

TASI Lectures on Flavor Physics

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Abstract

This review is based on lectures on flavor physics given at TASI 2008. First I summarize our present knowledge on the fundamental parameters of the flavor sector. Then I discuss various scenarios going beyond the standard model which attempt to explain aspects of the “flavor puzzle”. Relating quark masses and mixing angles via flavor symmetry is explored. Explaining the mass hierarchy via the Froggatt–Nielsen mechanism is reviewed and illustrated. Grand unification ideas are pursued to seek a pattern in the observed masses and mixings of quarks and leptons. Generating light fermion masses as radiative corrections is explained and illustrated. The popular solutions to the strong CP problem are summarized. Finally, specific processes in B meson system where significant new flavor contributions can arise are discussed.

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1 Overview

This set of lectures will focus on the flavor sector of the Standard Model (SM). As you know, most of the free parameters of the SM reside in this sector. In case you have not thought about it lately, let me remind you of the counting of parameters of the SM. Not including neutrino masses, there are 19 parameters in the SM. Five of these are flavor universal – the three gauge couplings (g_1, g_2, g_3), one Higgs quartic coupling λ , and one Higgs mass-squared μ^2 , while the remaining fourteen are parameters associated with the flavor sector. Six quark masses, three charged lepton masses, four quark mixing angles (including one weak CP violating phase) make up thirteen, while the strong CP violating parameter $\bar{\theta}$, which is intimately related to the quark masses, is the fourteenth flavor parameter. If we include small neutrino masses and mixing angles into the SM, as needed to explain neutrino oscillation data from a variety of experiments, an additional nine parameters will have to be introduced (three neutrino masses, three neutrino mixing angles and three CP violating phases, in the case of Majorana neutrinos). You see that twenty three of the twenty eight parameters describe flavor physics in the SM.

While there is abundant information on the numerical values these parameters take, a fundamental understanding of the origin of these parameters is currently lacking. Why are there three families of quarks and leptons in the first place? Are the flavor parameters all arbitrary, or are they inter-connected? Why do the charged fermion masses exhibit a strong hierarchical structure spanning some six orders of magnitude? Why are the mixing angles in the quark sector hierarchical? Are the mixing parameters related to the mass ratios? Why is $\bar{\theta} < 10^{-9}$? Why are neutrino masses so much smaller than the charged fermion masses? What causes (at least two of) the neutrino mixing angles to be much larger than the corresponding quark mixing angles? What is the origin of CP violation? The lack of a fundamental understanding of such issues is often referred to as the “flavor puzzle”.

Various solutions to this puzzle have been proposed, inevitably leading to physics beyond the Standard Model, for within the SM these parameters can only be accommodated, and not explained. Forthcoming experiments, especially at the LHC, have the potential to confirm or refute some, but not all, of these proposed non-standard scenarios. If the new flavor dynamics occurs near the TeV scale, it is potentially accessible to the LHC, but if it occurs at a much higher scale, then it will not be directly accessible. It should be mentioned at the outset that there is no compelling reason for the flavor dynamics to occur near the TeV scale, most puzzles can be explained even when the dynamics takes place near the Planck scale. This is because the small parameters of the flavor sector are quite stable under radiative corrections, owing to chiral symmetries. If the smallness of a certain parameter has an explanation from Planck scale physics, it is an equally good explanation at the low energy scale. Testing such high

scale theories would be more difficult in general. In some cases, for example, with low energy supersymmetry, information from the high scale flavor dynamics will be carried by particles which survive to the TeV scale (the SUSY particles), in which case flavor physics may be tested at colliders. Processes such as lepton flavor violating $\mu \rightarrow e\gamma$ decay and $b \rightarrow s\gamma$ transition appear to be promising setups to test such scenarios.

The Higgs boson is waiting to be discovered at the LHC. Its production and decay rates can be significantly modified relative to the SM expectations in some of the flavor–extensions of the SM. I will describe explicit models in this category. Very little is known about the top quark properties currently. LHC will serve as a top quark factory where modifications in the top sector arising from flavor–extensions can be studied. These include flavor changing decays of the top and its possible anomalous couplings to the gauge bosons. We have learned a lot about the B meson system from the B factories lately, but there are still many open issues and some puzzles which will be probed at the LHC. These include precise determination of the CP violating parameters, rare processes allowed in the SM but not yet observed, and new physics processes in B decays that require modification of the SM structure.

In Sec. 2, we will take a tour of the flavor parameters of the SM and review how these are measured and interpreted. Various ideas attempting to understand aspects of the flavor puzzle will then be introduced and their experimental consequences outlined. In Sec. 3 we will seek inter–relations between quark masses and mixing angle. Sec. 4 will be devoted to an understanding of the fermion mass and mixing hierarchies based on the Frogatt–Nielsen mechanism. In Sec. 5 we will develop grand unification as a possible clue to the flavor puzzle. Sec. 6 discusses radiative fermion mass generation, Sec. 7 summarizes the suggested solutions to the strong CP problem, and in Sec. 8 we introduce specific beyond the SM scenarios for the flavor sector and study their experimental manifestations at the LHC.

2 Flavor structure of the Standard Model

Because of the chiral structure of weak interactions, bare fermion masses are not allowed in the Standard Model. Fermion masses arise via Yukawa interactions given by the Lagrangian

$$\mathcal{L}_{\text{Yukawa}} = Q^T Y_u u^c H - Q^T Y_d d^c \tilde{H} - L^T Y_e e^c \tilde{H} + h.c. \quad (1)$$

Here I have used the standard notation for quark (Q, u^c, d^c) and lepton (L, e^c) fields. (Q, L) are $SU(2)_L$ doublets, as is the Higgs field H and its conjugate $\tilde{H} = i\tau_2 H^*$, while the (u^c, d^c, e^c) fields are $SU(2)_L$ singlets. All fermion fields are left–handed, a charge conjugation matrix C is understood to be sandwiched between all of the fermion bi–linears in Eq. (1). Contraction

of the color indices is not displayed, but should be obvious. $Y_{u,d,\ell}$ are the Yukawa couplings matrices spanning generation space which are complex and non-Hermitian. $SU(2)_L$ contraction between the fermion doublet and Higgs doublet involves the matrix $i\tau_2$. Explicitly, we have (for a family labeled by index i)

$$Q_i = \begin{pmatrix} u_i \\ d_i \end{pmatrix}; \quad L_i = \begin{pmatrix} \nu_i \\ e_i \end{pmatrix}; \quad H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix}; \quad \tilde{H} = \begin{pmatrix} H^{0*} \\ -H^- \end{pmatrix}, \quad (2)$$

so that Eq. (1) expands to

$$\mathcal{L}_{\text{Yukawa}} = (Y_u)_{ij}[u_i u_j^c H^0 - d_i u_j^c H^+] + (Y_d)_{ij}[u_i d_j^c H^- + d_i d_j^c H^{0*}] + (Y_\ell)_{ij}[\nu_i e_j^c H^- + e_i e_j^c H^{0*}] + h.c. \quad (3)$$

The neutral component of H acquires a vacuum expectation value (VEV) $\langle H^0 \rangle = v$, spontaneously breaking the electroweak symmetry ($v \simeq 174$ GeV). The Higgs field can then be parametrized in the unitary gauge as $H^0 = (\frac{h}{\sqrt{2}} + v)$ where h is a real physical field (the Higgs boson). In this gauge H^\pm , which are eaten up by the W^\pm gauge bosons, and the phase of H^0 , which is eaten up by the Z^0 gauge boson, do not appear.

The VEV of H^0 generates the following fermion mass matrices:

$$M_u = Y_u v, \quad M_d = Y_d v, \quad M_\ell = Y_\ell v. \quad (4)$$

The Yukawa coupling matrices contained in $(Y_u)_{ij}/\sqrt{2}(uu^c h)$, etc in each of the up, down and charged lepton sector becomes proportional to the corresponding mass matrix. Once the mass matrices are brought to diagonal forms, the Yukawa coupling matrices will be simultaneously diagonal. There is thus no tree-level flavor changing current mediated by the neutral Higgs boson in the Standard Model. This is a feature that is generally lost as we extend the SM to address the flavor issue (for example by introducing multiple Higgs doublets or extra fermions).

We make unitary rotations on the quark fields in family space. Unitarity of these rotations will ensure that the quark kinetic terms remain canonical. Specifically, we define mass eigenstates $(u^0, u^{c0}, d^0, d^{c0})$ via

$$\begin{aligned} u &= V_u u^0, & u^c &= V_{u^c} u^{c0}, \\ d &= V_d d^0, & d^c &= V_{d^c} d^{c0}, \end{aligned} \quad (5)$$

and we choose the unitary matrices such that

$$V_u^T (Y_u v) V_{u^c} = \begin{pmatrix} m_u & & \\ & m_c & \\ & & m_t \end{pmatrix}, \quad V_d^T (Y_d v) V_{d^c} = \begin{pmatrix} m_d & & \\ & m_s & \\ & & m_b \end{pmatrix}. \quad (6)$$

We have assumed here that the number of families is three, but the procedure applies to any number of families. Bi-unitary transformations such as the ones in Eq. (6) can diagonalize non-Hermitian matrices. The same transformations should be applied to all interactions of the quarks. As already noted, these transformations will bring the Yukawa interactions of quarks with the Higgs boson into diagonal forms. The couplings of the Z^0 boson and the photon to quarks will have the original diagonal form even after this rotation. For example, $(\bar{u}\gamma_\mu I u)Z^\mu$ where I is the identity matrix acting on family space will transform to $(\bar{u}^0\gamma_\mu(V_u^\dagger I V_u)u^0)Z^\mu$, which is identical to $(\bar{u}^0\gamma_\mu I u^0)Z^\mu$. Similarly, $(\bar{u}^c\gamma_\mu I u^c)Z^\mu$ will transform to $(\bar{u}^{c0}\gamma_\mu I u^{c0})Z^\mu$. We see that there is no tree level flavor changing neutral current (FCNC) mediated by the Z^0 boson and the photon in the SM.

Most significantly, the transformations of Eq. (6) will bring the charged current quark interaction, which originally is of the form $\mathcal{L}_{cc} = g/\sqrt{2}(\bar{u}\gamma_\mu d)W^{+\mu} + h.c.$, into the form

$$\mathcal{L}_{cc} = \frac{g}{\sqrt{2}}[\bar{u}^0\gamma_\mu V d^0] W^{\mu+} + h.c. \quad (7)$$

where

$$V = V_u^\dagger V_d \quad (8)$$

is the quark mixing matrix, or the Cabibbo-Kobayashi-Maskawa (CKM) matrix [1, 2]. In the SM, all the flavor violation is contained in V . Being product of unitary matrices, V is itself unitary. This feature has thus far withstood experimental scrutiny, with further scrutiny expected from LHC experiments.

Note that the right-handed rotation matrices V_{uc} and V_{dc} have completely disappeared, a result of the purely left-handed nature of charged weak current.

We can repeat this process in the leptonic sector. We define, in analogy with Eq. (5),

$$\nu = V_\nu \nu^0, \quad e = V_e e^0, \quad e^c = V_{ec} e^{c0}. \quad (9)$$

We choose Y_e and Y_{ec} such that

$$Y_e^T(Y_\ell \nu)Y_{ec} = \begin{pmatrix} m_e & & \\ & m_\mu & \\ & & m_\tau \end{pmatrix}. \quad (10)$$

Note that there is no right-handed neutrino in the SM. If the Yukawa Lagrangian is as given in Eq. (1), there is no neutrino mass. In that case one can choose $V_\nu = V_e$, so that the charged current weak interactions will remain flavor diagonal. However, it is now well established that neutrinos have small masses. Additional terms must be added to Eq. (1) in

order to accommodate them. The simplest possibility is to add a non-renormalizable term

$$\mathcal{L}_{\nu\text{-mass}} = \frac{(L^T Y_\nu L) H H}{2M_*} + h.c. \quad (11)$$

where the $SU(2)_L$ contraction between the H fields is in the triplet channel and Y_ν is a complex symmetric matrix in generation space. Here M_* is a mass scale much above the weak interaction scale. Eq. (11) can arise by integrating out some heavy fields with mass of order M_* . The most celebrated realization of this is the seesaw mechanism, where M_* corresponds to the mass of the right-handed neutrino [3]. The neutrino masses are suppressed, compared to the charged fermion masses, because of the inverse dependence on the heavy scale M_* . Right-handed neutrinos, if they exist, are complete singlets of the SM gauge symmetry, and can possess bare SM invariant mass terms, unlike any other fermion of the SM. This is an elegant explanation of why the neutrinos are much lighter than other fermions, relying only on symmetry principles and dimensional analysis. Eq. (11) leads to a light neutrino mass matrix given by

$$M_\nu = Y_\nu \frac{v^2}{M_*}. \quad (12)$$

Now we choose V_ν so that

$$V_\nu^T Y_\nu \frac{v^2}{M_*} V_\nu = \begin{pmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{pmatrix}, \quad (13)$$

with $m_{1,2,3}$ being the tiny masses of the three light neutrinos. The leptonic charge current interaction now becomes

$$\mathcal{L}_{cc}^\ell = \frac{g}{\sqrt{2}} [\bar{e}^0 \gamma_\mu U \nu^0] W^{-\mu} + h.c. \quad (14)$$

where

$$U = V_e^\dagger V_\nu \quad (15)$$

is the leptonic mixing matrix, or the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix [4]. As V , U is also unitary. Neutrino oscillations observed in experiments are attributed to the off-diagonal entries of the matrix U . We assumed here that the neutrino mass generation mechanism violated total lepton number by two units. While this is very attractive, it should be mentioned that neutrinos could acquire masses very much like the quarks. That would require the right-handed ν^c states to be part of the low energy theory. M_ν will then be similar to M_ℓ of Eq. (10). Neutrino oscillation phenomenology will be identical to the case of L -violating neutrino masses. In this case, however, the neutrino Yukawa couplings will have to be extremely tiny to accommodate the observed masses. Furthermore, some global symmetries,

such as total lepton number, will have to be assumed in order to forbid gauge invariant mass terms for the right-handed neutrinos.

