbol{Y}+\boldsymbol{Z}^T\boldsymbol{Z}$ given in [66], the above inequality (5.65) can be reduced to

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r^e}\mu_i\mu_j\mu_k^e \begin{bmatrix} \tilde{\boldsymbol{\Lambda}}_{ijk}^*(t,\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) + \boldsymbol{X}^T(\boldsymbol{\theta})\boldsymbol{Q}^*\boldsymbol{X}(\boldsymbol{\theta}) & * \\ \boldsymbol{D}_{ik}^{*T} & -\rho^2\boldsymbol{I} \end{bmatrix} < 0 \tag{5.66}$$

where

$$\begin{aligned} \tilde{\boldsymbol{\Lambda}}_{ijk}^*(t,\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) &= \boldsymbol{\Lambda}_{ijk}^*(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) + \Delta\boldsymbol{\Lambda}_{ijk}^*(t,\boldsymbol{\theta}) \\ \boldsymbol{\Lambda}_{ijk}^*(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) &= \boldsymbol{X}^T(\boldsymbol{\theta})(\boldsymbol{A}_i^*(\boldsymbol{\theta})+\boldsymbol{B}_i^*(\boldsymbol{\theta})\boldsymbol{K}_{jk}^*)^T + (*) - \boldsymbol{E}^*(\boldsymbol{X}(\dot{\boldsymbol{\theta}})-\boldsymbol{X}_0) \\ \Delta\boldsymbol{\Lambda}_{ijk}^*(t,\boldsymbol{\theta}) &= \boldsymbol{X}^T(\boldsymbol{\theta})(\Delta\boldsymbol{A}_i^*(t)+\Delta\boldsymbol{B}_i^*(t)\boldsymbol{K}_{jk}^*)^T + (*) \end{aligned}$$

Pre-multiplying (5.66) by diag$(\boldsymbol{X}^{-T}(\boldsymbol{\theta}), \boldsymbol{I})$ and post-multiplying by diag$(\boldsymbol{X}^{-1}(\boldsymbol{\theta}), \boldsymbol{I})$, gives

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r^e} \mu_i\mu_j\mu_k^e \begin{bmatrix} \boldsymbol{X}^{-T}(\boldsymbol{\theta})\tilde{\boldsymbol{\Lambda}}_{ijk}^{*}(t,\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\boldsymbol{X}^{-1}(\boldsymbol{\theta}) + \boldsymbol{Q}^* & * \\ \boldsymbol{D}_{ik}^{*T}\boldsymbol{X}^{-1}(\boldsymbol{\theta}) & -\rho^2\boldsymbol{I} \end{bmatrix} < 0 \tag{5.67}$$

Let us consider the parametric Lyapunov function given by (5.57),

$$V(t,\boldsymbol{\theta}) = \boldsymbol{x}^{*T}(t)\boldsymbol{E}^{*T}\boldsymbol{X}^{-1}(\boldsymbol{\theta})\boldsymbol{x}^*(t). \tag{5.68}$$

The time derivative of $V(t,\boldsymbol{\theta})$ along the trajectory of (5.29) is

$$\begin{aligned} \dot{V}(t,\boldsymbol{\theta}) &= \dot{\boldsymbol{x}}^{*T}(t)\boldsymbol{E}^{*T}\boldsymbol{X}^{-1}(\boldsymbol{\theta})\boldsymbol{x}^*(t) + \boldsymbol{x}^{*T}(t)\boldsymbol{E}^{*T}\boldsymbol{X}^{-1}(\boldsymbol{\theta})\dot{\boldsymbol{x}}^*(t) \\ &\quad + \boldsymbol{x}^{*T}(t)\boldsymbol{E}^{*T}\frac{d}{dt}\boldsymbol{X}^{-1}(\boldsymbol{\theta})\boldsymbol{x}^*(t) \end{aligned} \tag{5.69}$$

$$\boldsymbol{E}^{*T}\frac{d}{dt}\boldsymbol{X}^{-1}(\boldsymbol{\theta}) = -\boldsymbol{E}^{*T}\boldsymbol{X}^{-1}(\boldsymbol{\theta})\left(\frac{d}{dt}\boldsymbol{X}(\boldsymbol{\theta})\right)\boldsymbol{X}^{-1}(\boldsymbol{\theta}) \tag{5.70}$$

$$= -\boldsymbol{E}^{*T}\boldsymbol{X}^{-1}(\boldsymbol{\theta})\left(\boldsymbol{X}(\dot{\boldsymbol{\theta}}) - \boldsymbol{X}_0\right)\boldsymbol{X}^{-1}(\boldsymbol{\theta}) \tag{5.71}$$

$$= -\boldsymbol{X}^{-T}(\boldsymbol{\theta})\boldsymbol{E}^*\left(\boldsymbol{X}(\dot{\boldsymbol{\theta}}) - \boldsymbol{X}_0\right)\boldsymbol{X}^{-1}(\boldsymbol{\theta}) \tag{5.72}$$

$$\begin{aligned} \dot{V}(t,\boldsymbol{\theta}) &= \dot{\boldsymbol{x}}^{*T}(t)\boldsymbol{E}^{*T}\boldsymbol{X}^{-1}(\boldsymbol{\theta})\boldsymbol{x}^*(t) + \boldsymbol{x}^{*T}(t)\boldsymbol{X}^{-T}(\boldsymbol{\theta})\boldsymbol{E}^*\dot{\boldsymbol{x}}^*(t) \\ &\quad - \boldsymbol{x}^{*T}(t)\boldsymbol{X}^{-T}(\boldsymbol{\theta})\boldsymbol{E}^*\left(\boldsymbol{X}(\dot{\boldsymbol{\theta}}) - \boldsymbol{X}_0\right)\boldsymbol{X}^{-1}(\boldsymbol{\theta})\boldsymbol{x}^*(t) \end{aligned} \tag{5.73}$$

$$\begin{aligned} &= \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r^e}\mu_i\mu_j\mu_k^e\boldsymbol{x}^{*T}(t)\Big\{(\boldsymbol{A}_i^*(\boldsymbol{\theta}) + \Delta\boldsymbol{A}_i^*(t) + (\boldsymbol{B}_i^*(\boldsymbol{\theta}) + \Delta\boldsymbol{B}_i^*(t))\boldsymbol{K}_{jk}^*)^T\boldsymbol{X}^{-1}(\boldsymbol{\theta}) + (*) \\ &\quad - \boldsymbol{X}^{-T}(\boldsymbol{\theta})\boldsymbol{E}^*\left(\boldsymbol{X}(\dot{\boldsymbol{\theta}}) - \boldsymbol{X}_0\right)\boldsymbol{X}^{-1}(\boldsymbol{\theta})\Big\}\boldsymbol{x}^*(t) \end{aligned} \tag{5.74}$$

Then from $\dot{V}(t,\boldsymbol{\theta})$, it follows that

$$\begin{aligned} &\dot{V}(t,\boldsymbol{\theta}) + \boldsymbol{x}^{*T}(t)\boldsymbol{Q}^*\boldsymbol{x}^*(t) - \rho^2\boldsymbol{w}^{*T}(t)\boldsymbol{w}^*(t) \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r^e}\mu_i\mu_j\mu_k^e \begin{bmatrix} \boldsymbol{x}^*(t) \\ \boldsymbol{w}^*(t) \end{bmatrix}^T \begin{bmatrix} \boldsymbol{X}^{-T}(\boldsymbol{\theta})\tilde{\boldsymbol{\Lambda}}_{ijk}^{*}(t,\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\boldsymbol{X}^{-1}(\boldsymbol{\theta}) + \boldsymbol{Q}^* & * \\ \boldsymbol{D}_{i}^{*T}\boldsymbol{X}^{-1}(\boldsymbol{\theta}) & -\rho^2\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^*(t) \\ \boldsymbol{w}^*(t) \end{bmatrix} \end{aligned} \tag{5.75}$$

From (5.67) and (5.75), the following inequality condition can be obtained

$$\dot{V}(t,\boldsymbol{\theta}) + \boldsymbol{x}^{*T}(t)\boldsymbol{Q}^*\boldsymbol{x}^*(t) - \rho^2\boldsymbol{w}^{*T}(t)\boldsymbol{w}^*(t) < 0 \tag{5.76}$$

With zero initial condition, $V(0) = 0$. Integrating the above inequality from 0 to ∞, produces the inequality condition $\int_0^\infty \boldsymbol{e}^T(t)\boldsymbol{Q}\boldsymbol{e}(t) < \int_0^\infty \rho^2\boldsymbol{w}^{*T}(t)\boldsymbol{w}^*(t)$. Thus the proof is complete. □

5.6.1 Special case

In this subsection, a special case, where $\boldsymbol{B}_i(\boldsymbol{\theta}) = \boldsymbol{B}(\boldsymbol{\theta})$ $(i = 1, \ldots, r)$, $\Delta\boldsymbol{B} = \boldsymbol{0}$, $\mu_i(z(t)) = \mu_i^e(z(t))$ and $r = r^e$ are considered. The LMI conditions for designing the controller are given by the following theorem.

Theorem 5.8. Let us consider the fuzzy descriptor system (5.26) (special case with $\boldsymbol{B}_i(\boldsymbol{\theta}) = \boldsymbol{B}(\boldsymbol{\theta})$ $(i = 1, \ldots, r)$, $\Delta\boldsymbol{B} = \boldsymbol{0}$, $\mu_i(z(t)) = \mu_i^e(z(t))$ and $r = r^e$) with the control law given by (5.28). If there exist certain matrices $\boldsymbol{X}_0$, $\boldsymbol{X}_l$ $(l = 1, \cdots, L)$ as defined in (5.57) and $\boldsymbol{K}_{ik}$ $(i = 1, ..., r,\ k = 1, \ldots, r^e)$ such that the following matrix inequalities are satisfied,

$$\boldsymbol{S}^T(\boldsymbol{\nu}) = \boldsymbol{S}(\boldsymbol{\nu}) \;>\; 0, \;\; \forall\;\; \nu \in \mathcal{V} \tag{5.77}$$

$$\boldsymbol{\Gamma}_i^*(\boldsymbol{\nu}, \boldsymbol{\omega}) \;<\; 0, \;\; \forall\;\; (\nu, \omega) \in \mathcal{V} \times \mathcal{W},\ i = 1, 2, \cdots, r \tag{5.78}$$

$$\boldsymbol{\Pi}_{li}^* \;\geq\; 0, \;\; l = 1, ..., L,\ i = 1, 2, \ldots, r \tag{5.79}$$

where

$$\boldsymbol{S}(\boldsymbol{\nu}) = \begin{bmatrix} \boldsymbol{X}_{11}(\boldsymbol{\nu}) & \boldsymbol{X}_{21}^T(\boldsymbol{\nu}) \\ \boldsymbol{X}_{21}(\boldsymbol{\nu}) & \boldsymbol{X}_{22}(\boldsymbol{\nu}) \end{bmatrix}$$

$$\boldsymbol{\Gamma}_i^*(\boldsymbol{\nu}, \boldsymbol{\omega}) = \left[\begin{array}{ccc:cc:cc} \boldsymbol{\Lambda}_i^*(\nu, \omega) & * & * & * & * & * & * \\ \boldsymbol{D}_i^{*T} & -\rho^2\boldsymbol{I} & * & * & * & * & * \\ \boldsymbol{\mathcal{Y}}(\boldsymbol{\nu}) & \boldsymbol{0} & -\boldsymbol{Q}^{-1} & * & * & * & * \\ \hdashline \boldsymbol{\mathcal{A}}_i^{31} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{\epsilon}_i^a & * & * & * \\ \boldsymbol{\mathcal{A}}_i^{41}(\boldsymbol{\nu}) & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{\epsilon}_i^a & * & * \\ \hdashline \boldsymbol{\mathcal{A}}_i^{51} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{\epsilon}_i^e & * \\ \boldsymbol{\mathcal{A}}_i^{61}(\boldsymbol{\nu}) & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{\epsilon}_i^e \end{array}\right],$$

$$\boldsymbol{\mathcal{A}}_i^{31} = \begin{bmatrix} \epsilon_{i1}^a \boldsymbol{M}_{i1}^{a*T} \\ \vdots \\ \epsilon_{iL_a}^a \boldsymbol{M}_{iL_a}^{a*\,T} \end{bmatrix}, \; \boldsymbol{\mathcal{A}}_i^{41} = \begin{bmatrix} \boldsymbol{N}_{i11}^{a*}\boldsymbol{X}(\boldsymbol{\nu}) \\ \vdots \\ \boldsymbol{N}_{i1L_a}^{a*}\boldsymbol{X}(\boldsymbol{\nu}) \end{bmatrix},$$

$$\boldsymbol{\mathcal{A}}_i^{51} = \begin{bmatrix} \epsilon_{i1}^e \boldsymbol{M}_{i1}^{e*T} \\ \vdots \\ \epsilon_{iL_e}^e \boldsymbol{M}_{iL_e}^{e*\,T} \end{bmatrix}, \; \boldsymbol{\mathcal{A}}_i^{61} = \begin{bmatrix} \boldsymbol{N}_{i1}^{e*}\boldsymbol{X}(\boldsymbol{\nu}) \\ \vdots \\ \boldsymbol{N}_{iL_e}^{e*}\boldsymbol{X}(\boldsymbol{\nu}) \end{bmatrix}, \; \boldsymbol{\mathcal{Y}}(\boldsymbol{\nu}) = \boldsymbol{X}(\boldsymbol{\nu}) \left[\begin{array}{c:c:c} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \end{array}\right]$$

$\boldsymbol{\epsilon}_i^a = \mathrm{diag}(\epsilon_{i1}^a, \cdots, \epsilon_{iL_a}^a)$, $\boldsymbol{\epsilon}_i^e = \mathrm{diag}(\epsilon_{i1}^e, \cdots, \epsilon_{iL_e}^e)$, $\boldsymbol{\Pi}_{li}^* = \boldsymbol{X}_l^T(\boldsymbol{A}_{il}^* + \boldsymbol{B}_l^*\boldsymbol{K}_i^*)^T + (*)$ and $\boldsymbol{\Lambda}_i^*(\boldsymbol{\nu}, \boldsymbol{\omega}) = \boldsymbol{X}^T(\boldsymbol{\nu})(\boldsymbol{A}_i^*(\boldsymbol{\nu}) + \boldsymbol{B}^*(\boldsymbol{\nu})\boldsymbol{K}_{jk}^*)^T + (*) - \boldsymbol{E}^*(\boldsymbol{X}(\boldsymbol{\omega}) - \boldsymbol{X}_0)$, then the H_∞ performance given by (5.5) is guaranteed for the overall fuzzy system (5.26).

In Theorem 5.8, the controller design for $\boldsymbol{B}_i(\boldsymbol{\theta}) = \boldsymbol{B}(\boldsymbol{\theta})$, $\Delta\boldsymbol{B} = \boldsymbol{0}$, $\mu_i(z(t)) = \mu_i^e(z(t))$ $(i = 1, 2, ..., r)$ and $r = r^e$ is presented. The conditions for designing the controller for other special cases can be derived in a similar way. □

The inequalities in Theorems 5.7 and 5.8 and are not standard LMIs with respect to $\boldsymbol{X}_0$, $\boldsymbol{X}_l$, $\boldsymbol{K}_{jk}$ and it is difficult to solve them simultaneously. It is easily observed that, if the matrices $\boldsymbol{K}_{jk}$ are fixed then the inequalities in Theorems 5.7 and 5.8 will become standard LMIs and these can be easily solved. For finding the value of $\boldsymbol{K}_{jk}$, the constant Lyapunov function based approach discussed in Section 5.5 can be used. But the results obtained using constant Lyapunov function based approach may be conservative than the parametric Lyapunov function based approach. Hence instead of finding the value of $\boldsymbol{K}_{jk}$ with entire range of uncertainty, a fraction of the uncertainty can be considered for designing $\boldsymbol{K}_{jk}$ for the fixed Lyapunov function based approach. If this value of $\boldsymbol{K}_{jk}$ is feasible in the entire range of uncertainty with the parametric Lyapunov function based approach then the controller will satisfy the given H_∞ performance. Based on this, the following iterative LMI (ILMI) based algorithm is proposed for solving the inequalities in Theorems 5.7 and 5.8.

Algorithm 5.1:

Step 1: Derive the fuzzy model for the given uncertain nonlinear system.

Step 2: Set counter, c = 1. Choose ϵ a small positive fraction say 0.05.

Step 3: By fixed Lyapunov function based approach solve for feedback gain matrices $\boldsymbol{K}_{jk}$. If the LMIs are infeasible, goto Step 5.

Step 4: Substitute of value of $\boldsymbol{K}_{jk}$ in the inequalities given by Theorems 5.7 or 5.8 and solve for $\boldsymbol{Y}_0$, $\boldsymbol{Y}_l$. If the inequalities are infeasible, goto Step 6.

Step 5: If $(1 - c\epsilon) > 0$, reduce the bounds of the uncertainty range $\overline{\theta}_l$ and $\underline{\theta}_l$ by $(1 - c\epsilon)\overline{\theta}_l$ and $(1 - c\epsilon)\underline{\theta}_l$ for $l = 1, ..., L$ and derive the fuzzy model with the new range of uncertainty. Increment the counter c and goto Step 3.

Step 6: Choose the value of $\boldsymbol{K}_{jk}$ with the latest feasible solution obtained during the previous iterations in Step 4 and Stop. If the conditions in Step 4 are infeasible during all the iterations, then controller cannot be designed with the proposed method.

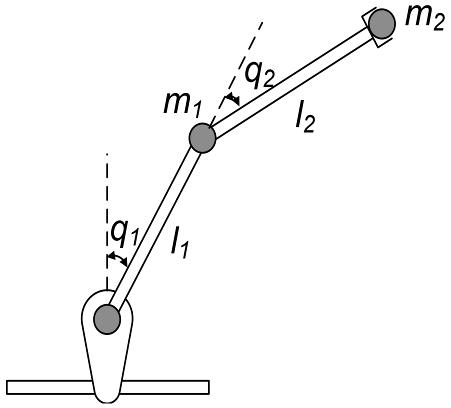


Fig. 5.2: Configuration of a two-link robotic manipulator

Table. 5.1: Premise variables for the fuzzy rules – Two link robotic manipulator

Rule i	N_{i1}	N_{i2}
1	Negative	Negative
2	Negative	Zero
3	Negative	Positive
4	Zero	Negative
5	Zero	Zero
6	Zero	Positive
7	Positive	Negative
8	Positive	Zero
9	Positive	Positive

5.7 Simulation Results

5.7.1 Two Link Robotic Manipulator

Let us consider the two-link robotic manipulator (Fig. 5.2) explained in Chapter 3. The dynamic equation of the two-link robotic manipulator [92] is expresses as,

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{\tau} \tag{5.80}$$

The nominal parameters of the system are the link masses $m_1 = m_2 = 1$kg, link lengths $l_1 = l_2 = 1$m and the gravitational acceleration $g_r = 9.81$m/s^2. In this example, structural uncertainties in masses are considered and the perturbation is assumed to be within $\pm5\%$ from their nominal value. The operating domain is considered as $x_1(t) \in [-\pi/3 \ \ \pi/3]$, $x_3(t) \in [-\pi/3 \ \ \pi/3]$, $x_2(t) \in [-5 \ \ 5]$, $x_4(t) \in [-5 \ \ 5]$ and the input $u_1(t) \in [-25 \ \ 25]$ and $u_2(t) \in [-15 \ \ 15]$.

Equidistant triangular membership functions with centers $-\pi/3$, 0 and $\pi/3$ are assumed for $x_1(t)$ and $x_3(t)$. With the uncertainties in mass m_1 and m_2, the uncertainties in the fuzzy model can be derived as $\theta_1(t) \in [-0.05 \ \ 0.05]$ and $\theta_2(t) \in [-0.05 \ \ 0.05]$. The fuzzy rules are considered to be in

Table. 5.2: Parameters of matrices $\boldsymbol{A}_i(\boldsymbol{\theta})$ and $\boldsymbol{B}_i(\boldsymbol{\theta})$ – Two link manipulator

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
a_{i021}	13.85	14.52	14.50	19.28	18.53	19.28	14.50	14.52	13.85
a_{i023}	2.41	1.155	-0.085	2.01	2.488	2.01	-0.085	1.155	2.41
a_{i041}	-2.34	-1.88	-1.259	-1.91	-1.78	-1.91	-1.259	-1.88	-2.34
a_{i043}	10.49	12.08	8.531	10.10	12.19	10.10	8.531	12.08	10.49
a_{i121}	9.244	7.775	7.25	10.29	10.98	10.29	7.25	7.775	9.244
a_{i123}	-0.975	-0.963	-0.881	0.196	0.385	0.196	-0.881	-0.963	-0.975
a_{i221}	19.99	4.645	2.719	13.68	10.95	13.68	2.719	4.645	19.99
a_{i223}	-11.72	-8.13	-3.93	3.913	6.144	3.913	-3.93	-8.13	-11.72
a_{i241}	17.37	-6.08	-0.069	5.564	-5.65	5.564	-0.069	-6.08	17.37
a_{i243}	-8.967	3.193	9.147	13.92	14.44	13.92	9.147	3.193	-8.967

the following form.

Plant rule i:

IF x_1 is N_{i1} and x_3 is N_{i2} *THEN*

$$\begin{aligned} (\boldsymbol{E}_i(\boldsymbol{\theta}) + \Delta\boldsymbol{E}_i(t))\dot{\boldsymbol{x}}(t) &= (\boldsymbol{A}_i(\boldsymbol{\theta}) + \Delta\boldsymbol{A}_i(t))\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{D}_i\boldsymbol{w}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}_i\boldsymbol{x}(t), \qquad i = 1, \cdots, 9 \end{aligned}$$

where

$$\boldsymbol{A}_{i0} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_{21i0} & 0 & a_{23i0} & 0 \\ 0 & 0 & 0 & 1 \\ a_{41i0} & 0 & a_{43i0} & 0 \end{bmatrix}, \quad \boldsymbol{A}_{il} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_{21il} & 0 & a_{23il} & 0 \\ 0 & 0 & 0 & 0 \\ a_{41il} & 0 & a_{43il} & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{C}_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{E}_{i0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & e_{24i0} \\ 0 & 0 & 1 & 0 \\ 0 & e_{42i0} & 0 & 1 \end{bmatrix}, \Delta\boldsymbol{A}_i(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \Delta a_{21i}(t) & \Delta a_{22i}(t) & \Delta a_{23i}(t) & \Delta a_{24i}(t) \\ 0 & 0 & 0 & 0 \\ \Delta a_{41i}(t) & \Delta a_{42i}(t) & \Delta a_{43i}(t) & \Delta a_{44i}(t) \end{bmatrix}, \boldsymbol{D}_i = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\boldsymbol{E}_{i1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{E}_{i2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & e_{i24} \\ 0 & 0 & 0 & 0 \\ 0 & e_{i42} & 0 & 1 \end{bmatrix}, \Delta\boldsymbol{E}_i(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta e_{24i}(t) \\ 0 & 0 & 0 & 0 \\ 0 & \Delta e_{42i}(t) & 0 & 0 \end{bmatrix}.$$

The fuzzy sets N_{i1} and N_{i2} for rule $i = 1, \ldots, 9$ are shown in Table 5.1. The parameters of the fuzzy model are obtained by the linear programming method discussed in Chapter 2. The parameters of $\boldsymbol{A}_i(\boldsymbol{\theta})$ and $\boldsymbol{B}_i(\boldsymbol{\theta})$ are shown in Table 5.2. The parameters of $\Delta\boldsymbol{A}_i(t)$ are shown in Table 5.3. For $\boldsymbol{E}_i(\boldsymbol{\theta})$ and $\Delta\boldsymbol{E}_i(t)$, the parameters are shown in Table 5.4.

Table. 5.3: Parameters of matrices $\delta \boldsymbol{A}_i$ – Two link manipulator

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
δa_{i21}	0.835	1.489	1.152	1.331	1.588	1.331	1.152	1.489	0.835
δa_{i22}	0.067	0	0	0	0	0	0	0	0.067
δa_{i23}	0.209	0.894	0.66	1.489	1.678	1.489	0.66	0.894	0.209
δa_{i24}	0.057	0	0	0	0	0	0	0	0.057
δa_{i41}	0.463	1.502	0.925	1.13	1.464	1.13	0.925	1.502	0.463
δa_{i42}	0.05	0	0	0	0.66	0	0	0	0.05
δa_{i43}	0.68	1.283	0.94	1.508	1.516	1.508	0.94	1.283	0.68
δa_{i44}	0.068	0	0	0	0	0	0	0	0.068

Table. 5.4: Parameters of matrices $\boldsymbol{E}_i(\boldsymbol{\theta})$, $\Delta \boldsymbol{E}_i$ – Two link manipulator

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
e_{240i}, e_{i420i}	1.061	0.641	-0.574	0.641	1.206	0.641	-0.574	0.641	1.061
e_{i224}, e_{i242}	1.024	0.653	0.037	0.653	1.149	0.653	0.037	0.653	1.024
$\Delta e_{i24}, \Delta e_{i42}$	0.256	0.386	0.386	0.386	0.386	0.386	0.386	0.386	0.256

Let us consider the following reference model:

$$\dot{\boldsymbol{x}}_r(t) = \boldsymbol{A}_r \boldsymbol{x}_r(t) + \boldsymbol{r}(t) \tag{5.81}$$

where

$$\boldsymbol{A}_r = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -6 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & -5 \end{bmatrix}$$

and

$$\boldsymbol{r}(t) = [0,\ 7\sin(t),\ 0,\ 7\cos(t)]^T$$

The H_∞ tracking controller design problem is considered with the above reference model given by (5.81). In this example, the descriptor fuzzy model satisfies the condition $\mu_i = \mu_i^e$ and $r = r^e$. Let us assume the mass of the links as $m_1 + \Delta m_1 = 1 + 0.05\sin(2t)$ and $m_2 + \Delta m_2 = 1 + 0.05\cos(2t)$. Here $0.05\sin(2t)$ and $0.05\cos(2t)$ represent the uncertainties. The external disturbances (e.g., cogging torque in the actuator) are assumed to be $w_1(t) = 0.4\cos(10t)\cos(2t) + 0.2\exp(-t)\sin(4t)$ and $w_2(t) = 0.3\sin(5t) + 0.25\exp(-2t)$.

With $\boldsymbol{Q} = 0.01\boldsymbol{I}$, the H_∞ tracking controller is designed for different values of ρ^2 using the Algorithm 5.1. With zero initial condition, the simulation results are shown in Figs. 5.3 – 5.5 for $\rho^2 = 0.001$ and $\rho^2 = 0.01$. In Fig. 5.3, the trajectories of the state variables $\boldsymbol{x}(t)$ and the reference trajectories $\boldsymbol{x}_r(t)$ for $\rho^2 = 0.001$ and $\rho^2 = 0.01$ are shown. The tracking error plots for these two values of ρ^2 are shown in Fig. 5.4. The corresponding control inputs $\boldsymbol{u}(t)$ are plotted in Fig. 5.5.