The fermionic states (e_i^0) are simply the physical electron, the muon, and the tau lepton states. Similarly, the quark fields with a superscript 0 are the mass eigenstates. It is conventional to drop these superscripts, which we shall do from now on.

2.1 Lepton masses

Conceptually charged lepton masses are the easiest to explain. Leptons are propagating states, and their masses are simply the poles in the propagators. Experimental information on charged lepton masses is rather accurate [5]:

$$\begin{aligned} m_e &= 0.510998902 \pm 0.000000021 \text{ MeV} , \\ m_\mu &= 105.658357 \pm 0.000005 \text{ MeV} , \\ m_\tau &= 1777.03_{-0.26}^{+0.30} \text{ MeV} . \end{aligned} \tag{16}$$

The direct kinematic limits on the three neutrino masses are [5]:

$$m_{\nu_e} \leq 3 \text{ eV} , \quad m_{\nu_\mu} \leq 0.19 \text{ MeV} , \quad m_{\nu_\tau} \leq 18.2 \text{ MeV} . \tag{17}$$

Neutrino oscillation experiments have provided much more accurate determinations of the squared mass differences $\Delta m_{ij}^2 = m_i^2 - m_j^2$. Solar and atmospheric neutrino oscillation experiments, when combined with accelerator and reactor neutrino experiments, suggest the following allowed values (with 2σ error quoted) [6]:

$$\begin{aligned} \Delta m_{21}^2 &= (7.25 - 8.11) \times 10^{-5} \text{ eV}^2 , \\ \Delta m_{31}^2 &= \pm(2.18 - 2.64) \times 10^{-3} \text{ eV}^2 . \end{aligned} \tag{18}$$

While this still leaves some room for the absolute masses, when combined with the direct limit on $m_{\nu_e} \leq 3 \text{ eV}$, the options become limited. Current data allow for two possible ordering of the mass hierarchies: (i) normal hierarchy where $m_1 \leq m_2 \ll m_3$, and (ii) inverted hierarchy where $m_1 \simeq m_2 \gg m_3$. More specifically, ν_e is mostly in the lightest eigenstate in the case of normal hierarchy, while it is mostly in the heavier eigenstate in the case of inverted hierarchy. The sign of Δm_{31}^2 is not known at the moment, which gives these two ordering options. On the other hand, the sign of Δm_{21}^2 is fixed from the condition that MSW resonance occurs inside the Sun.

2.2 Leptonic mixing matrix

The PMNS matrix U , being unitary, has N^2 independent components for N families of leptons. Out of these, $N(N-1)/2$ are Euler angles, while the remaining $N(N+1)/2$ are phases. Many of these phases can be absorbed into the fermionic fields and removed. If one writes $U = Q\hat{U}P$, where P and Q are diagonal phase matrices, then by redefining the phases of e fields as $e \rightarrow Qe$, the N phases in Q can be removed. P has only $N-1$ non-removable phases (an overall phase is irrelevant). For $N=3$, $P = \text{diag.}(e^{i\alpha}, e^{i\beta}, 1)$. α, β are called the Majorana phases. (If the neutrino masses are of the Dirac type, these phases can also be removed by redefining the ν^c fields.) \hat{U} will then have $N(N+1)/2 - (2N-1) = \frac{1}{2}(N-1)(N-2)$ phases. For $N=3$, there is a single ‘‘Dirac’’ phase in U . This single phase will be relevant for neutrino oscillation phenomenology. The two Majorana phases (α, β) do not affect neutrino oscillations, but will be relevant for neutrino-less double beta decay.

In general, the PMNS matrix for three families of leptons can be written as

$$U = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix}. \quad (19)$$

To enforce the unitarity relations it is convenient to adopt specific parametrizations. The Euler angles, as you know, can be parametrized in many different ways. Furthermore, the Dirac phase can be chosen to appear in different ways (by field redefinitions). The ‘‘standard parametrization’’ that is now widely used [5] has $U_{PMNS} = U.P$ where

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}. \quad (20)$$

Here $s_{ij} = \sin \theta_{ij}$, $c_{ij} = \cos \theta_{ij}$.

Our current understanding of these mixing angles arising from neutrino oscillations can be summarized as follows (2 σ error bars quoted) [6]:

$$\begin{aligned} \sin^2 \theta_{12} &= 0.27 - 0.35, \\ \sin^2 \theta_{23} &= 0.39 - 0.63, \\ \sin^2 \theta_{13} &\leq 0.040. \end{aligned} \quad (21)$$

Here θ_{12} limit arises from solar neutrino data (when combined with KamLand reactor neutrino data), θ_{23} from atmospheric neutrinos (when combined with MINOS accelerator neutrino data), and θ_{13} from reactor neutrino data.

It is intriguing that the current understanding of leptonic mixing can be parametrized by the unitary matrix

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} P. \quad (22)$$

This mixing is known as tri-bimaximal mixing [7]. This nomenclature is based on the numerology $\sin^2 \theta_{12} = 1/3$, $\sin^2 \theta_{23} = 1/2$, $\sin^2 \theta_{13} = 0$ that follows from Eq. (22). As we will see, such a geometric structure is far from being similar to the quark mixing matrix. Note that currently θ_{13} is allowed to be zero, in which case the Dirac phase δ becomes irrelevant. We also have no information on the Majorana phases (α, β in P), which can only be tested in neutrino-less double beta decay experiments.

There have been considerable activity in the literature in trying to reproduce the tri-bimaximal mixing matrix of Eq. (22) based on symmetries. The most popular idea has been to adopt the non-Abelian flavor symmetry A_4 , which is the symmetry group of a regular tetrahedron. It is also the group of even permutations of four letters. This finite group has twelve elements, which fall into one three-dimensional ($\mathbf{3}$) and three one-dimensional ($\mathbf{1} + \mathbf{1}' + \mathbf{1}''$) irreducible representations. A_4 is the simplest symmetry group with a triplet representation. Assigning the lepton doublets to the $\mathbf{3}$, and the three charged lepton singlets to the the ($\mathbf{1} + \mathbf{1}' + \mathbf{1}''$), it is possible, assuming a specific vacuum structure, to reproduce the “geometric” form of the leptonic mixing matrix [8].

2.3 Quark masses

Unlike the leptons, quarks are not propagating particles. So their masses have to be inferred indirectly from properties of hadrons. There are various techniques to do this. Let me illustrate this for the light quark masses (u, d, s) by the method of chiral perturbation theory [9].

Consider the QCD Lagrangian at low energy scales. Electroweak symmetry has already been broken, and heavy quarks (t, b, c) have decoupled. The Lagrangian for the light quarks (u, d, s) and the gluon fields takes the form

$$\mathcal{L} = \sum_{k=1}^{N_F=3} \bar{q}_k (i\not{D} - m_k) q_k - \frac{1}{4} G_{\mu\nu} G^{\mu\nu}, \quad (23)$$

where $G_{\mu\nu}$ is the gluon field strength and \not{D} is the covariant derivative. m_k is the mass of the k -th quark and q_k denotes the quark field. This Lagrangian has a chiral symmetry in the

limit where the quark masses vanish. The three left-handed quarks can be rotated into one another, and the three right-handed quarks can be rotated independently. The symmetry is $SU(3)_L \times SU(3)_R \times U(1)_V$, with the axial $U(1)_A$ (of the classical symmetry $U(3)_L \times U(3)_R$) explicitly broken by anomalies. The $U(1)_V$ is baryon number, which remains unbroken even after QCD dynamics. QCD dynamics breaks the $SU(3)_L \times SU(3)_R$ symmetry down to the diagonal subgroup $SU(3)_V$. In the limit of vanishing quark masses, there must be 8 Goldstone bosons corresponding to this symmetry breaking. These Goldstone bosons are identified as the pseudoscalar mesons, which are however, not exactly massless. The (small) quark masses actually break the chiral symmetry explicitly and thus generate small masses for the mesons.

Chiral perturbation theory is a systematic expansion in p/Λ_χ , where p is the particle momentum and $\Lambda_\chi \sim 1$ GeV is the chiral symmetry breaking scale. Since the masses of the light quarks (u , d , s) are smaller than Λ_χ , we can treat them as small perturbations and apply chiral expansion. The explicit breaking of chiral symmetry occurs via the mass term

$$M = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}. \quad (24)$$

M can be thought of as a spurion field which breaks the chiral symmetry spontaneously. Under $SU(3)_L \times SU(3)_R$ symmetry $q_L \rightarrow U_L q_L$, $q_R \rightarrow U_R q_R$, while $M \rightarrow U_L M U_R^\dagger$. That is, M transforms as a $(3, 3^*)$ of this group. Under the unbroken diagonal $SU(3)_V$ subgroup, both q_L and q_R transform as triplets, while M splits into a $\mathbf{1} + \mathbf{8}$. Thus M can be written as $M = M_1 + M_8$, where M_1 is a singlet of $SU(3)_V$, while M_8 is an octet:

$$\begin{aligned} M_1 &= \frac{(m_u + m_d + m_s)}{3} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \\ M_8 &= \frac{(m_u - m_d)}{2} \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} + \frac{(m_u - m_d - 2m_s)}{6} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}. \end{aligned} \quad (25)$$

The octet (under $SU(3)_V$) of mesons can be written down as a (normalized) matrix

$$\Phi = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta^0 \end{pmatrix}. \quad (26)$$

The lowest order invariants involving Φ bilinear and M are

$$A \text{Tr}(\Phi^2)M_1 + B \text{Tr}(\Phi^2 M_8). \quad (27)$$

Here A and B are arbitrary coefficients. Eq. (27) can be readily expanded, which will give relations for the masses of mesons. Now, in the limit of $m_u = 0, m_d = 0, m_s \neq 0$, the $SU(2)_L \times SU(2)_R$ chiral symmetry remains unbroken, and so the pion fields should be massless. Working out the mass terms, and demanding that the pion mass vanishes in this limit, one finds a relation $A = 2B$. Using this relation we can write down the pseudoscalar meson masses. In doing so, let us also recall that electromagnetic interactions will split the masses of the neutral and charged members. To lowest order, this splitting will be universal. Then we have

$$\begin{aligned} m_{\pi^0}^2 &= B(m_u + m_d) \\ m_{\pi^\pm}^2 &= B(m_u + m_d) + \Delta_{\text{em}} \\ m_{K^0}^2 &= m_{\bar{K}^0}^2 = B(m_d + m_s) \\ m_{K^\pm}^2 &= B(m_u + m_s) + \Delta_{\text{em}} \\ m_\eta^2 &= \frac{1}{3}B(m_u + m_d + 4m_s). \end{aligned} \quad (28)$$

Here small $\pi^0 - \eta^0$ mixing has been neglected, which vanishes in the limit $m_u - m_d$ vanishes.

Eliminating B and Δ_{em} from Eq. (28) we obtain two relations for quark mass ratios:

$$\begin{aligned} \frac{m_u}{m_d} &= \frac{2m_{\pi^0}^2 - m_{\pi^+}^2 + m_{K^+}^2 - m_{K^0}^2}{m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2} = 0.56 \\ \frac{m_s}{m_d} &= \frac{m_{K^0}^2 + m_{K^+}^2 - m_{\pi^+}^2}{m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2} = 20.1 \end{aligned} \quad (29)$$

This is the lowest order chiral perturbation theory result for the mass ratios. Second order chiral perturbation theory makes important corrections to these ratios as discussed in more detail in Ref. [10]. Note that the absolute masses cannot be determined in this way. Alternative techniques, such as QCD sum rules and lattice calculations which provide the most precise numbers have to be applied for this.

For heavy quarks (c and b), one can invoke another type of symmetry, the heavy quark effective theory (HQET) [11]. When the mass of the quark is heavier than the typical momentum of the partons $\Lambda \sim m_p/3 = 330 \text{ MeV}$, one can make another type of expansion. In analogy with atomic physics, where different isotopes exhibit similar chemical behavior, the behavior of charm hadrons and bottom hadrons will be similar. In fact, there will be an $SU(2)$ symmetry relating the two, to lowest order in HQET expansion. One consequence is that the mass splitting between the vector and scalar mesons in the b and c sector should be related. This leads

to a relations $M_{B^*} - M_B = \Lambda^2/m_b$ and $M_{D^*} - M_D = \Lambda^2/m_c$, leading to the prediction

$$\frac{M_{B^*} - M_B}{M_{D^*} - M_D} = \frac{m_c}{m_b}, \quad (30)$$

which is in good agreement with experiments.