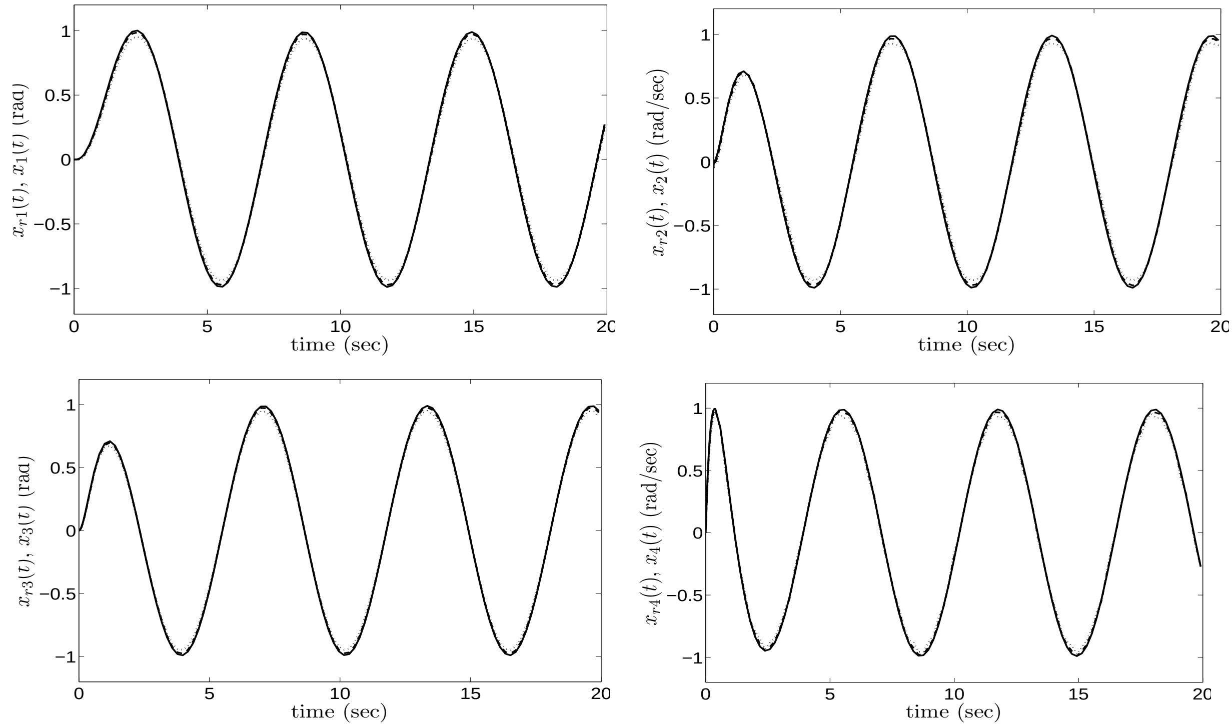


Fig. 5.3: Trajectories of state variables $\boldsymbol{x}(t)$ (dashed line and dotted line for $\rho^2 = 0.001$ and $\rho^2 = 0.01$ respectively) and the reference trajectories $\boldsymbol{x}_r(t)$ (solid line).

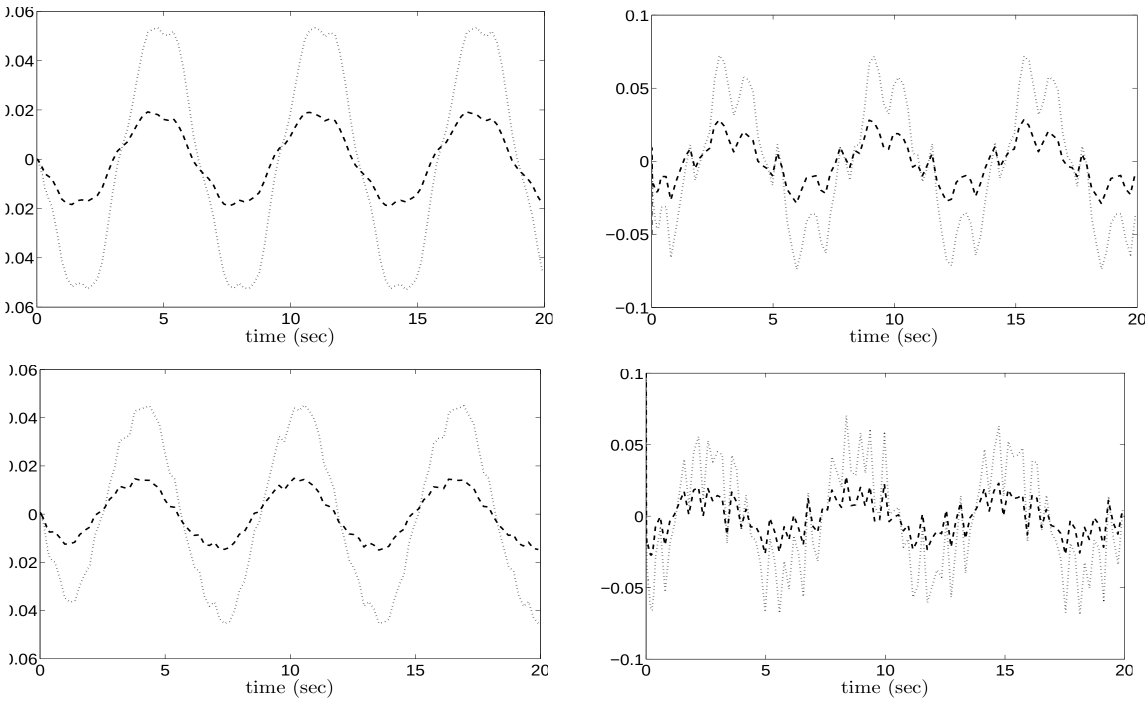


Fig. 5.4: Tracking error $\boldsymbol{e}(t)$ (dashed line and dotted line for $\rho^2 = 0.001$, and $\rho^2 = 0.01$ respectively).

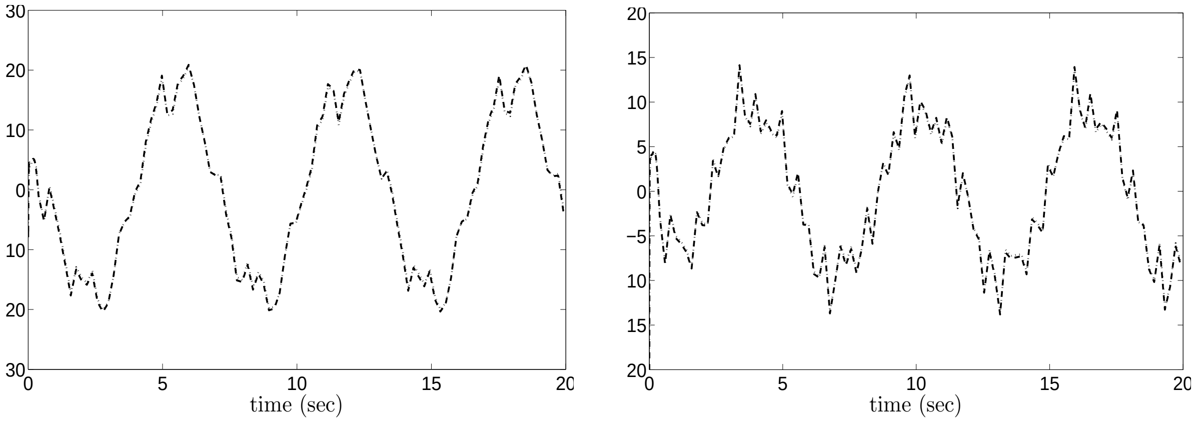


Fig. 5.5: Control input $\boldsymbol{u}(t)$ (dashed line and dotted line for $\rho^2 = 0.001$ and $\rho^2 = 0.01$ respectively).

Let us consider one of the above case with $\rho^2 = 0.001$. The parameters of the parametric Lyapunov function and the feedback gain matrices are given below.

$$\boldsymbol{X}_{110} = \begin{bmatrix} 31.58 & * & * & * \\ -89.05 & 3013 & * & * \\ 0.1910 & -0.8244 & 31.28 & * \\ -0.3849 & -1060 & -88.50 & 3770 \end{bmatrix} \quad \boldsymbol{X}_{210} = \begin{bmatrix} -19.48 & 69.32 & -1.425 & -5.683 \\ 26.69 & -202.6 & 3.350 & 48.24 \\ -0.6686 & 0.0216 & -16.96 & 64.95 \\ -1.635 & -0.4088 & 19.38 & -177.9 \end{bmatrix}$$

$$\boldsymbol{X}_{220} = \begin{bmatrix} 5555 & * & * & * \\ -10467 & 40977 & * & * \\ 6.500 & -11.89 & 5692 & * \\ -11.30 & 43.15 & -11272 & 42848 \end{bmatrix} \quad \boldsymbol{X}_{310} = \begin{bmatrix} -79387 & 2923 & -0.7348 & -1078 \\ -136.5 & -208310 & 181.7 & 53738 \\ -0.2953 & -1082 & -79386 & 3706 \\ 151.0 & 64491 & -276.9 & -259494 \end{bmatrix}$$

$$\boldsymbol{X}_{320} = \begin{bmatrix} 43.21 & -163.7 & 0.4827 & -2.134 \\ -1494 & 9731 & 105.8 & 3391 \\ -6.524 & 52.24 & 43.47 & -157.1 \\ 552.5 & -974.7 & -1722 & 6579 \end{bmatrix} \quad \boldsymbol{X}_{330} = \begin{bmatrix} 79342 & * & * & * \\ -11.68 & 342555 & * & * \\ 0.0299 & -3.466 & 79342 & * \\ -4.752 & -185278 & -8.180 & 489497 \end{bmatrix}$$

$$\boldsymbol{X}_{111} = \begin{bmatrix} 0.1535 & * & * & * \\ 0.1777 & -198.4 & * & * \\ -0.0489 & -0.6233 & 0.062 & * \\ 0.4618 & 373.1 & 0.043 & -659.9 \end{bmatrix} \quad \boldsymbol{X}_{211} = \begin{bmatrix} 13.83 & 70.81 & 26.19 & 137.1 \\ -32.0 & -76.24 & -70.5 & -217.7 \\ -24.5 & -128.5 & 13.21 & 66.28 \\ 64.19 & 208.4 & -32.87 & -96.30 \end{bmatrix}$$

$$\boldsymbol{X}_{221} = \begin{bmatrix} -7.720 & * & * & * \\ 14.38 & -66.65 & * & * \\ 15.80 & -18.46 & -35.88 & * \\ -91.38 & 73.75 & 142.1 & -531.1 \end{bmatrix} \quad \boldsymbol{X}_{311} = \begin{bmatrix} 0.04473 & -171.0 & -0.5668 & 315.3 \\ 86.84 & 7927 & -160.8 & -25798 \\ 0.5195 & 318.0 & 0.0211 & -565.7 \\ -175.1 & 13624 & 333.2 & 7505 \end{bmatrix}$$

$$\boldsymbol{X}_{321} = \begin{bmatrix} 52.50 & -78.52 & -97.82 & 196.0 \\ 2293 & -4402 & 14046 & -27490 \\ 101.8 & -203.2 & 52.03 & -99.04 \\ 670.6 & 1504 & -11478 & 19024 \end{bmatrix} \quad \boldsymbol{X}_{331} = \begin{bmatrix} -0.042 & * & * & * \\ -4.465 & -49635 & * & * \\ 0.0159 & 5.389 & -0.0027 & * \\ 5.052 & 59452 & -4.524 & -96536 \end{bmatrix}$$

$$\boldsymbol{X}_{112} = \begin{bmatrix} 4.296 & * & * & * \\ 5.364 & 6974 & * & * \\ -3.150 & -11.38 & 2.616 & * \\ 4.677 & -8983 & 1.840 & 13072 \end{bmatrix} \quad \boldsymbol{X}_{212} = \begin{bmatrix} 18.13 & 137.8 & -21.64 & -167.7 \\ -86.53 & -508.9 & 106.0 & 648.3 \\ -28.44 & -181.2 & 59.78 & 326.7 \\ 109.0 & 561.9 & -180.5 & -711.6 \end{bmatrix}$$

$$\boldsymbol{X}_{222} = \begin{bmatrix} 4.499 & * & * & * \\ -80.16 & 682.1 & * & * \\ 0.07371 & 120.9 & -14.32 & * \\ 19.9 & -563.2 & -78.70 & 489.0 \end{bmatrix} \quad \boldsymbol{X}_{312} = \begin{bmatrix} 0.8063 & 6613 & -8.033 & -8354 \\ -5539 & -47038 & 7172 & 227228 \\ 8.033 & -8581 & -0.8477 & 12217 \\ 6612 & -53504 & -9766 & -123222 \end{bmatrix}$$

$$\boldsymbol{X}_{322} = \begin{bmatrix} 113.7 & -425.2 & -144.0 & 461.5 \\ -13484 & 52214 & 15979 & -53224 \\ -139.3 & 536.6 & 253.7 & -624.6 \\ 16363 & -50661 & -30390 & 102718 \end{bmatrix} \quad \boldsymbol{X}_{332} = \begin{bmatrix} -1.52 & * & * & * \\ -0.21 & 1124456 & * & * \\ 1.118 & -9.794 & -0.895 & * \\ -7.92 & -1374376 & 20.915 & 1872734 \end{bmatrix}$$