The most reliable determination of light quark masses come from lattice QCD. The QCD Lagrangian of Eq. (23) has only very few parameters, the strong coupling constant, and the three light quark masses. All the hadron masses and decay constants should in principle be calculable in terms of these parameters. Since QCD coupling is strong at low energies, perturbation theory is not reliable. Lattice QCD is formulated on discrete space time lattice points, rather than in the continuum. When the lattice spacing takes small value, lattice QCD should reproduce continuum QCD. No approximation is made as regards the value of the strong coupling constant α_s . It is thus a non-perturbative technique which, upon matching certain measured quantities, can be used to calculate the light quark mass parameters. In the last five years there has been tremendous advances in lattice QCD, owing to improved lattice action, as well as increased computing power. Early results on light quark masses assumed “quenched”, i.e., ignored fermions propagating inside loops, but now full three flavor un-quenched calculation with dynamical fermions are available. There have been several independent evaluations of the light quark masses, which generally are in good agreement with one another. Conventionally these masses are presented as running masses at $q = 2$ GeV in the $\overline{\text{MS}}$ scheme.

The MILC collaboration [12], which adopted a partially quenched approximation, finds for the light quark masses

$$\begin{aligned} m_u(2 \text{ GeV}) &= 1.7 \pm 0.3 \text{ MeV}, \\ m_d(2 \text{ GeV}) &= 3.9 \pm 0.46 \text{ MeV}, \\ m_s(2 \text{ GeV}) &= 76 \pm 7.6 \text{ MeV}. \end{aligned} \quad (31)$$

Here I have combined the various uncertainties (statistical, systematic, simulation, and electromagnetic) in quadrature. The ratios of light quark masses are thought to be more reliable, as many of the uncertainties cancel in the ratios. It is customary to define an average mass of up and down quarks $\hat{m} = (m_u + m_d)/2$. The results of MILC collaboration corresponds to the following mass ratios:

$$\begin{aligned} \frac{m_u}{m_d} &= 0.43 \pm 0.08, \\ \frac{m_s}{\hat{m}} &= 27.4 \pm 4.2. \end{aligned} \quad (32)$$

The JLQCD collaboration [13], which includes three flavors of dynamical quarks finds

$$\begin{aligned}
\hat{m}(2 \text{ GeV}) &= 3.55_{-0.28}^{+0.65} \text{ MeV} , \\
m_s(2 \text{ GeV}) &= 90.1_{-6.1}^{+17.2} \text{ MeV} , \\
\frac{m_u}{m_d} &= 0.577 \pm 0.025 .
\end{aligned} \tag{33}$$

The RBC & UKQCD collaboration [14], which includes 2+1 dynamical domain wall quarks finds

$$\begin{aligned}
\hat{m}(2 \text{ GeV}) &= 3.72 \pm 0.41 \text{ MeV} , \\
m_s(2 \text{ GeV}) &= 107.3 \pm 11.7 \text{ MeV} , \\
\hat{m} : m_s &= 1 : 28.8 \pm 1.65 .
\end{aligned} \tag{34}$$

And finally, the HPQCD collaboration finds [15]

$$\begin{aligned}
m_u(2 \text{ GeV}) &= 1.9 \pm 0.24 \text{ MeV} , \\
m_d(2 \text{ GeV}) &= 4.4 \pm 0.34 \text{ MeV} , \\
m_s(2 \text{ GeV}) &= 87 \pm 5.7 \text{ MeV} , \\
\hat{m}(2 \text{ GeV}) &= 3.2 \pm 0.89 \text{ MeV} , \\
\frac{m_u}{m_d} &= 0.43 \pm 0.08 .
\end{aligned} \tag{35}$$

One sees that the lattice calculations are settling down, and have become quite reliable. It should be mentioned that the same lattice QCD calculations also provide several of the hadronic form factors which enter into the determination of the CKM mixing angles.

The masses of the c and b quarks can be determined in a variety of ways. Charmonium and Upsilon spectroscopy, in conjunction with lattice calculations seem to be the most reliable. We summarize the masses of these quarks thus obtained, along with the ranges for the light quark masses [5].

$$\begin{aligned}
m_u(2 \text{ GeV}) &= 1.5 \text{ to } 3.3 \text{ MeV} , \\
m_d(2 \text{ GeV}) &= 3.5 \text{ to } 6.0 \text{ MeV} , \\
m_s(2 \text{ GeV}) &= 105_{-35}^{+25} \text{ MeV} , \\
\frac{m_u}{m_d} &= 0.35 \text{ to } 0.60 , \\
\frac{m_s}{m_d} &= 17 \text{ to } 22 , \\
\frac{m_s}{(m_u + m_d)/2} &= 25 \text{ to } 30 , \\
m_c(m_c) &= 1.27_{-0.11}^{+0.07} \text{ GeV} , \\
m_b(m_b) &= 4.20_{-0.07}^{+0.17} \text{ GeV} .
\end{aligned} \tag{36}$$

Sometimes the light quark masses are quoted at $q = 1$ GeV, rather at $q = 2$ GeV. There are significant differences in these two sets of values due to the rapid running of the strong coupling in this regime. Typically one finds for example, $m_u(1 \text{ GeV}) \simeq 1.35 m_u(2 \text{ GeV})$.

The top quark mass is more directly determined leading to the value [5]

$$m_t = 171.3 \pm 1.1 \pm 1.2 \text{ GeV} . \quad (37)$$

Any ambitious theory of flavor should aim to address these observed values of quark masses.

2.4 Running quark and lepton masses

In attempting to explain the observed masses of fermions, it will be convenient to compare their masses at a common momentum scale μ . Usually this scale is taken to be much heavier than the QCD scale of about 1 GeV, or even the weak scale of 246 GeV, since new flavor dynamics cannot happen at lower scales. The measured quark and lepton masses then have to be extrapolated to a common momentum scale μ . Below the weak scale, this extrapolation would require the renormalization group evolution of the mass parameters caused by QCD and QED loops. The beta functions and the gamma functions necessary to do this have been computed to three-loop (and in some cases four-loop) accuracy [16]. In the \overline{MS} scheme, which is widely used, the contributions to the beta functions and gamma functions from a specific flavor of fermion will decouple for momenta μ less than the mass of the particle. Before discussing this evolution, it is necessary to remark on the differences between “pole mass” and “running mass” of a fermion. For heavy quarks (c, b, t) the pole mass M_q is the physical mass, which appears as the pole in the propagator. (For light quarks (u, d, s) pole mass is not defined because of the non-perturbative nature of strong interactions at their mass scales.) The running mass $m_q(M_q)$ includes corrections from QCD and QED loops. The two are related for quarks via

$$M_q = m_q(M_q) \left[1 + \frac{4}{3} \frac{\alpha_s(M_q)}{\pi} + \kappa_q^{(2)} \left(\frac{\alpha_s(M_q)}{\pi} \right)^2 + \kappa_q^{(3)} \left(\frac{\alpha_s(M_q)}{\pi} \right)^3 \right], \quad (38)$$

where terms of order α_s^4 and higher have been neglected. The two-loop and the three-loop QCD correction factors are $\{\kappa_c^{(2)}, \kappa_b^{(2)}, \kappa_t^{(2)}\} = \{11, 21, 10.17, 9.13\}$ and $\{\kappa_c^{(3)}, \kappa_b^{(3)}, \kappa_t^{(3)}\} = \{123.8, 101.5, 80.4\}$. There can be significant differences between M_q and $m_q(M_q)$. For example, using $\alpha_s(M_Z) = 0.1176$ and $M_t = 172.5$ GeV, one obtains, with QCD evolution of α_s from M_Z to M_t , $\alpha_s(M_t) = 0.108$, and then from Eq. (38) $m_t(M_t) = 162.8$ GeV. For c and b quarks the differences are even bigger.

The running masses of leptons can be defined analogously, but now the QCD corrections are replaced by QED corrections. Consequently the differences between the pole mass M_ℓ and

running mass $m_\ell(M_\ell)$ are less significant. The two masses are related via

$$m_\ell(\mu) = M_\ell \left[1 - \frac{\alpha}{\pi} \left\{ 1 + \frac{3}{2} \ln \frac{\mu}{m_\ell(\mu)} \right\} \right]. \quad (39)$$

For momentum scales higher than the electroweak symmetry breaking scale, one should evolve the Yukawa couplings of the fermions, rather than their masses. One can define the running mass in this momentum regime as

$$m_i(\mu) = Y_i(\mu) v. \quad (40)$$

Here $v = 174$ GeV is the VEV of the Higgs doublet evaluated at the weak scale. Since the VEV v also is a function of momentum (owing to wave function renormalization of the Higgs field), one could in principle define the running mass as $m_i(\mu) = Y_i(\mu)v(\mu)$. But this is usually not necessary, and will not be adopted here. The renormalization group evolution equations for the Yukawa couplings of the SM have been worked out to two-loop accuracy [16].

While extrapolating the Yukawa coupling above the weak scale one has to specify the theory valid in that regime. Often it will be assumed to be the minimal supersymmetric standard model (MSSM). In the fermion Yukawa sector there are significant differences between the MSSM and the SM. The main difference is that supersymmetry requires two Higgs doublets, H_u with $(Y/2) = +1/2$ and H_d with $(Y/2) = -1/2$. The extra doublet is needed for anomaly cancelation and also for generating all fermion masses. Recall that in the SM Yukawa interaction of Eq. (1) we used H for generating the up-type quark masses and its conjugate \tilde{H} for the down-type quark and charged lepton masses. Supersymmetric Yukawa couplings must be derived from a superpotential W , which is required to be holomorphic. This means that if H appears in W , then H^* cannot appear. The MSSM Yukawa interactions arise from the following superpotential.

$$\mathcal{W}_{\text{Yukawa}}^{\text{MSSM}} = Q^T Y_u u^c H_u - Q^T Y_d d^c H_d - L^T Y_\ell e^c H_d. \quad (41)$$

If we denote the VEVs of H_u and H_d as v_u and v_d , then the mass matrices for the three charged fermion sectors are

$$M_u = Y_u v_u, \quad M_d = Y_d v_d, \quad M_\ell = Y_\ell v_d. \quad (42)$$

The diagonalization procedure follows as in the SM. Notably, there is no Higgs boson mediated flavor changing couplings at tree level, in spite of having two Higgs doublets. The constraints of supersymmetry is the reason for its absence. (Only a single Higgs doublet couples to each one of the three sectors.) A new parameter appears, which is the ratio of the two Higgs vacuum expectation values:

$$\tan \beta = \frac{v_u}{v_d}. \quad (43)$$

$m_i \setminus \mu$	$m_c(m_c)$	2 GeV	$m_b(m_b)$	$m_t(m_t)$	1 TeV	$\Lambda_{\text{GUT}}^{\tan\beta=10}$	$\Lambda_{\text{GUT}}^{\tan\beta=50}$
$m_u(\text{MeV})$	2.57	2.2	1.86	1.22	1.10	0.49	0.48
$m_d(\text{MeV})$	5.85	5.0	4.22	2.76	2.50	0.70	0.51
$m_s(\text{MeV})$	111	95	80	52	47	13	10
$m_c(\text{GeV})$	1.25	1.07	0.901	0.590	0.532	0.236	0.237
$m_b(\text{GeV})$	5.99	5.05	4.20	2.75	2.43	0.79	0.61
$m_t(\text{GeV})$	<i>384.8</i>	<i>318.4</i>	<i>259.8</i>	162.9	150.7	92.2	94.7
$m_e(\text{MeV})$	0.4955	\sim	0.4931	0.4853	0.4959	0.2838	0.206
$m_\mu(\text{MeV})$	104.474	\sim	103.995	102.467	104.688	59.903	43.502
$m_\tau(\text{MeV})$	1774.90	\sim	1767.08	1742.15	1779.74	1021.95	773.44

Table 1: The running masses of quarks and leptons as a function of momentum μ . The last two columns correspond to the running masses at $\Lambda_{\text{GUT}} = 2 \times 10^{16}$ GeV assuming low energy MSSM spectrum with $\tan\beta = 10$ and 50.

This parameter will influence many physical processes. $\tan\beta$ plays an important role in the RGE evolution of the Yukawa couplings. The range of $\tan\beta$ preferred in the MSSM is $\tan\beta = (1.7 - 60)$. When $\tan\beta < 1.7$ the top quark Yukawa couplings blows up before the momentum scale $\mu = \Lambda_{\text{GUT}} \approx 2 \times 10^{16}$ GeV. Λ_{GUT} is associated with the scale of grand unification, where the three gauge couplings of the SM appear to meet, if there is low energy supersymmetry. For $\tan\beta > 60$ the b -quark and τ -lepton Yukawa couplings become non-perturbative before reaching Λ_{GUT} .