$$\boldsymbol{K}_{11} = \begin{bmatrix} -811.3 & -282.5 & -256.1 & -89.72 \\ -235.4 & -84.05 & -628.8 & -219.2 \end{bmatrix} \quad \boldsymbol{K}_{21} = \begin{bmatrix} -25.65 & -9.785 & -8.670 & -5.192 \\ -3.919 & -5.192 & -16.38 & -4.891 \end{bmatrix}$$

$$\boldsymbol{K}_{12} = \begin{bmatrix} -814.2 & -283.3 & -126.3 & -44.20 \\ -107.3 & -38.53 & -628.2 & -218.4 \end{bmatrix} \quad \boldsymbol{K}_{22} = \begin{bmatrix} -26.31 & -9.782 & -4.937 & -3.137 \\ -1.900 & -3.137 & -17.98 & -4.888 \end{bmatrix}$$

$$\boldsymbol{K}_{13} = \begin{bmatrix} -820.7 & -285.6 & 247.0 & 87.48 \\ 264.1 & 93.15 & -618.1 & -216.1 \end{bmatrix} \quad \boldsymbol{K}_{23} = \begin{bmatrix} -26.30 & -9.780 & 3.472 & 2.808 \\ 4.646 & 2.809 & -14.42 & -4.886 \end{bmatrix}$$

$$\boldsymbol{K}_{14} = \begin{bmatrix} -819.0 & -283.3 & -127.1 & -44.20 \\ -107.2 & -38.53 & -626.2 & -218.4 \end{bmatrix} \quad \boldsymbol{K}_{24} = \begin{bmatrix} -31.08 & -9.782 & -5.793 & -3.137 \\ -1.876 & -3.137 & -15.99 & -4.888 \end{bmatrix}$$

$$\boldsymbol{K}_{15} = \begin{bmatrix} -815.2 & -282.2 & -300.4 & -105.4 \\ -280.2 & -99.71 & -631.3 & -219.5 \end{bmatrix} \quad \boldsymbol{K}_{25} = \begin{bmatrix} -30.33 & -9.787 & -9.603 & -5.900 \\ -5.336 & -5.900 & -18.09 & -4.892 \end{bmatrix}$$

$$\boldsymbol{K}_{16} = \begin{bmatrix} -819.0 & -283.3 & -127.1 & -44.20 \\ -107.3 & -38.53 & -626.2 & -218.4 \end{bmatrix} \quad \boldsymbol{K}_{26} = \begin{bmatrix} -31.08 & -9.782 & -5.793 & -3.137 \\ -1.876 & -3.137 & -15.99 & -4.888 \end{bmatrix}$$

$$\boldsymbol{K}_{17} = \begin{bmatrix} -820.7 & -285.6 & 247.0 & 87.48 \\ 264.1 & 93.15 & -618.1 & -216.1 \end{bmatrix} \quad \boldsymbol{K}_{27} = \begin{bmatrix} -26.30 & -9.800 & 3.472 & 2.808 \\ 4.646 & 2.809 & -14.42 & -4.886 \end{bmatrix}$$

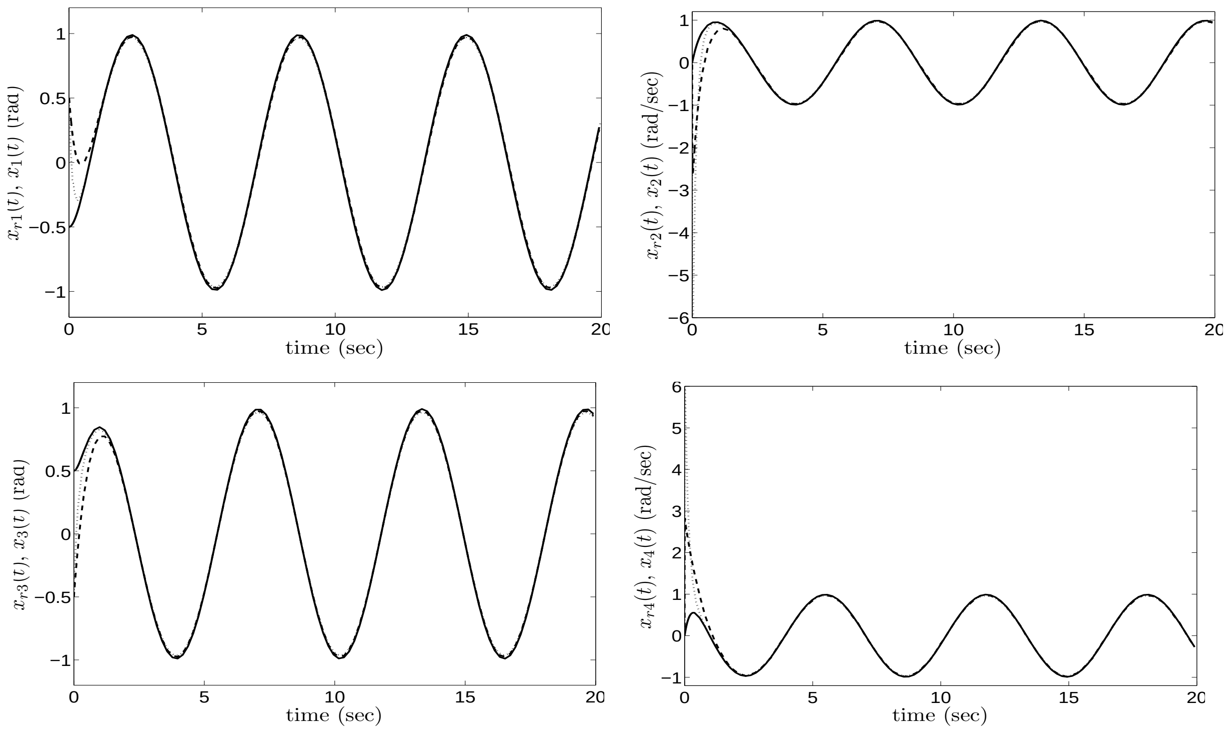


Fig. 5.6: Trajectories of state variables $\boldsymbol{x}(t)$ (dashed line and dotted line for parametric and fixed Lyapunov function based approach respectively) and the reference trajectories $\boldsymbol{x}_r(t)$ (solid line) for $\rho^2 = 0.001$.

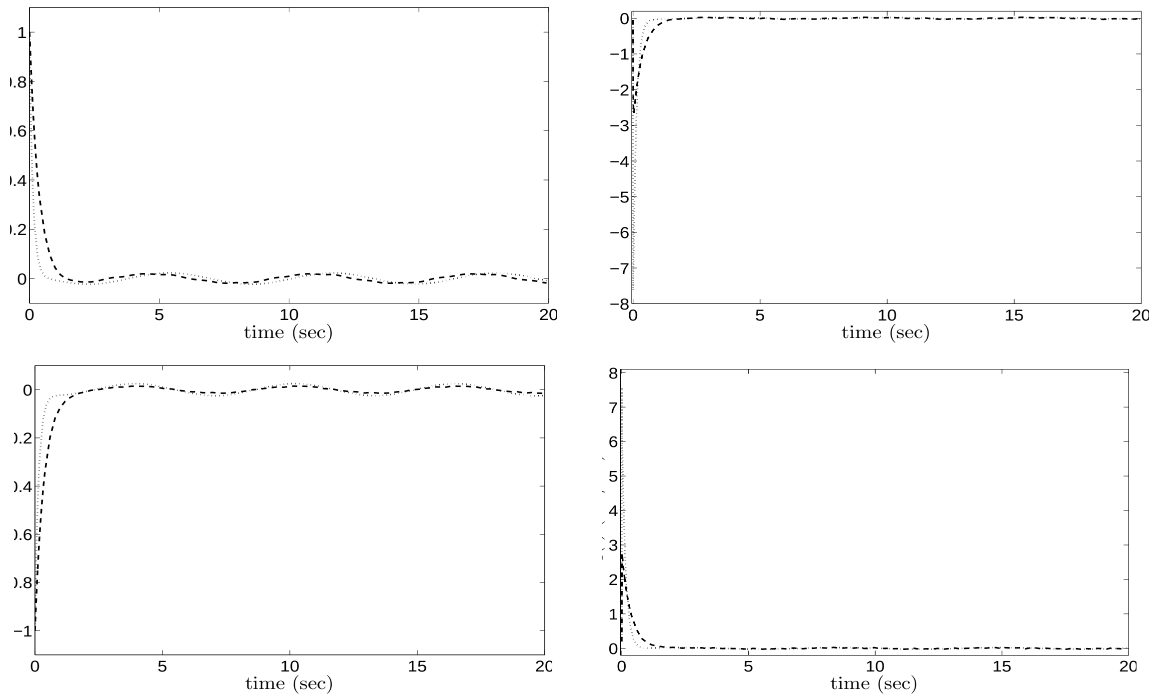


Fig. 5.7: Tracking error $\boldsymbol{e}(t)$ (dashed line and dotted line for parametric and fixed Lyapunov function based approach respectively) for $\rho^2 = 0.001$.

$$\boldsymbol{K}_{18} = \begin{bmatrix} -814.2 & -283.3 & -126.3 & -44.20 \\ -107.3 & -38.53 & -628.2 & -218.4 \end{bmatrix} \quad \boldsymbol{K}_{28} = \begin{bmatrix} -26.31 & -9.782 & -4.937 & -3.137 \\ -1.900 & -3.137 & -17.98 & -4.888 \end{bmatrix}$$

$$\boldsymbol{K}_{19} = \begin{bmatrix} -811.3 & -282.5 & -256.1 & -89.72 \\ -235.4 & -84.05 & -628.8 & -219.2 \end{bmatrix} \quad \boldsymbol{K}_{29} = \begin{bmatrix} -25.65 & -9.785 & -8.700 & -5.192 \\ -3.919 & -5.192 & -16.38 & -4.890 \end{bmatrix}$$

The initial condition $\boldsymbol{x}(0) = [0.5 \quad 0 \quad -0.5 \quad 0]^T$ and $\boldsymbol{x}_r(0) = [-0.5 \quad 0 \quad 0.5 \quad 0]^T$ are chosen for simulation.

The same problem is solved by using fixed Lyapunov function based approach and the following controller parameters are obtained:

$$\boldsymbol{K}_{11} = \begin{bmatrix} -41645 & -5611 & 5096 & 689.2 \\ 5168 & 696.5 & -42911 & -5797 \end{bmatrix} \quad \boldsymbol{K}_{21} = \begin{bmatrix} -166.5 & -34.51 & 9.151 & -1.881 \\ 13.26 & -2.127 & -169.1 & -33.34 \end{bmatrix}$$

$$\boldsymbol{K}_{12} = \begin{bmatrix} -40740 & -5488 & 8444 & 1141 \\ 8371 & 1128 & -41958 & -5668 \end{bmatrix} \quad \boldsymbol{K}_{22} = \begin{bmatrix} -163.7 & -33.47 & 24.68 & 2.191 \\ 25.93 & 1.532 & -166.7 & -32.17 \end{bmatrix}$$

$$\boldsymbol{K}_{13} = \begin{bmatrix} -40395 & -5442 & 19120 & 2584 \\ 18959 & 2555 & -41742 & -5640 \end{bmatrix} \quad \boldsymbol{K}_{23} = \begin{bmatrix} -162.2 & -33.06 & 70.99 & 15.17 \\ 68.16 & 13.62 & -162.2 & -31.90 \end{bmatrix}$$

$$\boldsymbol{K}_{14} = \begin{bmatrix} -40770 & -5492 & 8431 & 1139 \\ 8386 & 1130 & -41926 & -5664 \end{bmatrix} \quad \boldsymbol{K}_{24} = \begin{bmatrix} -168.5 & -33.50 & 23.84 & 2.175 \\ 25.99 & 1.549 & -164.6 & -32.14 \end{bmatrix}$$

$$\boldsymbol{K}_{15} = \begin{bmatrix} -41141 & -5542 & 3755 & 508 \\ 3542 & 477.3 & -43014 & -5811 \end{bmatrix} \quad \boldsymbol{K}_{25} = \begin{bmatrix} -169.7 & -33.93 & 3.713 & -3.511 \\ 6.451 & -3.984 & -171.6 & -33.46 \end{bmatrix}$$

$$\boldsymbol{K}_{16} = \begin{bmatrix} -40770 & -5492 & 8431 & 1139 \\ 8386 & 1130 & -41926 & -5664 \end{bmatrix} \quad \boldsymbol{K}_{26} = \begin{bmatrix} -168.5 & -33.50 & 23.84 & 2.175 \\ 25.99 & 1.549 & -164.6 & -32.14 \end{bmatrix}$$

$$\boldsymbol{K}_{17} = \begin{bmatrix} -40395 & -5442 & 19120 & 2584 \\ 18959 & 2555 & -41742 & -5640 \end{bmatrix} \quad \boldsymbol{K}_{27} = \begin{bmatrix} -162.2 & -33.06 & 70.99 & 15.17 \\ 68.16 & 13.62 & -162.2 & -31.90 \end{bmatrix}$$

$$\boldsymbol{K}_{18} = \begin{bmatrix} -40740 & -5488 & 8444 & 1141 \\ 8371 & 1128 & -41958 & -5668 \end{bmatrix} \quad \boldsymbol{K}_{28} = \begin{bmatrix} -163.7 & -33.47 & 24.68 & 2.191 \\ 25.93 & 1.532 & -166.7 & -32.17 \end{bmatrix}$$

$$\boldsymbol{K}_{19} = \begin{bmatrix} -41645 & -5611 & 5096 & 689.2 \\ 5168 & 696.5 & -42911 & -5797 \end{bmatrix} \quad \boldsymbol{K}_{29} = \begin{bmatrix} -166.5 & -34.51 & 9.151 & -1.881 \\ 13.26 & -2.127 & -169.1 & -33.34 \end{bmatrix}$$

Same initial conditions $\boldsymbol{x}(0) = [0.5 \quad 0 \quad -0.5 \quad 0]^T$ and $\boldsymbol{x}_r(0) = [-0.5 \quad 0 \quad 0.5 \quad 0]^T$ are chosen for simulation. Fig. 5.6 shows the state trajectories $\boldsymbol{x}(t)$ and the reference trajectories $\boldsymbol{x}_r(t)$ for these two cases. The tracking error plots obtained by using these two approaches are shown in Fig. 5.7.

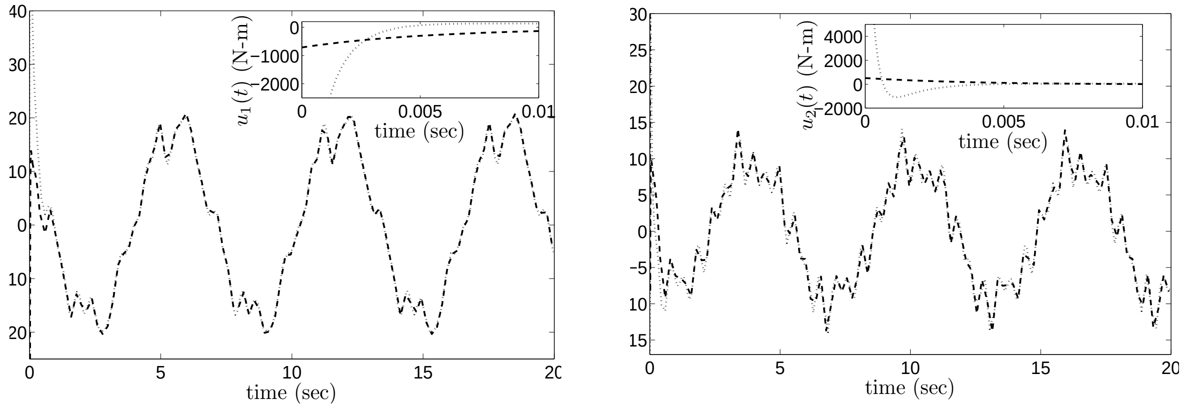


Fig. 5.8: Control inputs $\boldsymbol{u}(t)$ (dashed line and dotted line for parametric and fixed Lyapunov function based approach respectively) for $\rho^2 = 0.001$.

Fig. 5.8 shows the control inputs $\boldsymbol{u}(t)$ needed for these two approaches. From Fig. 5.8, it is observed that the magnitude of the feedback gains obtained by using fixed Lyapunov function based approach is very high compared to the parametric Lyapunov function based approach. The effect of this high gain can be observed in the control input (transient region in Fig. 5.8). The control input is very high in this region which is undesirable in practical applications. The high gain in the fixed Lyapunov function based approach is due to the conservatism existing in the design. In the case of parametric Lyapunov function based approach, the gain obtained is of lower value and it is achieved by relaxing the conservatism in the design.

By Algorithm 5.1, the feasible solution is obtained with $(1 - c\epsilon) = 0$ and hence for the uncertain fuzzy descriptor systems, the parametric Lyapunov function based approach is less conservative than the fixed Lyapunov function based approach.

5.8 Summary

This chapter has examined the problem of designing a model reference trajectory controller satisfying H_∞ performance criterion for uncertain fuzzy descriptor systems. Sufficient conditions for controller design which meet the given H_∞ performance criterion are formulated in terms of matrix inequalities. The proposed descriptor system approach yields lesser number of inequality conditions than those obtained using the standard state-space approach. It is shown that, by the proposed design method, the desired tracking controller can be obtained by solving a set of inequalities and the specified H_∞ disturbance attenuation level can be obtained. To show the effectiveness of the proposed controller design, tracking control of a two link robotic manipulator is considered and the simulation results show that the system states closely track the reference trajectory.

Chapter 6

Conclusions and Future Work

In this chapter, the main contributions of this research work are summarized and a few directions for future research are outlined.

6.1 Concluding Remarks

The central objective of this thesis is to build a systematic framework for stabilization and controller design for a class of uncertain nonlinear systems represented by T-S type fuzzy model. The research work carried out in this thesis resulted in the following contributions:

i) A fuzzy identification method for deriving the fuzzy model of an uncertain nonlinear system in a form suitable for robust fuzzy control is developed.

ii) A robust fuzzy guaranteed cost controller for trajectory tracking in uncertain nonlinear systems is built.

iii) Robust stabilization, H_∞ stabilization and H_∞ tracking control for uncertain nonlinear systems using parametric Lyapunov function are analyzed.

iv) Parametric Lyapunov function based robust H_∞ tracking control design for uncertain descriptor fuzzy systems is proposed.

The fuzzy logic based identification technique for modeling an uncertain nonlinear system is presented in Chapter 2. Here, the antecedent part of the T-S fuzzy model used for modeling is assumed to be available and the method of finding the consequent part is investigated. The identification of the consequent part involves identifying the nominal model and the uncertain terms of the T-S fuzzy model. The suitability of the derived model for robust fuzzy control is substantiated illustrating the robust stability conditions.

A robust fuzzy guaranteed cost controller design for trajectory tracking in uncertain nonlinear systems is proposed in Chapter 3. A quadratic performance function is considered and the controller design method for finding the feedback gain matrices of the fuzzy controller is derived in terms of matrix inequalities. These matrix inequalities are then transformed into standard LMIs which can be solved easily and efficiently. The derived controller satisfies the defined performance measure and also ensures closed loop stability.

Chapter 4 focuses on designing a T-S fuzzy controller for uncertain nonlinear systems using a richer class of Lyapunov function called parametric Lyapunov function. The proposed method is aimed at designing a fuzzy controller for stabilization and tracking control of nonlinear systems with slowly varying uncertainties. The conditions for designing the controller are derived in terms of matrix inequalities involving uncertain terms which are then reduced to finite dimensional inequalities by applying multi-convexity concept. These inequalities are not standard LMIs and hence, an iterative LMI based algorithm is proposed for solving these inequalities in order to obtain the parameters of the parametric Lyapunov function as well as the feedback gain matrices. Use of the parametric Lyapunov function reduces conservatism in the design which is achieved because of varying the Lyapunov function along the uncertainties. Simulation results show that the parametric Lyapunov function based approach is less conservative than the fixed Lyapunov function based approach and hence it can admit a wider range of uncertainties.

Chapter 5 deals with the uncertain fuzzy descriptor system which is an extension to standard T-S fuzzy system. Initially a fixed Lyapunov function based approach is considered and controller design for this rich class of fuzzy descriptor systems is formulated as a problem of solving a set of LMIs. Finally, the concepts of fixed Lyapunov function based approach are combined with the parametric Lyapunov function based approach presented in Chapter 4 for designing a controller to control the uncertain descriptor fuzzy system. The design conditions for the descriptor fuzzy system are more complicated than the standard state-space based systems. However, the descriptor fuzzy system based approach has the advantage of possessing fewer number of matrix inequality conditions for certain special cases [8]. Hence, it is suitable for complex systems represented in descriptor form which is often observed in nonlinear mechanical systems.

6.2 Directions for Future Work

Following the design methods described in this thesis, a few tracks for future research are listed below that can be taken up for robust control of uncertain nonlinear systems:

- Identification of the antecedent part of the fuzzy model with uncertain data by tuning the parameters using min-max optimization may yield a better model with less number of rules.

- In this thesis, all the design are carried out for state feedback case. But in practice, all the system states may not be available for measurement and in that case we need to design an observer to reconstruct the missing states for observer based output feedback control. This can be carried out using the separation principle [8].

- The parametric Lyapunov function based approach can be extended by making the Lyapunov function dependent on the membership function which can further reduce the conservatism in the design.

- Many complicated nonlinear systems can be approximated by fuzzy large-scale systems which are composed of a number of T-S fuzzy subsystems. The suitability of the parametric Lyapunov function based approach for robust fuzzy control of uncertain fuzzy large scale systems can be examined.

- In the parametric Lyapunov function based approach, the matrix inequalities involving the parametric uncertainties are reduced to finite dimensional matrix inequalities by using the multi-convexity concept. It is required to check the inequalities at the corners of the parameter box and the rate box and this will result in large number of inequalities for systems having more uncertain terms. Hence, finding a better way to solve the parametric dependent inequalities is another interesting problem.

- In the T-S fuzzy model considered in this thesis, time delay is not considered in the system. However, time delay often occurs in many dynamical systems such as chemical and biological systems. The existence of time delay may result in instability and poor performance. The problem of robust controller design for this class of nonlinear time-delay systems is another interesting problem that deserves attention.

Appendix A

Supplementary Materials

Lemma A.1. [105]: The parameterized linear matrix inequality,

$$\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\, \mu_j \mathcal{M}_{ij} < 0 \tag{A.1}$$

is fulfilled, if the following condition holds:

$$\mathcal{M}_{ii} \quad < \quad 0, \quad i = 1, 2, \ldots, r \tag{A.2}$$

$$\frac{1}{r-1}\mathcal{M}_{ii} + \frac{1}{2}(\mathcal{M}_{ij} + \mathcal{M}_{ji}) \quad < \quad 0, \quad 1 \leq i \neq j \leq r \tag{A.3}$$

□

A.1 Output Tracking control

This section reviews the method of finding the feedback gain matrices for output tracking control design problem in [91]. Let us consider the fuzzy model of a nonlinear system in the following form:

Plant rule i:

IF $z_1(t)$ is F_{1i} and $\cdots$ and $z_p(t)$ is F_{pi} *THEN*

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i \boldsymbol{x}(t) + \boldsymbol{B}_i \boldsymbol{u}(t), \qquad i = 1, 2, \ldots, r \tag{A.4}$$

where $z_1(t)$, ... , $z_p(t)$ are the premise variables; $F_{ji}(j = 1, \ldots, p)$ are the fuzzy sets; r is the number of fuzzy rules; $\boldsymbol{x}$ is the state variable; $\boldsymbol{u}(t)$ is the input; $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ are the system matrices of appropriate dimensions. The overall fuzzy system is inferred as

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} \mu_i \left\{ \boldsymbol{A}_i \boldsymbol{x}(t) + \boldsymbol{B}_i \boldsymbol{u}(t) \right\} \tag{A.5}$$

For output tracking control, the control law is required to satisfy

$$\overline{\boldsymbol{y}}(t) - \overline{\boldsymbol{r}}(t) \to \boldsymbol{0} \qquad \text{as} \qquad t \to \infty \tag{A.6}$$

where $\overline{\boldsymbol{r}}$ denotes the desired trajectory or reference signal and $\overline{\boldsymbol{y}}(t)$ is the output variable. In order to convert the output tracking problem into a stabilization problem, a set of virtual desired variable $\boldsymbol{x}_d$ is introduced which will be tracked by the state variable $\boldsymbol{x}(t)$. Let $\tilde{\boldsymbol{x}}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_d(t)$ denote the tracking error for the state variables. A fuzzy PDC controller is considered with the following form:

$$\boldsymbol{\tau}(t) = -\sum_{i=1}^{r} \mu_i \boldsymbol{K}_i \tilde{\boldsymbol{x}}(t). \tag{A.7}$$

With the above fuzzy PDC controller (A.7), the closed loop system can be obtained as

$$\tilde{\boldsymbol{x}}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j (\boldsymbol{A}_i - \boldsymbol{B}_i \boldsymbol{K}_j)\tilde{\boldsymbol{x}}(t). \tag{A.8}$$

Theorem A.1. [91]: Suppose that the virtual desired variable $\boldsymbol{x}_d$ and its derivative $\dot{\boldsymbol{x}}_d$ are bounded. The augmented error system (A.8) is exponentially stable if there exist a common positive-definite matrix $\boldsymbol{P} = \boldsymbol{P}^T > 0$ and symmetric positive-definite matrices $\boldsymbol{D}$ and $\boldsymbol{Q}_{ij}$ such that

$$\boldsymbol{\Lambda}_{ii}^T \boldsymbol{P} + \boldsymbol{P}\boldsymbol{\Lambda}_{ii} + \boldsymbol{Q}_{ii} + \boldsymbol{DPD} < \mathbf{0}, \qquad i = 1, \dots, r \tag{A.9}$$

$$\boldsymbol{\Lambda}_{ij}^T \boldsymbol{P} + \boldsymbol{P}\boldsymbol{\Lambda}_{ij} + \boldsymbol{Q}_{ij} \leq \mathbf{0}, \qquad i < j \leq r \tag{A.10}$$

$$\begin{bmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} & \dots & \boldsymbol{Q}_{1r} \\ \boldsymbol{Q}_{12} & \boldsymbol{Q}_{22} & \dots & \boldsymbol{Q}_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{Q}_{1r} & \boldsymbol{Q}_{2r} & \dots & \boldsymbol{Q}_{rr} \end{bmatrix} = \tilde{\boldsymbol{Q}} > \mathbf{0} \tag{A.11}$$

where $\boldsymbol{G}_{ij} = \boldsymbol{A}_i - \boldsymbol{B}_i\boldsymbol{K}_j$, $\boldsymbol{\Lambda}_{ii} = \boldsymbol{G}_{ii}$ and $\boldsymbol{\Lambda}_{ij} = (\boldsymbol{G}_{ij} + \boldsymbol{G}_{ji})/2$.