In Table 1 we list the running masses of quarks and leptons as a function of the momentum scale μ . We have adopted the numbers listed from Ref. [17], but our independent calculations show general agreement at the level of few per cent with Ref. [17]. The input values for (c, b, t) quarks are the running masses indicated in bold. For this Table we have used light quark masses at $\mu = 2$ GeV as indicated in bold. For the charged lepton, we have used as input the masses given in Eq. (16). The masses of all fermions are listed at momentum scale $\mu = m_t$ and $\mu = 1$ TeV assuming the validity of the SM up to 1 TeV. Also listed are the running masses at $\mu = \Lambda_{\text{GUT}} = 2 \times 10^{16}$ GeV assuming MSSM spectrum, for two values of $\tan\beta$ (10 and 50). The following input values have been used. $\alpha_s(M_Z) = 0.1176$, $\alpha^{-1}(M_Z) = 127.918$, and $\sin^2\theta_W(M_Z) = 0.23122$.

There are various noteworthy features in Table 1. The light quark masses (m_u, m_d, m_s)

decrease by about a factor of two in going from $\mu = 2$ GeV to $\mu = 1$ TeV. This decrease is a result of QCD corrections. The d and s -quark masses decrease by about another factor of 4 in going from $\mu = 1$ TeV to $\mu = \Lambda_{\text{GUT}}$, while m_u decreases by a factor of 2.3. The net change in the values of (m_u, m_d, m_s) in going from $\mu = 2$ GeV to $\mu = \Lambda_{\text{GUT}}$ for the case of $\tan\beta = 10$ is a factor (4.9, 7.9, 7.3). The value of b -quark mass decreases considerably, by a factor of 6.9, in going from $\mu = m_b$ to $\mu = \Lambda_{\text{GUT}}$ for $\tan\beta = 10$. $m_b(\mu = \Lambda_{\text{GUT}})$ is close to the τ -lepton mass $m_\tau(\mu = \Lambda_{\text{GUT}})$ (to within about 20%). The lepton masses decrease by about a factor of 2 in going from low energies to Λ_{GUT} . This decrease occurs because of the $SU(2)_L \times U(1)_Y$ contributions to the beta functions of Y_ℓ . These features will be relevant when we discuss predictions for fermion masses from Grand Unified theories in Sec. 5.

Sometimes the light quark masses are quoted at $\mu = 1$ GeV. In going from $\mu = 2$ GeV down to $\mu = 1$ GeV, the masses increase by a factor of 1.31, if $\alpha_s(M_Z) = 0.1176$ is used. The running factor to go from $\mu = 2$ GeV down to $\mu = m_c$ is indicated in Table 1, while the additional running factor to go from $\mu = m_c$ to $\mu = 1$ GeV is found to be 1.12. Thus, $(m_u, m_d, m_s) = (2.2, 5, 95)$ MeV at $\mu = 2$ GeV correspond to $(m_u, m_d, m_s) = (2.88, 6.58, 124)$ MeV at $\mu = 1$ GeV.

In Table 1 we have also included the top quark mass at momentum scales below M_t (indicated in italics). These values, which are un-physical, since the top quark decouples at its mass, will be rarely used.

2.5 Quark mixing and CP violation

The unitary matrix V of Eq. (8) which appears in the charged current interactions of Eq. (7) enters in a variety of processes. A lot of information has been gained on the matrix elements of V . The general matrix can be written as

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \quad (44)$$

The standard parametrization of V is as in Eq. (20), but now understood to be for the quark sector. V has a single un-removable phase for three families of quarks and leptons. (The phases (α, β) which appeared in the case of Majorana neutrinos can be removed by right-handed quark field redefinition.) The single un-removable phase in V allows for the violation of CP symmetry in the quark sector. Unlike in the leptonic sector, the quark mixing angles turn out to be small. This enables one to make a perturbative expansion of the mixing matrix a la Wolfenstein [18].

The small parameter is taken to be $\lambda = |V_{us}|$ in terms of which one has

$$V = \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 - \frac{1}{8}\lambda^4 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 - \frac{1}{8}\lambda^4(1 + 4A^2) & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 + \frac{1}{2}A\lambda^4(1 - 2(\rho + i\eta)) & 1 - \frac{1}{2}A^2\lambda^4 \end{pmatrix} + \mathcal{O}(\lambda^5). \quad (45)$$

Here the exact correspondence with Eq. (20) is given by

$$s_{12} \equiv \lambda, \quad s_{23} \equiv A\lambda^2, \quad s_{13}e^{-i\delta} \equiv A\lambda^3(\rho - i\eta). \quad (46)$$

Matrix elements of V are determined usually via semileptonic decays of quarks. In Fig. 1 we have displayed the dominant processes enabling determination of these elements. Fig. 1 (a) is the diagram for nuclear beta decay, from which $|V_{ud}|$ has been extracted rather accurately [19]:

$$|V_{ud}| = 0.97377 \pm 0.00027. \quad (47)$$

Fig. 1 (b) shows semileptonic K decay from which the Cabibbo angle $|V_{us}|$ can be extracted. The decays $K_L^0 \rightarrow \pi\ell\nu$ and $K^\pm \rightarrow \pi^0\ell^\pm\nu$ ($\ell = e, \mu$) have been averaged to obtain for the product $|V_{us}|f_+(0) = 0.21668 \pm 0.00045$. Here $f_+(0)$ is the form factor associated with this semileptonic decay evaluated at $q^2 = 0$. Using $f_+(0) = 0.961 \pm 0.008$ (obtained from QCD calculations, which are in agreement with lattice QCD evaluations), one obtains

$$|V_{us}| = 0.2257 \pm 0.0021. \quad (48)$$

$|V_{cd}|$ is extracted from $D \rightarrow K\ell\nu$ and $D \rightarrow \pi\ell\nu$ decays with assistance from lattice QCD for the computation of the relevant form factors. V_{cs} is determined from semileptonic D decays and from leptonic D_s decay ($D_s^+ \rightarrow \mu^+\nu$), combined with lattice calculation of the decay form factor f_{D_s} . Both $|V_{cd}|$ and $|V_{cs}|$ have rather large errors currently:

$$\begin{aligned} |V_{cd}| &= 0.230 \pm 0.011, \\ |V_{cs}| &= 0.957 \pm 0.010. \end{aligned} \quad (49)$$

$|V_{cb}|$ is determined from both inclusive and exclusive decays of B hadrons into charm, yielding a value

$$|V_{cb}| = (41.6 \pm 0.6) \times 10^{-3}. \quad (50)$$

$|V_{ub}|$ is determined from charmless B decays and gives

$$|V_{ub}| = (4.31 \pm 0.30) \times 10^{-3}. \quad (51)$$

Elements $|V_{td}|$ and $|V_{ts}|$ cannot be currently determined, for a lack of top quark events, but can be inferred from B meson mixings where these elements appear through the box diagram. The result is

$$|V_{td}| = (7.4 \pm 0.8) \times 10^{-3},$$

$$\frac{|V_{td}|}{|V_{ts}|} = 0.208 \pm 0.008. \quad (52)$$

Fig. 1 (f) depicts the decay of top quark into $W + b$. It can also decay into $W + q$ where q is d, s, b . By taking the ratio of branching ratios $R = B(t \rightarrow Wb) / \sum_q B(t \rightarrow Wq)$, CDF and D0 have arrived at a limit on $|V_{td}| > 0.74$ [5].

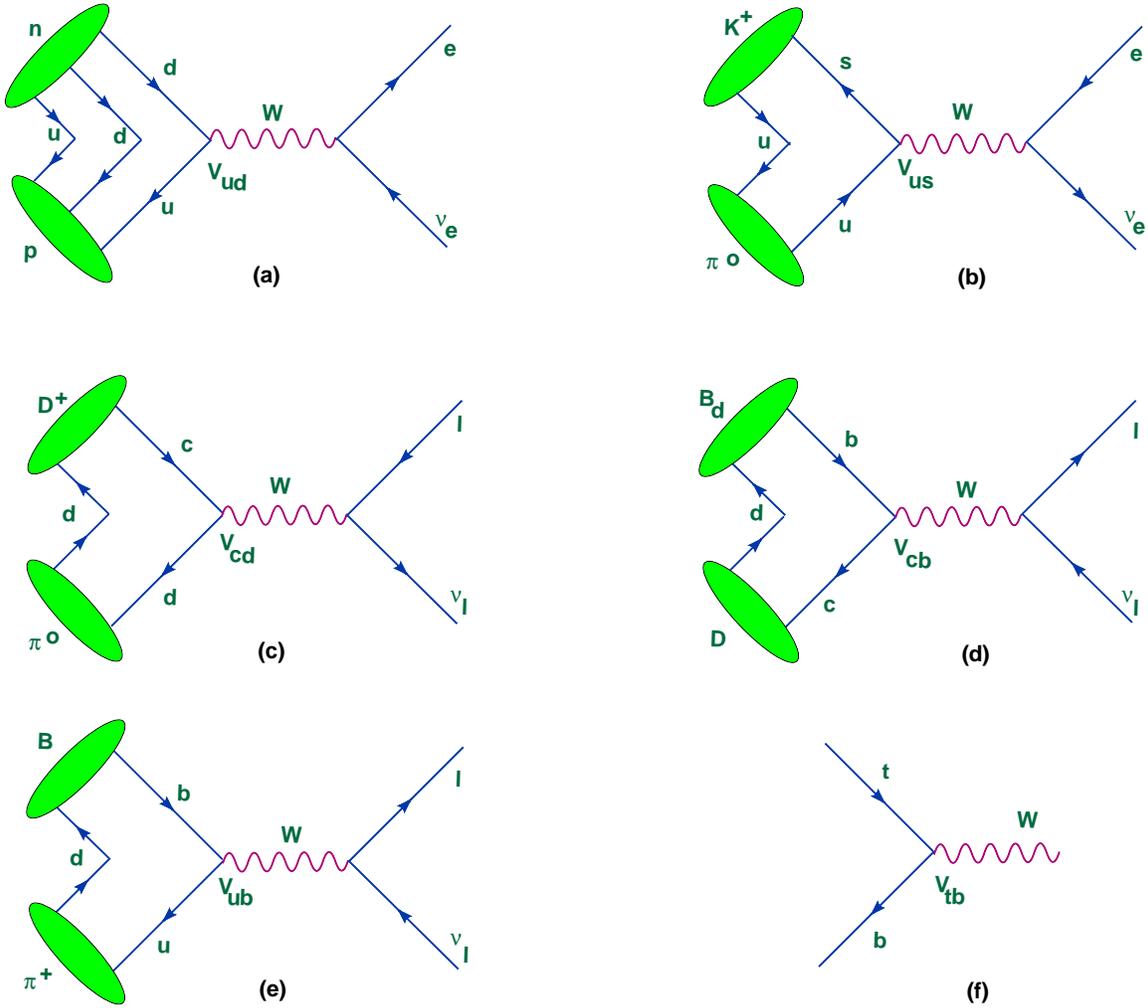


Figure 1: Processes determining $|V_{ij}|$.

2.5.1 Heavy quark symmetry

Heavy quark symmetry plays an important role in the determination of $|V_{ub}|$ and $|V_{cb}|$. While a thorough discussion of HQET (Heavy Quark Effective Theory) is outside the scope of this write-up, it would be useful to get a feeling of some of the ideas involved. We refer the reader to Ref. [11] for a thorough review, and Ref. [20] for a pedagogical exposure.

Consider first the purely leptonic decay $B^- \rightarrow \ell^- \bar{\nu}_\ell$ for $\ell = e, \mu, \tau$. The transition amplitude for this decay is

$$T_{fi} = \frac{G_F}{\sqrt{2}} V_{ub} [\bar{u}_\ell \gamma_\mu (1 - \gamma_5) u_\nu] \langle 0 | \bar{u} \gamma^\mu (1 - \gamma_5) b | B^- \rangle. \quad (53)$$

Here G_F is the Fermi coupling. To compute the decay rate, the hadronic matrix element for the transition of B meson to vacuum needs to be evaluated. Note that the matrix element of vector current between pseudoscalar meson and vacuum vanishes: $\langle 0 | \bar{u} \gamma^\mu b | B^- \rangle = 0$, while the axial vector current matrix element is parametrized as $\langle 0 | \bar{u} \gamma^\mu \gamma_5 b | B^- \rangle = i f_B q^\mu$, with f_B being the B meson decay constant and q^μ the B meson momentum. With this matrix element, the decay rate can be readily computed. One obtains

$$\Gamma(B^- \rightarrow \ell^- \bar{\nu}_\ell) = \frac{G_F^2}{8\pi} f_B^2 |V_{ub}|^2 M_B m_\ell^2 \left(1 - \frac{m_\ell^2}{M_B^2}\right)^2. \quad (54)$$

Note the helicity suppression, which implies that the number of events in this channel will be small. Recently BELLE collaboration has observed the decay $B^- \rightarrow \tau^- \bar{\nu}$ with a 3.5 sigma statistical significance. Their results can be converted to a value for the product $|V_{ub}| f_B$ as

$$|V_{ub}| f_B = [10.1_{-1.4}^{+1.6}(\text{stat})_{-1.4}^{+1.3}(\text{syst})] \times 10^{-4} \text{ GeV}. \quad (55)$$

Using lattice evaluations of f_B , one can obtain the value of $|V_{ub}|$ from Eq. (55). The accuracy of this determination, which is rather direct, suffers from the lack of events for this helicity suppressed decay.

Semileptonic decays do not suffer from the helicity suppression, and are therefore more promising. Unlike a single form factor that appears in the purely leptonic decay, now there will be two form factors. These two can be related via heavy quark symmetry, as we outline below. Consider the decay $\bar{B}_d^0 \rightarrow D^+ \ell \bar{\nu}_\ell$ which proceeds via Fig. 1 (d). The transition amplitude for this decay has the form

$$T_{fi} = \frac{G_F}{\sqrt{2}} V_{cb} [\bar{u}_\ell \gamma_\mu (1 - \gamma_5) u_\nu] \langle D^+ | \bar{c} \gamma^\mu (1 - \gamma_5) b | \bar{B}_d^0 \rangle. \quad (56)$$

A similar expression is obtained for the decay $\bar{B}_d^0 \rightarrow \pi^+ \ell \bar{\nu}_\ell$, with $|V_{cb}|$ replaced by $|V_{ub}|$ in Eq. (56). The matrix element of axial vector current between two pseudoscalar mesons vanishes:

$\langle D^+ | \bar{u} \gamma^\mu \gamma_5 b | \bar{B}_d^0 \rangle = 0$. The vector current matrix element between two pseudoscalar mesons contains two form factors:

$$\langle D^+(k) | \bar{u} \gamma^\mu b | \bar{B}_d^0(p) \rangle = F_1(q^2) \left[(p+k)_\mu - \frac{M_B^2 - M_D^2}{q^2} q_\mu \right] + F_0(q^2) \frac{M_B^2 - M_D^2}{q^2} q_\mu, \quad (57)$$

where $q = p - k$.

To see how HQET can relate the two form factors $F_1(q^2)$ and $F_0(q^2)$, let me briefly review the crucial elements of HQET. In a hadron composed of one heavy (b) quark and one light anti-quark \bar{u} or \bar{d} , the mass of b is much larger than the scale of QCD dynamics, Λ_{QCD} . The b quark is then almost on-shell, moving with a velocity close to the hadron's four velocity. We write this as

$$p_Q^\mu = m_Q v^\mu + k^\mu, \quad (58)$$

where $k \ll m_Q$ is the residual momentum, and $v^2 = 1$. The b quark interacts with the light degrees of freedom, but such interactions can cause a change in the residual momentum by $\Delta k \sim \Lambda_{\text{QCD}} \ll m_Q$. Thus $\Delta v \rightarrow 0$ as $\Lambda_{\text{QCD}}/m_Q \rightarrow 0$.

In the heavy quark symmetry limit ($\Lambda_{\text{QCD}}/m_Q \rightarrow 0$), the elastic scattering process $\bar{B}(v) \rightarrow \bar{B}(v')$ has the amplitude

$$\frac{1}{M_B} \langle \bar{B}(v') | \bar{b}(v') \gamma_\mu b(v) | \bar{B}(v) \rangle = \xi(v'.v) (v + v')_\mu. \quad (59)$$

A term of the type $(v - v')_\mu$ cannot appear on the right-hand side of Eq. (59) since $\not{v} b_v = b_v$ and $\bar{b}_{v'} \not{v}' = \bar{b}_{v'}$. The $1/M_B$ factor in Eq. (59) is associated with normalization of states, so the right-hand side of Eq. (59) has no dependence on the heavy quark flavor. Current conservation implies $\xi(v'.v = 1) = 1$, so that the function $\xi(v'.v)$, the Isgur–Wise function [21], is independent of the heavy quark flavor. Thus, in the heavy quark symmetry limit, we have

$$\frac{1}{\sqrt{M_D M_B}} \langle D(v') | \bar{c}_{v'} \gamma_\mu b_v | \bar{B}(v) \rangle = \xi(v.v') (v + v')_\mu. \quad (60)$$

This transition is now governed by a single form factor, $\xi(v'.v)$ with $\xi(1) = 1$. Comparing with Eq. (57), one finds

$$\begin{aligned} F_1(q^2) &= \frac{M_D + M_B}{2\sqrt{M_D M_B}} \xi(w) \\ F_0(q^2) &= \frac{2\sqrt{M_D M_B}}{M_D + M_B} \left(\frac{1+w}{2} \right) \xi(w) \end{aligned} \quad (61)$$

where

$$w = v_D.v_B = \frac{M_D^2 + M_B^2 - q^2}{2M_D M_B}. \quad (62)$$

As an application of these ideas, consider the decay $\overline{B} \rightarrow D^* \ell \nu$. The differential decay rate for this process can be written as

$$\frac{d\Gamma}{dw} = G_F^2 K F(w)^2 |V_{cb}|^2, \quad (63)$$

where K is a known kinematic function and $F(w)$ is related to the Isgur–Wise function (up to perturbative QCD corrections). It should obey the normalization

$$F(1) = \eta_A(\alpha_s) \left[1 + \frac{0}{m_c} + \frac{0}{m_b} + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{m_{b,c}^2}\right) \right]. \quad (64)$$

Here $\eta_A(\alpha_s)$ is a perturbatively calculable function. Note that $\mathcal{O}(\Lambda_{\text{QCD}}/m_{c,b})$ corrections vanish [22]. This decay distribution can be measured as a function of w , from which $F(w)|V_{cb}|$ can be extracted. Now, when extrapolated to zero recoil limit ($w = 1$), whence the decay rate vanishes), from Eq. (64), one obtains a value of $|V_{cb}|$.

2.5.2 CP violation

Charge conjugation (C) takes a particle to its antiparticle, Parity (spatial reflection) changes the helicity of the particle. Under CP, e_L^- will transform to e_R^+ . Both C and P are broken symmetries in the SM, but the product CP is approximately conserved. Violation of CP has been seen only in weak interactions. The CKM mechanism predicts CP violation through a single complex phase that appears in the CKM matrix. Thus in the SM, various CP violating processes in K , B and other systems get correlated. So far such correlations have been consistent with CKM predictions, but more precise determinations in the B and D systems at the LHC may open up new physics possibilities.

In the $K^0 - \overline{K}^0$ system, CP violation has been observed both in mixing and in direct decays. CP violation in mixing arises in the SM via the W -boson box diagram shown in Fig. 2. The CP asymmetry in mixing is parametrized by ϵ , which is a measure of the mixing between the CP even and CP odd states $K_{1,2}^0 = (K^0 \pm \overline{K}^0)/\sqrt{2}$. It has been measured to be

$$|\epsilon| = (2.229 \pm 0.010) \times 10^{-3}. \quad (65)$$

The measured value is in excellent agreement with expectations from the SM, and enables us to determine the single phase of the CKM matrix. The box diagram contribution to ϵ is given by

$$\begin{aligned} |\epsilon| &= \frac{G_F^2 f_k^2 m_K m_W^2}{12\sqrt{2}\pi^2 \Delta m_K} \hat{B}_K \{ \eta_c S(x_c) \text{Im}[(V_{cs} V_{cd}^*)^2] \\ &+ \eta_t S(x_t) \text{Im}[(V_{ts} V_{td}^*)^2] + 2\eta_{ct} S(x_c, x_t) \text{Im}[V_{cs} V_{cd}^* V_{ts} V_{td}^*] \}. \end{aligned} \quad (66)$$

Here $S(x)$ and $S(x, y)$ are Inami–Lim functions [23] with $x_{c,t} = m_{c,t}^2/M_W^2$, and the η factors are QCD correction factors for the running of the effective $\Delta S = 2$ Hamiltonian from M_W to the hadron mass scale.

The direct CP violation parameter that leads to the decay $K \rightarrow \pi\pi$ has also been measured, leading to the value

$$Re(\epsilon'/\epsilon) = (1.65 \pm 0.26) \times 10^{-3}. \quad (67)$$

These decays occur via the penguin diagrams shown in Fig. 3. There are electromagnetic penguins and gluonic penguins, which tend to cancel each other. While the KM model predicts non-zero value of ϵ'/ϵ , estimating this value reliably has been difficult, partly because of this cancelation. Most estimates are in agreement with observations.

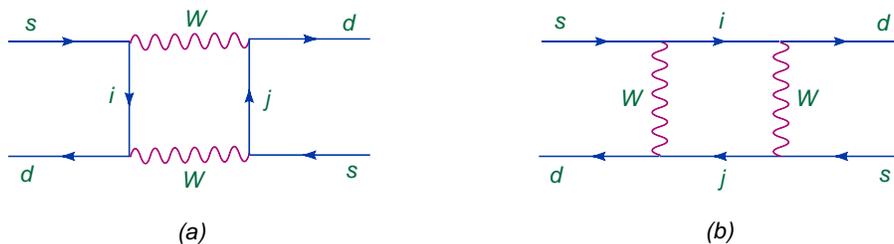


Figure 2: Box diagram inducing $K^0 - \overline{K^0}$ transition in the SM.

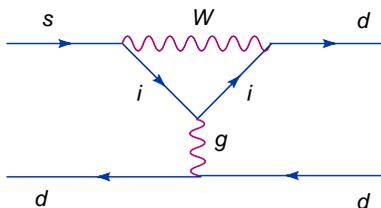


Figure 3: One loop penguin diagram that generates CP violation in direct $K \rightarrow \pi\pi$ decay.

A wealth of information has been gained about CP violation from the B factories over the last decade. CP violation in B meson system is now well established. Several CP violating quantities have been measured in B_d meson system [20], all of which show consistency with the CKM mixing matrix. Unitarity of the CKM matrix implies that $\sum_i V_{ij}V_{ik}^* = \delta_{jk}$ and $\sum_j V_{ij}V_{kj}^* = \delta_{ik}$. There are six vanishing combinations, which can be expressed as triangles in the complex plane. The areas of all of these triangles are the same. The most commonly used

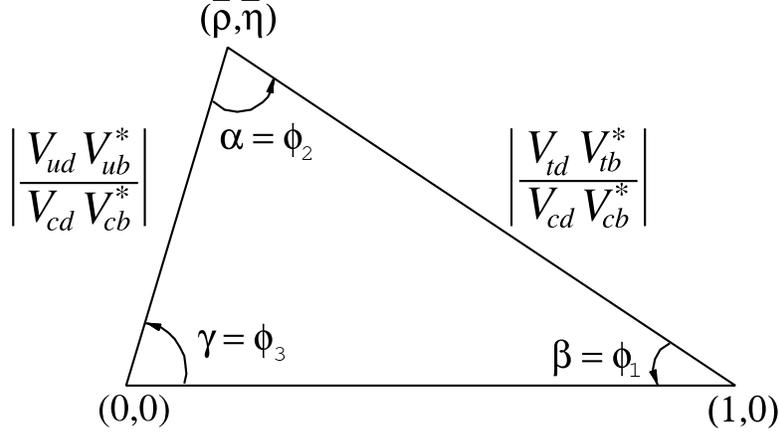


Figure 4: Unitarity triangle in the CKM model.

triangle arises from the relation

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0. \quad (68)$$

In the complex plane, the resulting triangle has sides of similar length (of order λ^3). This unitarity triangle relation is shown in Fig. 4. The three interior angles (α, β, γ) , also referred to as (ϕ_2, ϕ_1, ϕ_3) , can be written in the CKM model as

$$\begin{aligned} \alpha &= \arg\left(\frac{-V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right) \simeq \arg\left(-\frac{1-\rho-i\eta}{\rho+i\eta}\right), \\ \beta &= \arg\left(\frac{-V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right) \simeq \arg\left(\frac{1}{1-\rho-i\eta}\right), \\ \gamma &= \arg\left(\frac{-V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right) \simeq \arg(\rho+i\eta). \end{aligned} \quad (69)$$

One experimental test of the CKM mechanism is the measurement of $\alpha + \beta + \gamma = 180^\circ$.

The angle β can be measured with the least theoretical uncertainty from the decay of $B_d \rightarrow J/\psi K_S$. It is found to be

$$\sin 2\beta = 0.68 \pm 0.03. \quad (70)$$

This value is in good agreement with the CKM prediction.

The angle α is measured from decay modes where $b \rightarrow u\bar{u}d$ is dominant. Such decays include $B \rightarrow \pi\pi$, $B \rightarrow \rho\rho$ and $B \rightarrow \pi\rho$. The value of α extracted is

$$\alpha = (88_{-5}^{+6})^\circ. \quad (71)$$

The angle γ does not depend on the top quark, and can in principle be measured from tree-level decays of B meson. Strong interaction uncertainties are rather large in decays such

as $B^\pm \rightarrow D^0 K^\pm$. The current value of the angle γ is

$$\gamma = (77^{+30}_{-32})^\circ. \quad (72)$$

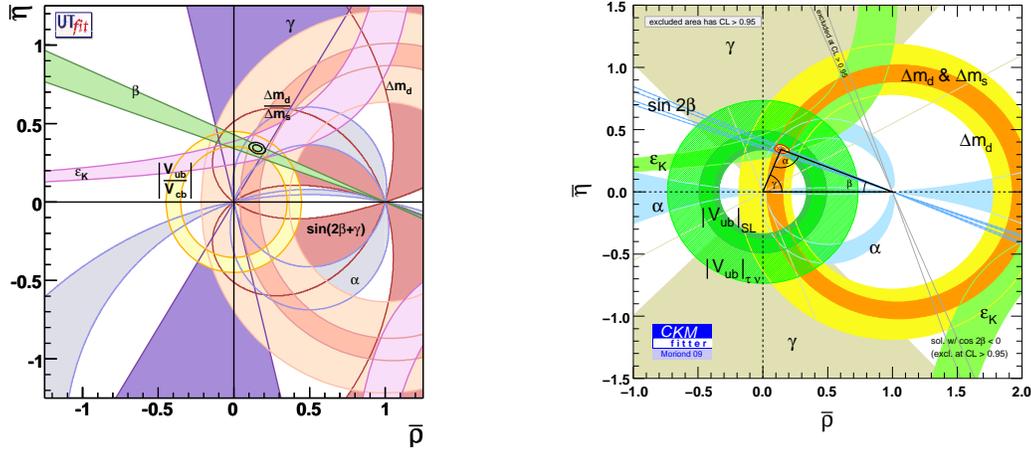


Figure 5: Global fit to the mixing and CP violation data from the UTfit collaboration (left panel) [24] and the CKMFitter collaboration (right panel) [25].