□

Proof of Theorem 2.1: Consider the Lyapunov function candidate $V(t) = \boldsymbol{x}^T(t)\boldsymbol{P}\,\boldsymbol{x}(t)$. Then,

$$\dot{V}(t) = \dot{\boldsymbol{x}}^T(t)\boldsymbol{P}\boldsymbol{x}(t) + \boldsymbol{x}^T(t)\boldsymbol{P}\dot{\boldsymbol{x}}(t) \tag{A.12}$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j \Big\{ \boldsymbol{x}^T(t)\Big(\Big((\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{K}_j)^T\boldsymbol{P} + (\Delta\boldsymbol{A}_i(t) + \Delta\boldsymbol{B}_i(t)\boldsymbol{K}_j)^T\boldsymbol{P}\Big) + \Big(* \Big)^T \Big)\boldsymbol{x}(t)\Big\} \tag{A.13}$$

Since $\boldsymbol{F}_{xi}(t)$ in (2.36) and $\boldsymbol{\epsilon}_i$ in (2.46) are diagonal matrices, $\frac{1}{\sqrt{\boldsymbol{\epsilon}_i}}\boldsymbol{F}_{xi}(t) = \boldsymbol{F}_{xi}(t)\frac{1}{\sqrt{\boldsymbol{\epsilon}_i}}$. With this condition and the inequality $\boldsymbol{X}^T\boldsymbol{Z} + \boldsymbol{Z}^T\boldsymbol{X} \leq \boldsymbol{X}^T\boldsymbol{X} + \boldsymbol{Z}\boldsymbol{Z}^T$ given in [66], it can be derived as,

$$\begin{aligned} &\boldsymbol{x}^T(t)\left(\Big((\Delta\boldsymbol{A}_i(t) + \Delta\boldsymbol{B}_i(t)\boldsymbol{K}_j)^T\boldsymbol{P}\Big) + \Big(* \Big)^T \right)\boldsymbol{x}(t) \\ &\quad = \boldsymbol{x}^T(t)\left(\Big((\boldsymbol{M}_{xi}\boldsymbol{F}_{xi}(t)(\boldsymbol{N}_{x1i} + \boldsymbol{N}_{x2i}\boldsymbol{K}_j))^T\boldsymbol{P}\Big) + \Big(* \Big)^T \right)\boldsymbol{x}(t) \\ &\quad = \boldsymbol{x}^T(t)\left(\Big((\boldsymbol{M}_{xi}\sqrt{\boldsymbol{\epsilon}_i}\frac{1}{\sqrt{\boldsymbol{\epsilon}_i}}\boldsymbol{F}_{xi}(t)(\boldsymbol{N}_{x1i} + \boldsymbol{N}_{x2i}\boldsymbol{K}_j))^T\boldsymbol{P}\Big) + \Big(* \Big)^T \right)\boldsymbol{x}(t) \end{aligned}$$

$$
\begin{aligned}
&= \boldsymbol{x}^T(t)\left(\left((\boldsymbol{M}_{xi}\sqrt{\boldsymbol{\epsilon}_i}\boldsymbol{F}_{xi}(t)\frac{1}{\sqrt{\boldsymbol{\epsilon}_i}}(\boldsymbol{N}_{x1i}+\boldsymbol{N}_{x2i}\boldsymbol{K}_j))^T\boldsymbol{P}\right)+\left(\,*\,\right)^T\right)\boldsymbol{x}(t)\\
&\leq \boldsymbol{x}^T(t)\left((\boldsymbol{N}_{x1i}+\boldsymbol{N}_{x2i}\boldsymbol{K}_j)^T\frac{1}{\sqrt{\boldsymbol{\epsilon}_i}}\boldsymbol{F}_{xi}^T(t)\boldsymbol{F}_{xi}(t)\frac{1}{\sqrt{\boldsymbol{\epsilon}_i}}(\boldsymbol{N}_{x1i}+\boldsymbol{N}_{x2i}\boldsymbol{K}_j)\right.\\
&\qquad\qquad \left.+\boldsymbol{P}\boldsymbol{M}_{xi}\boldsymbol{\epsilon}_i\boldsymbol{M}_{xi}^T\boldsymbol{P}\right)\boldsymbol{x}(t)\\
&\leq \boldsymbol{x}^T(t)\left((\boldsymbol{N}_{x1i}+\boldsymbol{N}_{x2i}\boldsymbol{K}_j)^T\boldsymbol{\epsilon}_i^{-1}(\boldsymbol{N}_{x1i}+\boldsymbol{N}_{x2i}\boldsymbol{K}_j)+\boldsymbol{P}\boldsymbol{M}_{xi}\boldsymbol{\epsilon}_i\boldsymbol{M}_{xi}^T\boldsymbol{P}\right)\boldsymbol{x}(t) \quad (A.14)
\end{aligned}
$$

Substituting (A.14) into (A.13) yields:

$$
\begin{aligned}
\dot{V}(t) \quad \leq \quad & \sum_{i=1}^{r}\sum_{j=1}^{r}\ \mu_i\,\mu_j\Big\{\bar{\boldsymbol{x}}^T(t)\Big(((\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{K}_j)^T\boldsymbol{P})+(*)^T+\boldsymbol{P}\boldsymbol{M}_{xi}\boldsymbol{\epsilon}_i\boldsymbol{M}_{xi}^T\boldsymbol{P}\\
&+(\boldsymbol{N}_{x1i}+\boldsymbol{N}_{x2i}\boldsymbol{K}_j)^T\boldsymbol{\epsilon}_i^{-1}(\boldsymbol{N}_{x1i}+\boldsymbol{N}_{x2i}\boldsymbol{K}_j)\Big)\bar{\boldsymbol{x}}(t)\Big\} \qquad (A.15)
\end{aligned}
$$

Hence proved. □

Proof of Theorem 4.1: Consider a Lyapunov function $V(t) = \boldsymbol{x}^T(t)\boldsymbol{P}\boldsymbol{x}(t)$. The time derivative of $V(t)$ along the trajectory is given by

$$
\begin{aligned}
\dot{V}(t) \quad &= \quad \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\boldsymbol{x}^T(t)\Big(((\boldsymbol{A}_i+\Delta\boldsymbol{A}_i(t))+(\boldsymbol{B}_i+\Delta\boldsymbol{B}_i(t))\boldsymbol{K}_j)^T\boldsymbol{P}+(*)\Big)\boldsymbol{x}(t) \qquad (A.16)\\
&= \quad \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\boldsymbol{x}^T(t)\left((\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{K}_j)^T\boldsymbol{P}+\left(\sum_{l=1}^{L}\boldsymbol{M}_{il}\Delta_{il}(t)(\boldsymbol{N}_{i1l}+\boldsymbol{N}_{i2l}\boldsymbol{K}_j)\right)^T\boldsymbol{P}+(*)\right)\boldsymbol{x}(t) \qquad (A.17)
\end{aligned}
$$

If the following inequality is satisfied, then $\dot{V}(\boldsymbol{x}) \leq 0$ and the fuzzy system (4.2) will be stable.

$$
\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\left((\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{K}_j)^T\boldsymbol{P}+\left(\sum_{l=1}^{L}\boldsymbol{M}_{il}\Delta_{il}(t)(\boldsymbol{N}_{i1l}+\boldsymbol{N}_{i2l}\boldsymbol{K}_j)\right)^T\boldsymbol{P}+(*)\right) \;<\; 0 \qquad (A.18)
$$

With this condition and the inequality $\boldsymbol{X}^T\boldsymbol{Y}+\boldsymbol{Y}^T\boldsymbol{X} \leq \boldsymbol{X}^T\boldsymbol{X}+\boldsymbol{Y}\boldsymbol{Y}^T$ given in [66], the following inequality condition can be obtained

$$
\begin{aligned}
&\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\left((\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{K}_j)^T\boldsymbol{P}+(*)+\sum_{l=1}^{L}\boldsymbol{P}\boldsymbol{M}_{il}\epsilon_l\boldsymbol{M}_{il}^T\boldsymbol{P}\right.\\
&\qquad\left.+\sum_{l=1}^{L}(\boldsymbol{N}_{i1l}+\boldsymbol{N}_{i2l}\boldsymbol{K}_j)^T\epsilon_l^{-1}(\boldsymbol{N}_{i1l}+\boldsymbol{N}_{i2l}\boldsymbol{K}_j)\right) \;<\; 0 \qquad (A.19)
\end{aligned}
$$

Pre-multiplying and post-multiplying the above inequality by $\boldsymbol{Y}$, where $\boldsymbol{Y} = \boldsymbol{P}^{-1}$ and by Schur complement, the inequality can be transformed as

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\left[\begin{array}{c|ccc|ccc} \boldsymbol{Y}\boldsymbol{A}_i^T+\boldsymbol{B}_i^T\boldsymbol{W}_j^T+(*) & * & \dots & * & * & \dots & * \\ \hline \epsilon_1\boldsymbol{M}_{i1}^T & \epsilon_1 & & & & & \\ \vdots & & \ddots & & & \boldsymbol{0} & \\ \epsilon_l\boldsymbol{M}_{il}^T & & & \epsilon_l & & & \\ \hline \boldsymbol{N}_{i11}\boldsymbol{Y}+\boldsymbol{N}_{i21}\boldsymbol{W}_j & & & & \epsilon_1 & & \\ \vdots & & \boldsymbol{0} & & & \ddots & \\ \boldsymbol{N}_{i1l}\boldsymbol{Y}+\boldsymbol{N}_{i2l}\boldsymbol{W}_j & & & & & & \epsilon_l \end{array}\right]<0. \tag{A.20}$$

Applying Lemma A.1 to the above inequality, the inequalities in Theorem 4.1 are obtained. Hence proved. □

References

[1] I. Škrjanc, S. Blažič, and O. Agamennoni, "Interval fuzzy model identification using l_∞-norm," *IEEE Trans. Fuzzy Syst.*, vol. 13, no. 5, pp. 561–568, Oct. 2005.

[2] J.-C. Lo and M.-L. Lin, "Robust H_∞ control for fuzzy systems with Frobenius norm-bounded uncertainties," *IEEE Trans. Fuzzy Syst.*, vol. 14, no. 1, pp. 1–15, Feb. 2006.

[3] O. Castillo and P. Melin, *Type-2 Fuzzy Logic: Theory and Applications.* Springer, 2007.

[4] K. Self, "Designing with fuzzy logic," *IEEE Spectrum Mag.*, vol. 27, no. 11, pp. 42–44, 105, Nov. 1990.

[5] K. M. Passino and S. Yurkovich, *Fuzzy Control.* Addison Wesley Longman, 1998.

[6] S. Y. Yi and M. J. Chung, "Robustness of fuzzy logic control for an uncertain dynamic system," *IEEE Trans. Fuzzy Syst.*, vol. 6, no. 2, pp. 216–225, May. 1998.

[7] M. Sugeno, "On stability of fuzzy systems expressed by fuzzy rules with singleton consequents," *IEEE Trans. Fuzzy Syst.*, vol. 7, no. 2, pp. 201–224, Apr. 1999.

[8] K. Tanaka and H. O. Wang, *Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach.* Wiley-Interscience Publication, John Wiley & Sons, Inc, New York, 2001.

[9] W. Gueaieb, F. Karray, and S. Al-Sharhan, "A robust adaptive fuzzy position/force control scheme for cooperative manipulators," *IEEE Trans. Control Syst. Technol.*, vol. 11, no. 4, pp. 516–528, Jul. 2003.

[10] J. Espinosa, J. Vandewalle, and V. Wertz, *Fuzzy Logic, Identification and Predictive Control.* Springer, 2005.

[11] C. W. Tao and J. S. Taur, "Robust fuzzy control for a plant with fuzzy linear model," *IEEE Trans. Fuzzy Syst.*, vol. 13, no. 1, pp. 30–41, Feb. 2005.

[12] C. Lin, Qing-GuoWang, T. H. Lee, and Y. He, *LMI Approach to Analysis and Control of Takagi-Sugeno Fuzzy Systems with Time Delay*, ser. Lecture Notes in Control and Information Sciences, M. Thoma and M. Morari, Eds. Springer, 2007.

[13] P. J. King and E. H. Mamdani, "The application of fuzzy control systems to industrial processes," *Automatica*, vol. 13, no. 3, pp. 235–242, May. 1977.

[14] T. J. Procyk and E. H. Mamdani, "A linguistic self-organizing process controller," *Automatica*, vol. 15, no. 1, pp. 15–30, Jan. 1979.

[15] F. V. D. Rhee, H. R. V. N. Lemke, and J. G. Dijkman, “Knowledge based fuzzy control of systems,” *IEEE Trans. Automat. Contr.*, vol. 35, no. 2, pp. 148–155, Feb. 1990.

[16] C.-H. Chou and H.-C. Lu, “A heuristic self-tuning fuzzy controller,” *Fuzzy Sets Syst.*, vol. 61, pp. 249–264, 1994.

[17] G. Feng, “A survey on analysis and design of model-based fuzzy control systems,” *IEEE Trans. Fuzzy Syst.*, vol. 14, no. 5, pp. 676–697, Oct. 2006.

[18] H. A. Malki, H. Li, and G. Chen, “New design and stability analysis of fuzzy proportional-derivative control systems,” *IEEE Trans. Fuzzy Syst.*, vol. 2, pp. 245–254, 1994.