The current situation with the CKM mixing angles and CP violation phase is depicted in Fig. 5. The left panel is the result of a global analysis of flavor mixing and CP violation data by the UTfit [24] collaboration, while the right panel depicts the results from an independent CKMFitter [25] collaboration. The Wolfenstein parameters $\bar{\eta}$ is plotted against $\bar{\rho}$ in these figures. Here $\bar{\eta} = \eta(1 - \lambda^2/2)$ and $\bar{\rho} = \rho(1 - \lambda^2/2)$. A variety of input parameters have gone into these fits. Some of the constraints used are explicitly indicated in these figures. It is very non-trivial that the various constraint curves have a common intersection. This demonstrates the success of the CKM mechanism of flavor mixing and CP violation. The intersection of the various ellipses gives the best fit value for the Wolfenstein parameters (λ, A, ρ, η) , which are as follows [5]:

$$\lambda = 0.2272 \pm 0.0010, \quad A = 0.818^{+0.007}_{-0.017}, \quad \rho = 0.221^{+0.064}_{-0.028}, \quad \eta = 0.340^{+0.017}_{-0.045}. \quad (73)$$

Theories of flavor should provide an understanding of these fundamental parameters.

3 Relating quark mixings and mass ratios

Having reviewed the fundamental flavor parameters of the Standard Model, now we turn to attempts which explain some of the observed features. Necessarily one needs to invoke non-

standard physics, which can be potentially tested at colliders.

We begin with a simple idea of relating quark masses and mixings by virtue of flavor symmetries. In the quark sector we have seen that the mass ratios such as m_d/m_s , m_u/m_c , etc are strongly hierarchical, while the mixing angles, such as V_{us} are also hierarchical, although the hierarchy here is not as strong. Can the quark mixing angle be computed in terms of the quark mass ratios? Clearly such attempts have to go beyond the SM. Here I give a simple two-family example which assumes a flavor $U(1)$ symmetry that distinguishes the two families.

3.1 Prediction for Cabibbo angle in a two family model

Consider the mass matrices for (u, c) and (d, s) quarks given by [26]

$$M_u = \begin{pmatrix} 0 & A_u \\ A_u^* & B_u \end{pmatrix}, \quad M_d = \begin{pmatrix} 0 & A_d \\ A_d^* & B_d \end{pmatrix}. \quad (74)$$

The crucial features of these matrices are (i) the zeros in the (1,1) entries, and (ii) their hermiticity. Neither of these features can be realized within the SM. Recall that the SM symmetry would have arbitrary non-hermitian matrices for M_u and M_d . The zero entries in Eq. (74) can be enforced by a flavor $U(1)$ symmetry, the hermitian nature can be obtained if the gauge sector is left-right symmetric. Before constructing such a model, let us examine the consequences of Eq. (74). Matrices in Eq. (74) have factorizable phases. That is, $M_u = P_u \hat{M}_u P_u^*$, where \hat{M}_u has the same form as M_u but with all entries real, and where $P_u = \text{diag}(e^{i\alpha_u}, 1)$ is a diagonal phase matrix. A similar factorization applies to M_d with a phase matrix $P_d = \text{diag}(e^{i\alpha_d}, 1)$. We can absorb these phase matrices into the quark fields, but since $\alpha_u \neq \alpha_d$, the matrix $P_u^* P_d = \text{diag}(e^{i\psi}, 1)$ will appear in the charged current matrix ($\psi = \alpha_d - \alpha_u$). The matrices \hat{M}_u and \hat{M}_d , which have all real entries, can be diagonalized readily, yielding for the mixing angles θ_u and θ_d

$$\begin{aligned} \tan^2 \theta_u &= \frac{m_u}{m_c}, \\ \tan^2 \theta_d &= \frac{m_d}{m_s}. \end{aligned} \quad (75)$$

This yields a prediction for the Cabibbo angle [26]

$$|\sin \theta_C| \simeq \left| \sqrt{\frac{m_d}{m_s}} - e^{i\psi} \sqrt{\frac{m_u}{m_c}} \right|. \quad (76)$$

This formula works rather well, especially since even without the second term, the Cabibbo angle is correctly reproduced. The phase ψ is a parameter, however, its effect is rather restricted.

For example, since $\sqrt{m_d/m_s} \simeq 0.22$ and $\sqrt{m_u/m_c} \simeq 0.07$, $|\sin \theta_C|$ must lie between 0.15 and 0.29, independent of the value of ψ .

Now to a possible derivation of Eq. (74). Since SM interactions do not conserve Parity, it is useful to extend the gauge sector to the left-right symmetric group $G \equiv SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$, wherein Parity invariance can be imposed [27]. The (1,2) and (2,1) elements of $M_{u,d}$ being complex conjugates of each other will then result. The left-handed and the right-handed quarks transform as $Q_{iL}(3, 2, 1, 1/3) + Q_{iR}(3, 1, 2, 1/3)$ under G . Under discrete parity operation $Q_{iL} \leftrightarrow Q_{iR}$. This symmetry can be consistently imposed, as $W_L \leftrightarrow W_R$ in the gauge sector under Parity. The leptons transform as $\psi_{iL}(1, 2, 1, -1) + \psi_{iR}(1, 1, 2, -1)$ under the gauge symmetry. Note that ψ_R , which is a doublet of $SU(2)_R$, contains the right-handed neutrino, as the partner of e_R . Thus there is a compelling reason for the existence of ν_R , unlike in the SM, where it is optional.

The Higgs field that couples to quarks should be $\Phi(1, 2, 2, 0)$, and under Parity $\Phi \rightarrow \Phi^\dagger$. In matrix form Q_{iL}, Q_{iR}, Φ read as

$$Q_{iL} = \begin{pmatrix} u_i \\ d_i \end{pmatrix}_L, \quad Q_{iR} = \begin{pmatrix} u_i \\ d_i \end{pmatrix}_R, \quad \Phi = \begin{pmatrix} \phi_1^0 & \phi_2^+ \\ \phi_1^- & \phi_2^0 \end{pmatrix}, \quad (77)$$

so that the Yukawa Lagrangian for quarks

$$\mathcal{L}_{\text{Yukawa}} = \overline{Q}_L \Phi Y Q_R + \overline{Q}_L \tilde{\Phi} \tilde{Y} Q_R + h.c. \quad (78)$$

is gauge invariant. Here $\tilde{\Phi} \equiv \tau_2 \Phi^* \tau_2$. Imposing Parity, we see that the Yukawa matrices Y and \tilde{Y} must be hermitian, $Y = Y^\dagger$ and $\tilde{Y} = \tilde{Y}^\dagger$. This is the desired result for deriving Eq. (74). The VEVs $\langle \phi_1^0 \rangle$ and $\langle \phi_2^0 \rangle$ can be complex in general, but this will not affect the prediction for the Cabibbo angle of Eq. (76), since that only requires $|(M_{u,d})_{12}| = |(M_{u,d})_{21}|$. Additional Higgs fields, eg., $\Delta_L(1, 3, 1, 2) + \Delta_R(1, 1, 3, 2)$, would be required for breaking the left-right symmetric gauge group down to the SM and for simultaneously generating large ν_R Majorana masses. However, these fields do not enter into the mass matrices of quarks.

To enforce zeros in the (1,1) entries of $M_{u,d}$ of Eq. (74), we can employ the following $U(1)$ flavor symmetry: $Q_{1L} : 2, Q_{1R} : -2, Q_{2L} : 1, Q_{2R} : -1, \Phi_1 : 2, \Phi_2 : 3$. Note that two Higgs bidoublet fields are needed. Φ_1 generates the (2,2) entries, while Φ_2 generates the (1,2) and (2,1) entries. There is no (1,1) entry generated, since there is no Higgs field with $U(1)$ charge of +4. Note also that the $\tilde{\Phi}_{1,2}$ fields, which have $U(1)$ charges $(-2, -3)$, do not couple to the quarks.

While we cannot determine the scale of flavor dynamics in this model, the $U(1)$ flavor symmetry and the left-right symmetry, which were crucial for the derivation of Eq. (76),

could show up as new particles at the LHC. In general, one would also expect multiple Higgs bosons. We should note that the full theory is more elaborate compared to the minimal left-right symmetry without the flavor symmetry (two, instead of one bi-doublet Higgs fields are needed), but the effective theory is simpler, with the mass matrices being predictive.

3.2 Three family generalization

Eq. (74) can be generalized for the case of three families, a la Fritzsch [28]. The up and down quark mass matrices have hermitian nearest neighbor interaction form:

$$M_{u,d} = \begin{pmatrix} 0 & A & 0 \\ A^* & 0 & B \\ 0 & B^* & C \end{pmatrix}_{u,d}. \quad (79)$$

Such matrices have factorizable phases, i.e., $M_{u,d} = P_{u,d} \hat{M}_{u,d} P_{u,d}^*$, where $\hat{M}_{u,d}$ are the same as in Eq. (79), but without any phases, and $P_{u,d}$ are diagonal phase matrices. Only two combinations of phases will enter into the CKM matrix, contained in the matrix $P = P_u^* P_d = \text{diag.}\{e^{i\alpha}, e^{i\beta}, 1\}$. The CKM matrix is then given by

$$V = O_u^T P O_d, \quad (80)$$

where $O_{u,d}$ are the orthogonal matrices that diagonalize $\hat{M}_{u,d}$ via

$$O_{u,d}^T \hat{M}_{u,d} \hat{M}_{u,d}^T O_{u,d} = \text{diag.}\{m_{u,d}^2, m_{c,s}^2, m_{t,b}^2\}. \quad (81)$$

In this model there are a total of eight parameters that describe quark masses, mixings and CP violation: six real parameters from $\hat{M}_{u,d}$ and the two phases (α, β). (Note that the six mixing angles that enter into O_u and O_d are determined in terms of the quark mass ratios.) These eight parameters must describe ten observables in the quark sector. There are thus two true predictions. Furthermore, since (α, β) are phases, they do not count as full parameters. One finds four relations between masses and mixings [29]:

$$\begin{aligned} |V_{us}| &\simeq \left| \sqrt{\frac{m_d}{m_s}} - e^{i\psi} \sqrt{\frac{m_u}{m_c}} \right|, \\ |V_{cb}| &\simeq \left| \sqrt{\frac{m_s}{m_b}} - e^{i\phi} \sqrt{\frac{m_c}{m_t}} \right|, \\ |V_{ub}| &\simeq \left| \frac{m_s}{m_b} \sqrt{\frac{m_d}{m_b}} + e^{i\psi} \sqrt{\frac{m_u}{m_c}} \left(\sqrt{\frac{m_s}{m_b}} - e^{i\phi} \sqrt{\frac{m_c}{m_t}} \right) \right|, \\ |V_{td}| &\simeq \left| \frac{m_c}{m_t} \sqrt{\frac{m_u}{m_t}} + e^{i\psi} \sqrt{\frac{m_d}{m_s}} \left(\sqrt{\frac{m_c}{m_t}} - e^{i\phi} \sqrt{\frac{m_s}{m_b}} \right) \right|. \end{aligned} \quad (82)$$

Here the two phases ψ and ϕ are related to the phases in the diagonal matrix P as $\psi = (\alpha - \beta)$ and $\phi = \beta$. Note that all independent elements of V are determined in this model in terms of the quark mass ratios and two phase parameters. In the expression for $|V_{ub}|$ in Eq. (82), the first term is numerically $\simeq 6 \times 10^{-4}$, which is about a factor of 8 less than the value of $|V_{ub}|$. Similarly, the first term in the expression for $|V_{td}|$ is $\sim 1 \times 10^{-5}$, which is negligible in relation to the value of $|V_{td}|$. (For these numerical estimates, I used the values of the running masses given in Table 1 evaluated at $\mu = 1$ TeV.) If these terms are neglected, one would have the following predictions:

$$\frac{|V_{ub}|}{|V_{cb}|} \simeq \sqrt{\frac{m_u}{m_c}}, \quad \frac{|V_{td}|}{|V_{ts}|} \simeq \sqrt{\frac{m_d}{m_s}}. \quad (83)$$

These predictions are consistent with experimental data.

While the prediction for $|V_{us}|$ in the Fritzsche ansatz is the same as in the two family model of Eq. (76), which is successful, the relation for $|V_{cb}|$ will predict the mass of the top quark to be in the range (40 – 80) GeV, which is now excluded by data.