[19] H. A. Malki, D. Misir, D. Feigenspan, and G. Chen, “Fuzzy PID control of a flexible-joint robot arm with uncertainties from time-varying loads,” *IEEE Trans. Control Syst. Technol.*, vol. 5, no. 3, pp. 371–378, May. 1997.

[20] C. W. Tao, M.-L. Chan, and T.-T. Lee, “Adaptive fuzzy sliding mode controller for linear systems with mismatched time-varying uncertainties,” *IEEE Trans. Syst., Man, Cybern. B*, vol. 33, no. 2, pp. 283–294, Apr. 2003.

[21] C. W. Tao, J. S. Taur, and M.-L. Chan, “Adaptive fuzzy terminal sliding mode controller for linear systems with mismatched time-varying uncertainties,” *IEEE Trans. Syst., Man, Cybern. B*, vol. 34, no. 1, pp. 255–262, Feb. 2004.

[22] T. Takagi and M. Sugeno, “Fuzzy identification of systems and its applications to modeling and control,” *IEEE Trans. Syst., Man, and Cybern.*, vol. 15, no. 1, pp. 116–132, 1985.

[23] T. Das and I. N. Kar, “Design and implementation of an adaptive fuzzy logic-based controller for wheeled mobile robots,” *IEEE Trans. Control Syst. Technol.*, vol. 14, no. 3, pp. 501–510, May 2006.

[24] K. Tanaka and M. Sugeno, “Stability analysis and design of fuzzy control systems,” *Fuzzy Sets Syst.*, vol. 45, no. 2, pp. 135–156, 1992.

[25] K. Tanaka and M. Sano, “A robust stabilization problem of fuzzy control systems and its application to backing up control of a truck trailer,” *IEEE Trans. Fuzzy Syst.*, vol. 2, no. 2, pp. 119–134, 1994.

[26] H. O. Wang, K. Tanaka, and M. F. Griffin, “Parallel distributed compensation of nonlinear systems by Takagi-Sugeno fuzzy model,” in *Proc. FUZZ-IEEE/IFES’95*, 1995, pp. 531–538.

[27] B.-S. Chen, C.-H. Lee, and Y.-C. Chang, “H_∞ tracking design of uncertain nonlinear SISO systems: Adaptive fuzzy approach,” *IEEE Trans. Fuzzy Syst.*, vol. 4, no. 1, pp. 32–43, Feb. 1996.

[28] H. K. Lam, F. H. F. Leung, and P. K. S. Tam, “Stable and robust fuzzy control for uncertain nonlinear systems,” *IEEE Trans. Syst., Man, Cybern. A*, vol. 30, no. 6, pp. 825–840, Nov. 2000.

[29] H. J. Lee, J. B. Park, and G. Chen, “Robust fuzzy control of nonlinear systems with parametric uncertainties,” *IEEE Trans. Fuzzy Syst.*, vol. 9, no. 2, pp. 369–379, Apr. 2001.

[30] K. R. Lee, E. T. Jeung, and H. B. Park, "Robust fuzzy H_∞ control for uncertain nonlinear systems via state feedback: an LMI approach," *Fuzzy Sets Syst.*, vol. 120, pp. 123–134, 2001.

[31] B. Marx, D. Koenig, and J. Ragot, "Design of observers for Takagi–Sugeno descriptor systems with unknown inputs and application to fault diagnosis," *IET Control Theory Appl.*, vol. 1, no. 5, pp. 1487–1495, Sep. 2007.

[32] C.-T. Pang and Y.-Y. Lur, "On the stability of Takagi-Sugeno fuzzy systems with time–varying uncertainties," *IEEE Trans. Fuzzy Syst.*, vol. 16, no. 1, pp. 162–170, Feb. 2008.

[33] H. H. Choi, "Robust stabilization of uncertain fuzzy systems using variable structure system approach," *IEEE Trans. Fuzzy Syst.*, vol. 16, no. 3, pp. 715–724, Jun. 2008.

[34] A. Sala and C. Ariño, "Relaxed stability and performance LMI conditions for Takagi–Sugeno fuzzy systems with polynomial constraints on membership function shapes," *IEEE Trans. Fuzzy Syst.*, vol. 16, no. 5, pp. 1328 –1336, Oct. 2008.

[35] C. Yang and Q. Zhang, "Multiobjective control for T-S fuzzy singularly perturbed systems," *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 1, pp. 104–115, Feb. 2009.

[36] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory.* SIAM, Philadelphia, 1994.

[37] P. Gahinet, A. Nemirovski, A. Laub, and M. Chilali, *LMI Control Toolbox.* Natick, MA: The Math Works, 1995.

[38] S. Boyd and L. Vandenberghe, *Convex Optimization.* Cambridge University press, 2004.

[39] W. Assawinchaichote, S. K. Nguang, and P. Shi, *Fuzzy Control and Filter Design for Uncertain Fuzzy Systems*, ser. Lecture Notes in Control and Information Sciences, M. Thoma and M. Morari, Eds. Springer, 2006.

[40] K. Tanaka, T. Ikeda, and H. O. Wang, "Fuzzy regulators and fuzzy observers: Relaxed stability conditions and LMI-based designs," *IEEE Trans. Fuzzy Syst.*, vol. 6, no. 2, pp. 250–265, May 1998.

[41] E. Kim, M. Park, S. Ji, and M. Park, "A new approach to fuzzy modeling," *IEEE Trans. Fuzzy Syst.*, vol. 5, no. 3, pp. 328–337, August 1997.

[42] T. Taniguchi, K. Tanaka, H. Ohtake, and H. O. Wang, "Model construction, rule reduction, and robust compensation for generalized form of Takagi-Sugeno fuzzy systems," *IEEE Trans. Fuzzy Syst.*, vol. 9, no. 4, pp. 525–538, Aug. 2001.

[43] J.-C. Lo and M.-L. Lin, "Robust H_∞ nonlinear modeling and control via uncertain fuzzy systems," *Fuzzy Sets Syst.*, vol. 143, no. 2, pp. 189–209, Apr. 2004.

[44] G. Tsekouras, H. Sarimveis, E. Kavakli, and G. Bafas, "A hierarchical fuzzy-clustering approach to fuzzy modeling," *Fuzzy Sets and Systems*, vol. 150, pp. 245–266, 2005.

[45] M. C. M. Teixeira and S. H. Żak, "Stabilizing controller design for uncertain nonlinear systems using fuzzy models," *IEEE Trans. Fuzzy Syst.*, vol. 7, no. 2, pp. 133–142, Apr. 1999.

[46] H. Ohtake, K. Tanaka, and H. O. Wang, "Fuzzy modeling via sector nonlinearity concept," in *IFSA World Congress and 20th NAFIPS International Conference, 2001. Joint 9th*, Jul. 2001, pp. 127–132.

[47] J. Abonyi, R. Babuška, and F. Szeifert, "Modified Gath-Geva fuzzy clustering for identification of Takagi-Sugeno fuzzy models," *IEEE Trans. Syst., Man, and Cybern. B*, vol. 32, no. 5, pp. 612–621, October 2002.

[48] T.-H. S. Li and S.-H. Tsai, "T–S fuzzy bilinear model and fuzzy controller design for a class of nonlinear systems," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 3, pp. 494–506, Jun. 2007.

[49] W.-H. Ho, J.-T. Tsai, and J.-H. Chou, "Robust-stable and quadratic-optimal control for TS-fuzzy-model-based control systems with elemental parametric uncertainties," *IET Control Theory Appl.*, vol. 1, no. 3, pp. 731–742, May. 2007.

[50] S. K. Nguang, P. Shi, and S. Ding, "Fault detection for uncertain fuzzy systems: An LMI approach," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 6, pp. 1251–1262, Dec. 2007.

[51] H. Ying, "General SISO Takagi-Sugeno fuzzy systems with linear rule consequent are universal approximators," *IEEE Trans. Fuzzy Syst.*, vol. 6, no. 4, pp. 582–587, Nov. 1998.

[52] I. Škrjanc, S. Blažič, and O. Agamennoni, "Interval fuzzy modeling applied to wiener models with uncertainties," *IEEE Trans. Syst., Man, Cybern. B*, vol. 35, no. 5, pp. 1092–1095, Oct. 2005.

[53] ——, "Identification of dynamical systems with a robust interval fuzzy model," *Automatica*, vol. 41, no. 2, pp. 327–332, Feb. 2005.

[54] M. Sugeno and G. T. Kang, "Fuzzy modeling and control of multilayer incinerator," *Fuzzy Sets Syst.*, vol. 18, pp. 329–346, 1986.

[55] K. Tanaka, T. Ikeda, and H. O. Wang, "Robust stabilization of a class of uncertain nonlinear systems via fuzzy control: Quadratic stabilizability, H_∞ control theory and linear matrix inequalities," *IEEE Trans. Fuzzy Syst.*, vol. 4, no. 1, pp. 1–13, Feb. 1996.

[56] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optimization Methods and Software*, vol. 11–12, pp. 625–653, 1999. [Online]. Available: http://sedumi.mcmaster.ca/

[57] S. S. L. Chang and T. K. C. Peng, "Adaptive guaranteed cost control of systems with uncertain parameters," *IEEE Trans. Automat. Contr.*, vol. 17, no. 4, pp. 474–483, 1972.

[58] L. Yu and J. Chu, "An LMI approach to guaranteed cost control of linear uncertain time-delay systems," *Automatica*, vol. 35, pp. 1155–1159, 1999.

[59] L. Behera and K. K. Anand, "Guaranteed tracking and regulatory performance of nonlinear dynamic systems using fuzzy neural networks," *IEE Proc. Control Theoy Appl.*, vol. 146, no. 5, pp. 484–491, Sep. 1999.

[60] N. Xie and G. Y. Tang, "Delay-dependent nonfragile guaranteed cost control for nonlinear time-delay systems," *Nonlinear Analysis*, vol. 64, pp. 2084–2097, 2006.

[61] X. Jiang and Q. L. Han, "Robust H_∞ control for uncertain Takagi-Sugeno fuzzy systems with interval time-varying delay," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 2, pp. 321–331, 2007.

[62] S.-W. Kau, H.-J. Lee, C.-M. Yang, C.-H. Lee, L. Hong, and C.-H. Fang, "Robust H_∞ fuzzy static output feedback control of T–S fuzzy systems with parametric uncertainties," *Fuzzy Sets Syst.*, vol. 158, no. 2, pp. 135–146, Jan. 2007.

[63] K. Tanaka, T. Taniguchi, and H. O. Wang, "Fuzzy control based on quadratic performance functions," in *37th IEEE Conference on Decision and Control*, Tampa, 1998, pp. 2914–2919.

[64] B. Chen and X. Liu, "Fuzzy guaranteed cost control for nonlinear systems with time-varying delay," *IEEE Trans. Fuzzy Syst.*, vol. 13, no. 2, pp. 238–249, 2005.

[65] S. Patra, S. Sen, and G. Ray, "Design of static H_∞ loop shaping controller in four-block framework using LMI approach," *Automatica*, vol. 44, pp. 2214–2220, 2008.

[66] S. Tong, T. Wang, and H. X. Li, "Fuzzy robust tracking control for uncertain nonlinear systems," *Int. J. Approx. Reason.*, vol. 30, no. 2, pp. 73–90, Jun. 2002.

[67] M. Johansson, A. Rantzer, and K.-E. Årzén, "Piecewise quadratic stability of fuzzy systems," *IEEE Trans. Fuzzy Syst.*, vol. 7, no. 6, pp. 713–722, Dec. 1999.

[68] M. Feng and C. J. Harris, "Piecewise Lyapunov stability conditions of fuzzy systems," *IEEE Trans. Syst., Man, Cybern. B*, vol. 31, no. 2, pp. 259–262, Apr. 2001.

[69] K. Kiriakidis, "Robust stabilization of the Takagi-Sugeno fuzzy model via bilinear matrix inequalities," *IEEE Trans. Fuzzy Syst.*, vol. 9, no. 2, pp. 269–277, Apr. 2001.

[70] G. Feng, "Controller synthesis of fuzzy dynamic systems based on piecewise Lyapunov functions," *IEEE Trans. Fuzzy Syst.*, vol. 11, no. 5, pp. 605–612, Oct. 2003.

[71] G. Feng, C.-L. Chen, D. Sun, and Y. Zhu, "H_∞ controller synthesis of fuzzy dynamic systems based on piecewise Lyapunov functions and bilinear matrix inequalities," *IEEE Trans. Fuzzy Syst.*, vol. 13, no. 1, pp. 94–103, Feb. 2005.

[72] H. Zhang, C. Li, and X. Liao, "Stability analysis and H_∞ controller design of fuzzy large-scale systems based on piecewise Lyapunov functions," *IEEE Trans. Syst., Man, Cybern. B*, vol. 36, no. 3, pp. 685–698, Jun. 2006.

[73] T. Zhang, G. Feng, and J. Lu, "Fuzzy constrained min–max model predictive control based on piecewise Lyapunov functions," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 4, pp. 686–698, Aug. 2007.

[74] K. Tanaka, T. Hori, and H. O. Wang, "A multiple Lyapunov function approach to stabilization of fuzzy control systems," *IEEE Trans. Fuzzy Syst.*, vol. 11, no. 4, pp. 582–589, Aug. 2003.

[75] K. Tanaka, H. Ohtake, and H. O. Wang, "A descriptor system approach to fuzzy control system design via fuzzy Lyapunov functions," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 3, pp. 333–341, Jun. 2007.