There have been attempts to fix the problem of Fritzsche mass matrices by modifying its form slightly. If small (2,2) elements are allowed in $M_{u,d}$, the troublesome relation for $|V_{cb}|$ will be removed. However, adding (2,2) entries in M_u and M_d introduces two more complex parameters, and such a model will have no true prediction. The relation of Eq. (76) will however be maintained, provided that the (2,2) entries are not too large. Furthermore, the relations of Eq. (83), $|V_{ub}|/|V_{cb}| \simeq \sqrt{m_u/m_c}$ and $|V_{td}|/|V_{ts}| \simeq \sqrt{m_d/m_s}$, will be preserved [30], if the new (2,2) entries are small perturbations.

A different alternative is to make the (2,3) and (3,2) entries of Eq. (79) different, while maintaining the relations between (1,2) and (2,1) entries. This can be achieved by non-Abelian discrete symmetries. Again, the number of parameters will increase by two compared to the original Fritzsche ansatz. A special case where there is still a true prediction is worth mentioning.

Consider a non-Abelian discrete subgroup G of $SU(2)$ serving as a family symmetry. G is assumed to have pseudo-real doublet representations, just as $SU(2)$. Let the first two families of quarks be pseudo-real doublets of G , while the third family quarks are singlets of G . A *real* Higgs doublet which is a true singlet of G will generate the (3,3) entries of $M_{u,d}$ as well as (1,2) and (2,1) entries. Note that invariance under G will lead to the (1,2) entry being the negative of the (2,1) entry, a property of the original $SU(2)$ family symmetry. Now, if G is broken by a Higgs field transforming as a doublet of G , then unequal (2,3) and (3,2) entries in $M_{u,d}$ can be generated. This is a concrete modification of the Fritzsche ansatz, with (1,2) and (2,1) entries having the same magnitude, but with the (2,3) and (3,2) entries unrelated.

A model of the type just described has been constructed in Ref. [31]. It is based on the dihedral group Q_6 which contains pseudo-real as well as real doublets. Most interestingly, if

the origin of CP violation is taken to be spontaneous, then the phase matrix P appearing in Eq. (80) will have the form $P = \text{diag.}\{e^{-i\phi}, e^{i\phi}, 1\}$. (This happens since M_u and M_d each will have a single phase, appearing in the (2,3) and (3,2) entries, apart from irrelevant overall phases, if all the Yukawa couplings are assumed to be real by virtue of CP invariance.) Such a model will have one true prediction, since now there are nine parameters describing ten observables. It was found in Ref. [31] that this prediction, which relates $\bar{\eta}$ with $\bar{\rho}$, is fully consistent with data.

We shall return to mass matrix “textures” of the type described here when discussing fermion masses in the context of Grand Unification in Sec. 5. The nearest neighbor interaction, not necessarily symmetrical, will find useful applications.

4 Froggatt–Nielsen mechanism for mass hierarchy

The hierarchy in the masses and mixings of quarks and leptons can be understood by assuming a flavor $U(1)$ symmetry under which the fermions are distinguished. In this approach developed by Froggatt and Nielsen [32], there is a “flavon” field S , which is a scalar, usually a SM singlet field, which acquires a VEV and breaks the $U(1)$ symmetry. This symmetry breaking is communicated to the fermions at different orders in a small parameter $\epsilon = \langle S \rangle / M_*$. Here M_* is the scale of flavor dynamics, and usually is associated with some heavy fermions which are integrated out. The nice feature of this approach is that the mass and mixing hierarchies will be explained as powers of the expansion parameter ϵ without assuming widely different Yukawa couplings. The effective theory below M_* is rather simple, while the full theory will have many heavy fermions, called Froggatt–Nielsen fields.

4.1 A two family model

Let me illustrate this idea with a two family example which is realistic when applied to the second and third families of quarks. Consider M_u and M_d for the (c , t) and (s , b) sectors given by

$$M_u = \begin{pmatrix} \epsilon^4 & \epsilon^2 \\ \epsilon^2 & 1 \end{pmatrix} v_u, \quad M_d = \begin{pmatrix} \epsilon^3 & \epsilon^3 \\ \epsilon & \epsilon \end{pmatrix} v_d. \quad (84)$$

Here $\epsilon \sim 0.2$ is a flavor symmetry breaking parameter. Every term in Eq. (84) has an order one coefficient which is not displayed. We obtain from Eq. (84) the following relations for quark

masses and $|V_{cb}|$:

$$\frac{m_c}{m_t} \sim \epsilon^4, \quad \frac{m_s}{m_b} \sim \epsilon^2, \quad |V_{cb}| \sim \epsilon^2. \quad (85)$$

All of these relations work well, for $\epsilon \sim 0.2$. Although precise predictions have not been made, one has a qualitative understanding of the hierarchies.

How do we arrive at Eq. (84)? We do it in two stages. First, let us look at the effective Yukawa couplings, which can be obtained from the Lagrangian:

$$\begin{aligned} \mathcal{L}_{FN}^{\text{eff}} = & [Q_3 u_3^c H_u + Q_2 u_3^c H_u S^2 + Q_3 u_2^c H_u S^2 + Q_2 u_2^c H_u S^4] \\ & + [Q_3 d_3^c H_d S + Q_3 d_2^c H_d S + Q_2 d_2^c H_d S^3 + Q_2 d_3^c H_d S^3] + h.c. \end{aligned} \quad (86)$$

Here I assumed supersymmetry, so that there are two Higgs doublets $H_{u,d}$. It is not necessary to assume SUSY, one can simply identify H_u as H of SM, and replace H_d by \tilde{H} . In Eq. (86) all couplings are taken to be of order one. The symmetry of Eq. (86) is a $U(1)$ with the following charge assignment.

$$\{Q_3, u_3^c\} : 0; \quad \{Q_2, u_2^c\} : 2; \quad \{d_2^c, d_3^c\} : 1; \quad \{H_u, H_d\} : 0; \quad S : -1. \quad (87)$$

Now we wish to obtain Eq. (86) by integrating out certain Froggatt–Nielsen fields. This is depicted in Fig. 6 via a set of “spaghetti” diagrams. As you can see, there are a variety of fields denoted by G_i, \overline{G}_i ($i = 1 - 4$) for the up–quark mass generation. G_i have the same gauge quantum numbers as the u^c quark of SM, while \overline{G}_i have the conjugate quantum numbers. F_i have the quantum numbers of d^c quark, while \overline{F}_i the conjugate quantum numbers.

You can readily read off the flavor $U(1)$ charges of the various F_i and G_i fields from the spaghetti diagrams. For example, the charge of G_1 is -2 , while that of \overline{G}_1 is $+2$. The charges of G_2 is -1 and that of \overline{G}_2 is $+1$.

All flavor dynamics in this class of models could occur near the Planck scale. As long as the hierarchy between $\langle S \rangle$ and the masses of the Froggatt–Nielsen fields is not too strong, realistic fermion masses will be generated. Consider for example, Fig. 6 (b) which induces the b –quark mass. The effective interaction from this diagram goes as $\mathcal{L}_b^{\text{eff}} = Y_1 Y_2 (Q_3 d_3^c H_d) (S/M_{F_1})$, where $Y_{1,2}$ are order one Yukawa couplings. If $\langle S \rangle / M_{F_1} \sim 0.2$ or so, realistic b –quark mass is obtained (with $\tan \beta \sim 10$). This allows for both $\langle S \rangle$ and M_{F_1} to be near the Planck scale. From Fig. 6 (f), one can read off the effective Lagrangian inducing the c –quark mass: $\mathcal{L}_c^{\text{eff}} = \Pi_{i=1}^5 Y'_i (Q_2 u_3^c H_u) (S^4/M_G^4)$. Here Y'_i are order one Yukawa couplings, and we assumed that all of G_i ($i = 1 - 4$) appearing in Fig. 6 (f) have a common mass M_G . With all couplings being order one, $m_c/m_t \sim 1/400$ can be reproduced, with $\epsilon \sim 0.2$. It should be emphasized that, although there are various Yukawa couplings, all of them can take order one values.

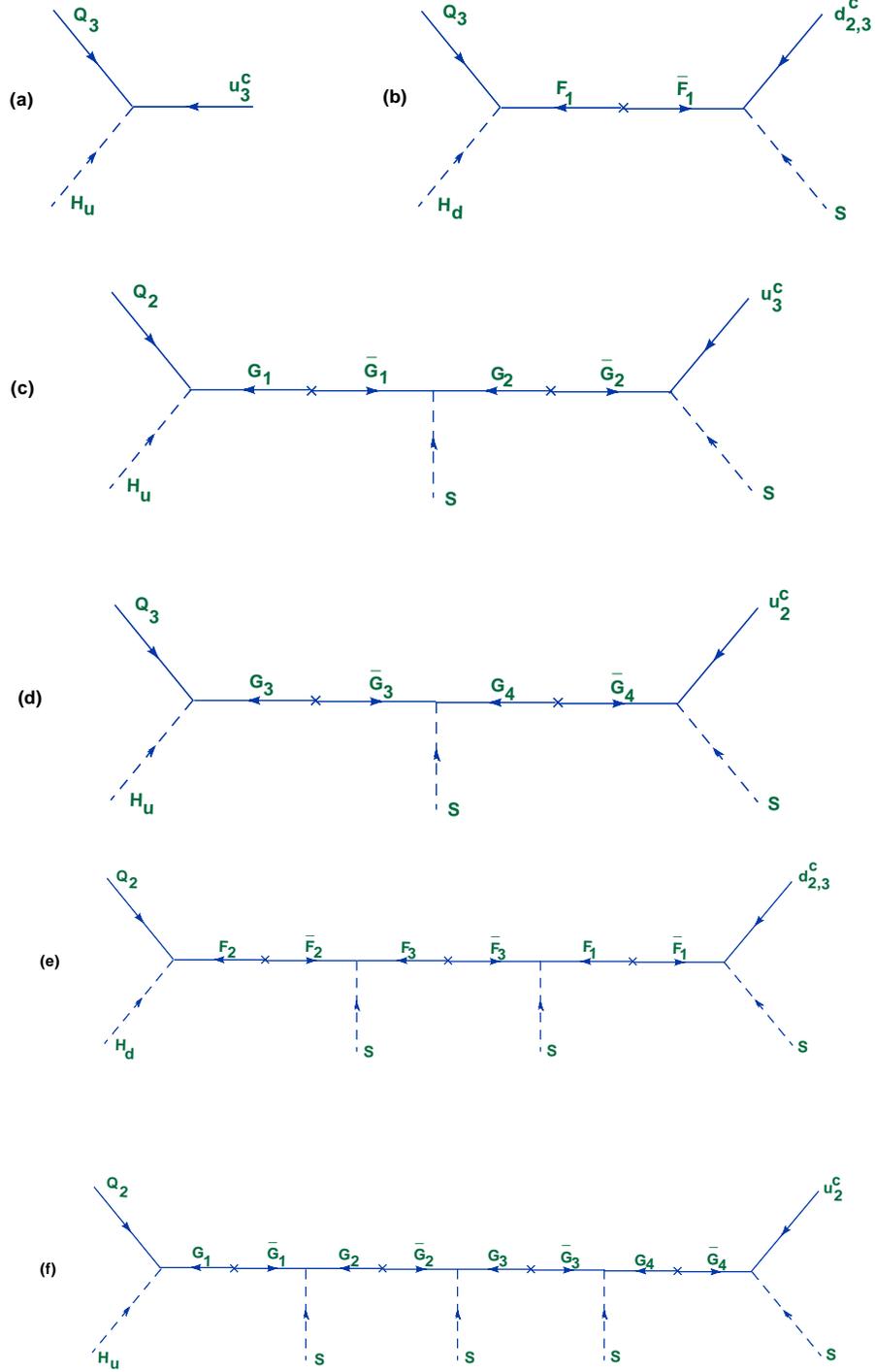


Figure 6: Froggatt–Nielsen fields generating effective Yukawa couplings of Eq. (86).

4.2 A realistic three family Froggatt–Nielsen model

Actually the flavor $U(1)$ that we used in the previous section is anomalous. String theory, when compactified to four dimension, generically gives an anomalous $U(1)_A$ with anomaly cancelation occuring by the Green–Schwarz mechanism [33]. In this case, we can get rid of the complicated Froggatt–Nielsen fields, and simply write down higher dimensional operators suppressed by the string scale. A bonus in this approach is that the small expansion parameter ϵ can be computed in specific models, where it tends to come out close to 0.2, of order the Cabibbo angle.