[76] J. Li, S. Zhou, and S. Xu, "Fuzzy control system design via fuzzy Lyapunov functions," *IEEE Trans. Syst., Man, Cybern. B*, vol. 38, no. 6, pp. 1657–1661, Dec. 2008.

[77] L. A. Mozelli, R. M. Palhares, and G. S. Avellar, "A systematic approach to improve multiple Lyapunov function stability and stabilization conditions for fuzzy systems," *Inform. Sci.*, vol. 179, no. 8, pp. 1149–1162, Mar. 2009.

[78] S. H. Kim, C. H. Lee, and P. Park, "H_∞ state–feedback control for fuzzy systems with input saturation via fuzzy weighting–dependent Lyapunov functions," *Computers Mathematics Appl.*, vol. 57, no. 6, pp. 981–990, Mar. 2009.

[79] L. Mozelli, R. Palhares, F. Souza, and E. Mendes, "Reducing conservativeness in recent stability conditions of TS fuzzy systems," *Automatica*, vol. 45, no. 6, pp. 1580–1583, Jun. 2009.

[80] K. Tanaka, T. Komatsu, H. Ohtake, and H. O. Wang, "Micro helicopter control: LMI approach vs SOS approach," in *Proc. FUZZ-IEEE 2008*, 2008, pp. 347–353.

[81] K. Tanaka, H. Ohtake, and H. O. Wang, "Guaranteed cost control of polynomial fuzzy systems via a sum of squares approach," *IEEE Trans. Syst., Man, Cybern. B*, vol. 39, no. 2, pp. 561–567, Apr. 2009.

[82] K. Tanaka, H. Yoshida, H. Ohtake, and H. O. Wang, "A sum of squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems," *IEEE Trans. Fuzzy Syst.*

[83] G. Feng, "H_∞ controller design of fuzzy dynamic systems based on piecewise Lyapunov functions," *IEEE Trans. Syst., Man, Cybern. B*, vol. 34, no. 1, pp. 283–292, Feb. 2004.

[84] K. Guelton, T. Bouarar, and N. Manamanni, "Robust dynamic output feedback fuzzy Lyapunov stabilization of Takagi–Sugeno systems– A descriptor redundancy approach," *Fuzzy Sets Syst.*, vol. 160, pp. 2796–2811, 2009.

[85] S. Prajna, A. Papachristodoulou, P. Seiler, and P. A. Parrilo, "SOSTOOLS: Sum of squares optimization toolbox for MATLAB," 2004. [Online]. Available: http://www.cds.caltech.edu/sostools

[86] T. Taniguchi, K. Tanaka, and H. O. Wang, "Fuzzy descriptor systems and nonlinear model following control," *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 4, pp. 442–452, Aug. 2000.

[87] W.-S. Yu, "Observer-based fuzzy tracking control design for nonlinear uncertain descriptor dynamic systems," in *IEEE International Conf. on Systems, Man and Cybernetics (ISIC 2007)*, Oct. 2007, pp. 127–132.

[88] Y. Wang, Q. L. Zhang, and W. Q. Liu, "Stability analysis and design for T–S fuzzy descriptor system," in *Proc. of 40th IEEE Conf. on Decision and Control*, Orlande, Florida, USA, Dec. 2001.

[89] W. Tian, H. Zhang, and X. Yang, "Robust H_∞ control for fuzzy descriptor systems with time-varying delay and parameter uncertainties," in *Third International Conference on Natural Computation (ICNC 2007)*, 2007.

[90] X.-J. Ma and Z.-Q. Sun, "Output tracking and regulation of nonlinear system based on takagisugeno fuzzy model," *IEEE Trans. Syst., Man, and Cybern. B*, vol. 30, no. 1, pp. 47–59, Feb. 2000.

[91] K.-Y. Lian and J.-J. Liou, "Output tracking control for fuzzy systems via output feedback design," *IEEE Trans. Fuzzy Syst.*, vol. 14, no. 5, pp. 628–639, Oct. 2006.

[92] C. S. Tseng, B. S. Chen, and H. J. Uang, "Fuzzy tracking control design for nonlinear dynamic systems via T-S fuzzy model," *IEEE Trans. Fuzzy Syst.*, vol. 9, no. 3, pp. 381–392, 2001.

[93] Y.-Y. Cao and Z. Lin, "Robust stability analysis and fuzzy-scheduling control for nonlinear systems subject to actuator saturation," *IEEE Trans. Fuzzy Syst.*, vol. 11, no. 1, pp. 57–67, Feb. 2003.

[94] H.-N. Wu and K.-Y. Cai, "Mode-independent robust stabilization for uncertain Markovian jump nonlinear systems via fuzzy control," *IEEE Trans. Syst., Man, Cybern. B*, vol. 36, no. 3, pp. 509–519, Jun. 2006.

[95] D. W. C. Ho and Y. Niu, "Robust fuzzy design for nonlinear uncertain stochastic systems via sliding-mode control," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 3, pp. 350–358, Jun. 2007.

[96] J.-C. Lo and M.-L. Lin, "Existence of similarity transformation converting BMIs to LMIs," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 5, pp. 840–851, Oct. 2007.

[97] A. Hamzaoui, N. Essounbouli, K. Benmahammed, and J. Zaytoon, "State observer based robust adaptive fuzzy controller for nonlinear uncertain and perturbed systems," *IEEE Trans. Syst., Man, Cybern. B*, vol. 34, no. 2, pp. 942–950, Apr. 2004.

[98] K. Ichida and Y. Fujii, "An interval arithmetic method for global optimization," *Computing*, vol. 23, no. 1, pp. 85–97, Mar. 1979.

[99] H. Munack and Tubingen, "On global optimization using interval arithmetic," *Computing*, vol. 48, pp. 319–336, 1992.

[100] G. Alefeld and G. Mayer, "Interval analysis: Theory and applications," *J. Comput. Appl. Math.*, vol. 121, no. 1-2, pp. 421–464, Sep. 2000.

[101] E. R. Hansen and G. W. Walster, *Global Optimization Using Interval Analysis.* CRC Press, 2004.

[102] C. Fantuzzi and R. Rovatti, "On the approximation capabilities of the homogeneous Takagi-Sugeno model," in *Proc. FUZZ-IEEE'98*, vol. 2, 8–11 Sep. 1996, pp. 1067–1072.

[103] X. Su, Q. Zhang, and J. Jin, "Stability analysis for interval descriptor systems: A matrix inequalities approach," in *Proc. 4th World Congress on Intelligent Control and Automation*, vol. 2, June 10–14 2002, pp. 1007–1011.

[104] B. Zhang, S. Zhou, and T. Li, "A new approach to robust and non-fragile H_∞ control for uncertain fuzzy systems," *Information Sciences*, vol. 177, no. 15, pp. 5118–5133, Nov. 2007.

[105] H. D. Tuan, P. Apkarian, T. Narikiyo, and Y. Yamamoto, "Parameterized linear matrix inequality techniques in fuzzy control system design," *IEEE Trans. Fuzzy Syst.*, vol. 9, no. 2, pp. 324–332, Apr. 2001.

[106] C. H. Fang, Y. S. Liu, S. W. Kau, L. Hong, and C. H. Lee, "A new LMI-based approach to relaxed quadratic stabilization of T–S fuzzy control systems," *IEEE Trans. Fuzzy Syst.*, vol. 14, no. 3, pp. 386–397, June 2006.

[107] B. Ding, H. Sun, and P. Yang, "Further studies on LMI-based relaxed stabilization conditions for nonlinear systems in Takagi–Sugenos form," *Automatica*, vol. 42, pp. 503–508, 2006.

[108] A. Sala and C. Ariño, "Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: Applications of Polya's theorem," *Fuzzy Sets and Syst.*, vol. 158, pp. 2671–2686, 2007.

[109] H. K. Lam and F. H. F. Leung, "Fuzzy combination of fuzzy and switching state-feedback controllers for nonlinear systems subject to parameter uncertainties," *IEEE Trans. Syst., Man, Cybern. B*, vol. 35, no. 2, pp. 269–281, Apr. 2005.

[110] J. Löfberg, "YALMIP: A toolbox for modeling and optimization in MATLAB," in *Proc. CACSD Conf.*, Taipei, Taiwan, 2004. [Online]. Available: http://control.ee.ethz.ch/~joloef/yalmip.php

[111] J. Sun and H. Grotstollen, "Averaged modelling of switching power converters: reformulation and theoretical basis," in *Power Electronics Specialists Conference, 1992. PESC '92 Record., 23rd Annual IEEE*, vol. 2, 1992, pp. 1165–1172.

[112] R. T. Bupp, D. S. Bernstein, and V. T. Coppola, "A benchmark problem for nonlinear control design: Problem statement, experimental testbed and passive nonlinear compensation," in *Proc. 1995 American Control Conference*, Seatle, 1995, pp. 4363–4367.

[113] M. Jankovic, D. Fontaine, and P. V. Kokotović, "TORA example: Cascade- and passivity-based control designs," *IEEE Trans. Control Syst. Technol.*, vol. 4, no. 3, pp. 292–297, May 1996.

[114] M. C. M. Teixeira, E. Assuno, and R. G. Avellar, "On relaxed LMI-based designs for fuzzy regulators and fuzzy observers," *IEEE Trans. Fuzzy Syst.*, vol. 11, no. 5, pp. 613–623, October 2003.

[115] L. Xiaodong and Z. Qingling, "New approaches to H_∞ controller designs based on fuzzy observers for T-S fuzzy systems via LMI," *Automatica*, vol. 39, pp. 1571–1582, 2003.

[116] T. M. Guerra and L. Vermeiren, "LMI-based relaxed nonquadratic stabilization conditions for nonlinear systems in the Takagi–Sugenos form," *Automatica*, vol. 40, pp. 823–829, 2004.

[117] J. Xu, Y. Wang, and H. Wu, "LMI based tracking guaranteed cost control," in *Proc. IEEE Int. Conf on Mechatronics and Automation*, Luoyang, China, June 25–28 2006, pp. 1037–1042.

[118] J. Tsinias, "Control Lyapunov functions, input-to-state stability and applications to global feedback stabilization for composite systems," *J. of Math. Estimation and Ctrl.*, vol. 7, no. 2, pp. 1–31, 1997.

[119] C. Ariño and A. Sala, "Relaxed LMI conditions for closed-loop fuzzy systems with tensor-product structure," *Engineering Applications of Artificial Intelligence*, vol. 20, pp. 1036–1046, 2007.

[120] F. Cuesta, F. Gordillo, J. Aracil, and A. Ollero, "Stability analysis of nonlinear multivariable Takagi–Sugeno fuzzy control systems," *IEEE Trans. Fuzzy Syst.*, vol. 7, no. 5, pp. 508–520, Oct. 1999.

[121] S. Ghosh, S. K. Das, and G. Ray, "Stabilizing adaptive controller for uncertain dynamical systems: An LMI approach," *Int. J. Control, Automation and Systems*, vol. 7, pp. 311–317, 2009.

[122] G. Ray, A. A. Khalate, S. K. Das, and T. K. Bhattacharya, "Stabilization of inverted cart–pendulum system based on T–S fuzzy model: Simulation and experimental results," *J of the Institution of Engineers (India)*, vol. 87, pp. 3–7, 2006.

[123] T. H. S. Li and K. J. Lin, "Composite fuzzy control of nonlinear singularly perturbed systems," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 2, pp. 176–187, 2007.

[124] P. Gahinet, P. Apkarian, and M. Chilali, "Affine parameter-dependent Lyapunov functions and real parametric uncertainty," *IEEE Trans. Automat. Contr.*, vol. 41, no. 6, pp. 436–442, Mar 1996.

[125] Y.-Y. Cao and Z. Lin, "A descriptor system approach to robust stability analysis and controller synthesis," *IEEE Trans. Automat. Contr.*, vol. 49, no. 11, pp. 2081–2084, Nov. 2004.

[126] J. Yu and A. Sideris, "H_∞ control with parametric Lyapunov functions," *Systems Control Lett.*, vol. 30, no. 2-3, pp. 57–69, Apr. 1997.

[127] B. R. Barmish and C. L. DeMarco, "A new method for improvement of robustness bound for linear state equations," in *Proc. Conf. Inform. Sci. Syst.* Princeton University, 1986.

[128] P. Apkarian and H. D. Tuan, "Parameterized LMIs in control theory," in *37th IEEE Conference on Decision and Control*, vol. 1, 1998, pp. 152–157.

List of Publications

Journal Publications

1. D. Senthilkumar and Chitralekha Mahanta, "Identification of uncertain nonlinear systems for robust fuzzy control", *ISA Transactions*, vol. 49, no. 1, pp. 27–38, Jan. 2010.

2. D. Senthilkumar and Chitralekha Mahanta, "Fuzzy guaranteed cost controller for trajectory tracking in nonlinear systems", *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 4, pp. 368–379, Nov. 2009.

Conference Publications

1. D. Senthilkumar and Chitralekha Mahanta, "H_∞ Trajectory tracking control for fuzzy descriptor systems" *Int. Conf. on Control, Automation, Communication and Energy Conservation, INCACEC - 2009*, Perundurai, India, pp. 211–216, 4–6 Jun. 2009.

2. D. Senthilkumar and Chitralekha Mahanta, "Modeling of uncertain nonlinear systems: Interval fuzzy model identification", *National Systems Conference NSC-2008*, IIT Roorkee, India, Dec. 17-19, 2008.

3. D. Senthilkumar and Chitralekha Mahanta, "Fuzzy guaranteed cost controller for robotic manipulator", *National Systems Conference NSC–2007*, Manipal, India, Dec. 14-15, 2007.