An explicit and complete anomalous $U(1)$ model that fits well all quark and lepton masses and mixings is constructed below. Consider the quark and lepton mass matrices of the following form [34]:

$$\begin{aligned}
 M_u &\sim \langle H_u \rangle \begin{pmatrix} \epsilon^8 & \epsilon^6 & \epsilon^4 \\ \epsilon^6 & \epsilon^4 & \epsilon^2 \\ \epsilon^4 & \epsilon^2 & 1 \end{pmatrix}, & M_d &\sim \langle H_d \rangle \epsilon^p \begin{pmatrix} \epsilon^5 & \epsilon^4 & \epsilon^4 \\ \epsilon^3 & \epsilon^2 & \epsilon^2 \\ \epsilon & 1 & 1 \end{pmatrix}, \\
 M_e &\sim \langle H_d \rangle \epsilon^p \begin{pmatrix} \epsilon^5 & \epsilon^3 & \epsilon \\ \epsilon^4 & \epsilon^2 & 1 \\ \epsilon^4 & \epsilon^2 & 1 \end{pmatrix}, & M_{\nu D} &\sim \langle H_u \rangle \epsilon^s \begin{pmatrix} \epsilon^2 & \epsilon & \epsilon \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix}, \\
 M_{\nu^c} &\sim M_R \begin{pmatrix} \epsilon^2 & \epsilon & \epsilon \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix} \Rightarrow M_{\nu}^{light} \sim \frac{\langle H_u \rangle^2}{M_R} \epsilon^{2s} \begin{pmatrix} \epsilon^2 & \epsilon & \epsilon \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix}. \tag{88}
 \end{aligned}$$

Here we work with the MSSM gauge group with supersymmetry realized at the TeV scale. Each entry has an order one pre-factor in the matrices of Eq. (88), which is not explicitly shown. These matrices can be obtained by the $U(1)$ charge assignment of Table 2. In Eq. (88), the integer p is allowed to take values 0, 1 or 2, corresponding to $\tan \beta$ taking large, medium or small values. The integer s only enters into neutrino masses. Green–Schwarz anomaly cancelation condition requires $s = p$ in the simplest scheme. With $s = p$, the charge assignment of Table 2 will be compatible with $SU(5)$ unification. That is to say that the $\{Q_i, u_i^c, e_i^c\}$ fields of a given generation all have the same $U(1)_A$ charge, and similarly, the $\{d_i^c, L_i\}$ fields of a given family have the same charge. As we discuss in Sec. 5, the former set of SM particles are grouped into a $\mathbf{10}$ of $SU(5)$, while the latter set forms a $\bar{\mathbf{5}}$.

In the last line of Eq. (88), M_{ν^c} stands for the heavy ν^c Majorana mass matrix. When the seesaw formula is applied one obtains the light neutrino mass matrix M_{ν}^{light} , shown also in the last line of Eq. (88).

Field	$U(1)_A$ Charge	Charge notation
Q_1, Q_2, Q_3	4, 2, 0	q_i^Q
L_1, L_2, L_3	$1 + s, s, s$	q_i^L
u_1^c, u_2^c, u_3^c	4, 2, 0	q_i^u
d_1^c, d_2^c, d_3^c	$1 + p, p, p$	q_i^d
e_1^c, e_2^c, e_3^c	$4 + p - s, 2 + p - s, p - s$	q_i^e
$\nu_1^c, \nu_2^c, \nu_3^c$	1, 0, 0	q_i^ν
H_u, H_d, S	0, 0, -1	(h, \bar{h}, q_s)

Table 2: The flavor $U(1)_A$ charge assignment for the MSSM fields and the flavon field S .

All the qualitative features of quark and lepton masses and mixings are reproduced by these matrices. These include small quark mixings and large neutrino mixings. The mass ratios in the up–quark sector scale as $m_u : m_c : m_t \sim \epsilon^8 : \epsilon^4 : 1$, while those in the down quarks scale as $m_d : m_s : m_b \sim \epsilon^5 : \epsilon^2 : 1$ with an identical scaling for the charged lepton mass ratios. (See the diagonal entries of $M_{u,d,e}$ in Eq. (88).) These are all consistent with experimental data. The quark mixing angles scale roughly as the down quark mass ratios, which is also reasonable. In the charged lepton sector, the mixing angles are larger, compared to the quark sector. This arises because of the lopsided structure of M_d and M_e with $M_d \sim M_e^T$. This is a feature of $SU(5)$ grand unification, where left–handed lepton doublets are paired with the conjugate of the right–handed down quarks. As a result, the left–handed leptonic mixing angles will be related to the right–handed down quark mixing angles, which are allowed to be large since they are unobservable in the SM [35]. Note also that the hierarchy between light neutrino masses is weaker, $(m_1 : m_2 : m_3) \sim (\epsilon^2 : 1 : 1)$, compared with the charged fermion mass hierarchy. This feature is also consistent with neutrino oscillation data.

A variety of models based on anomalous $U(1)$ flavor symmetry have been proposed in the literature. A cross section of these models can be found in Ref. [36, 37, 38].

4.2.1 More about anomalous $U(1)$ flavor symmetry

To see the consistency of the three family model described above, and to see how it may be subject to experimental scrutiny, let us explore the structure of anomalous $U(1)$ flavor symmetry and its applications a little further. This will also enable us to compute the small parameter ϵ in the model of Table 2.

In heterotic string theory the $U(1)_A$ anomalies are canceled by the Green–Schwarz mecha-

nism [33] which requires

$$\frac{A_1}{k_1} = \frac{A_2}{k_2} = \frac{A_3}{k_3} = \frac{A_F}{3k_F} = \frac{A_{gravity}}{24}. \quad (89)$$

Here A_1 , A_2 , A_3 , A_F and $A_{gravity}$ are the $U(1)_Y^2 \times U(1)_A$, $SU(2)_L^2 \times U(1)_A$, $SU(3)_C^2 \times U(1)_A$, $U(1)_A^3$ and $(Gravity)^2 \times U(1)_A$ anomaly coefficients. (The subscript F is used to indicate the anomalous $U(1)$ flavor symmetry group.) All other anomalies (such as $U(1)_A^2 \times U(1)_Y$) must vanish. k_i ($i = 1, 2, 3$), k_F are the Kac-Moody levels. The non-Abelian levels k_2 and k_3 must be integers. The factor $1/3$ in front of the cubic anomaly A_F has a combinatorial origin owing to the three identical $U(1)_A$ gauge boson legs.

Even without a covering grand unified group, string theory predicts unification of all gauge couplings, including that of the $U(1)_A$ and g_F , at the fundamental scale M_{st} [39, 40]:

$$k_i g_i^2 = k_F g_F^2 = 2g_{st}^2. \quad (90)$$

Here g_i are the $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$ gauge couplings for $i = 1, 2, 3$.

With $k_2 = k_3 = 1$ we find from Table 2, $A_2 = (19 + 3s)/2$ and $A_3 = (19 + 3p)/2$. Eq. (89) then requires $p = s$, i.e., a common exponent for the charged lepton and the neutrino Dirac Yukawa coupling matrices. With $p = s$, the condition $A_1/k_1 = A_2/k_2$ fixes k_1 to be $5/3$, which is consistent with $SU(5)$ unification. Note also that the charges given in Table 2 become compatible with $SU(5)$ unification. Since $\text{Tr}(Y) = 0$ for the fermion multiplets of $SU(5)$, and since the Higgs doublets carry zero $U(1)_A$ charge, the anomaly coefficient $[U(1)_A]^2 \times U(1)_Y$ vanishes, as required. The last equality in Eq. (89) requires

$$A_{gravity} = \text{Tr}(q) = 12(19 + 3p). \quad (91)$$

This cannot be satisfied with the MSSM fields alone, since $\text{Tr}(q)_{MSSM} = 5(13 + 3p)$, which does not match Eq. (91). We cancel this anomaly by introducing MSSM singlet fields X_k obeying $\text{Tr}(q)_X = A_{gravity} - \text{Tr}(q)_{MSSM} = 163 + 21p$. If all the X_k fields have the same charge equal to $+1$, they will acquire masses of order $M_{st}\epsilon^2$ through the coupling $X_k X_k S^2/M_{st}$ and will decouple from low energy theory. We will assume that these fields X_k have charge $+1$.

With the charges of all fields fixed, we are now in a position to determine the $U(1)_A$ charge normalization so that $g_F^2 = g_2^2 = g_3^2$ at the string scale, (We take $k_2 = k_3 = 1$.) This normalization factor, which we denote as $|q_s|$, is given by $|q_s| = 1/\sqrt{k_F}$. All the charges given in Table 2 are to be multiplied by $|q_s|$. From the Green-Schwarz anomaly cancelation condition $A_F/(3k_F) = A_2/k_2$, we have

$$\frac{\text{Tr}(q^3)}{3k_F} = \frac{19 + 3p}{2k_2}, \quad (92)$$

from which we find the normalization of the $U(1)_A$ charge $|q_s| = 1/\sqrt{k_F}$ to be

$$|q_s| = (0.179, 0.186, 0.181) \text{ for } p = (0, 1, 2). \quad (93)$$

The Fayet–Iliopoulos term for the anomalous $U(1)_A$, generated through the gravitational anomaly, is given by [41]

$$\xi = \frac{g_{st}^2 M_{st}^2}{192\pi^2} |q_s| A_{gravity}, \quad (94)$$

where g_{st} is the unified gauge coupling at the string scale (see Eq. (90)). By minimizing the potential from the $U(1)_A$ D -term

$$V = \frac{|q_s|^2 g_F^2}{8} \left(\frac{\xi}{|q_s|} - |S|^2 + \sum_a q_a^f |\tilde{f}_a|^2 + \sum_k q_k^X |X_k|^2 \right)^2, \quad (95)$$

in such a way that supersymmetry remains unbroken (\tilde{f}_a are the MSSM sfermions and X_k are the singlet fields, which do not acquire VEVs), one finds for the VEV of S

$$\epsilon = \langle S \rangle / M_{st} = \sqrt{g_{st}^2 A_{gravity} / 192\pi^2}. \quad (96)$$

For the fermion mass texture in Eq. (88), corresponding to the $U(1)_A$ charges given in Table 2, we find

$$\epsilon = (0.177, 0.191, 0.204) \text{ for } p = (0, 1, 2). \quad (97)$$

This shows that the small expansion parameter can indeed be calculated in string–inspired models. It should be noted that this is a bottom–up approach to model building, it would of course be desirable to start from string theory and arrive at the spectrum and charges listed in Table 2.

The masses of the $U(1)_A$ gauge boson and the corresponding gaugino are obtained from $M_F = |q_s| g_F \langle S \rangle / \sqrt{2}$ and found to be

$$M_F = \left(\frac{M_{st}}{54.5}, \frac{M_{st}}{52.5}, \frac{M_{st}}{53.9} \right) \text{ for } p = (0, 1, 2). \quad (98)$$

In the momentum range below M_{st} and above M_F , these gauge particles will be active and will induce flavor dependent corrections to the sfermion soft masses and the A -terms. Implications of these effects have been studied in Ref. [34, 38], where it has been shown that the process $\mu \rightarrow e\gamma$ in this class of models is very close to the current experimental limits. Ongoing MEG experiment should be able to probe the entire allowed parameter space of these models, provided that the SUSY particles have masses not exceeding about 1 TeV.

4.3 The SM Higgs boson as the flavon

Can the SM Higgs field itself be the flavon field? Clearly, then new flavor dynamics must happen near the TeV scale. This is apparently possible with significant consequences for Higgs boson physics, as I shall now outline [42, 43].

Consider an expansion in $H^\dagger H/M^2$, which is a SM singlet that can play the role of S . Here H is the SM Higgs doublet and M is the scale of new physics. Immediately you may wonder how this is possible, since $H^\dagger H$ cannot carry any $U(1)$ quantum number. But think of SUSY at the TeV scale. SUSY has two Higgs doublets, H_u and H_d , in which case the combination $H_u H_d$ can carry $U(1)$ charge. When reduced to SM this expansion in terms of $H^\dagger H$ can be consistent.

Consider the following mass matrices for quarks in terms of the expansion parameter

$$\epsilon = \frac{v}{M}. \quad (99)$$

$$M_u = \begin{pmatrix} h_{11}^u \epsilon^6 & h_{12}^u \epsilon^4 & h_{13}^u \epsilon^4 \\ h_{21}^u \epsilon^4 & h_{22}^u \epsilon^2 & h_{23}^u \epsilon^2 \\ h_{31}^u \epsilon^4 & h_{32}^u \epsilon^2 & h_{33}^u \end{pmatrix} v, \quad M_d = \begin{pmatrix} h_{11}^d \epsilon^6 & h_{12}^d \epsilon^6 & h_{13}^d \epsilon^6 \\ h_{21}^d \epsilon^6 & h_{22}^d \epsilon^4 & h_{23}^d \epsilon^4 \\ h_{31}^d \epsilon^6 & h_{32}^d \epsilon^4 & h_{33}^d \epsilon^2 \end{pmatrix} v. \quad (100)$$

The charged lepton mass matrix is taken to have a form similar to M_d , with the couplings h_{ij}^d replaced by h_{ij}^ℓ . These matrices give good fit to masses and mixings, as in the case of anomalous $U(1)$ model with $\epsilon \sim 1/7$ and all the couplings $h_{ij}^{u,d}$ being of order one. The masses of the quarks and leptons can be read off from Eq. (100) in the approximation $\epsilon \ll 1$:

$$\begin{aligned} \{m_t, m_c, m_u\} &\simeq \{|h_{33}^u|, |h_{22}^u| \epsilon^2, |h_{11}^u - h_{12}^u h_{21}^u / h_{22}^u| \epsilon^6\} v, \\ \{m_b, m_s, m_d\} &\simeq \{|h_{33}^d| \epsilon^2, |h_{22}^d| \epsilon^4, |h_{11}^d| \epsilon^6\} v, \\ \{m_\tau, m_\mu, m_e\} &\simeq \{|h_{33}^\ell| \epsilon^2, |h_{22}^\ell| \epsilon^4, |h_{11}^\ell| \epsilon^6\} v. \end{aligned} \quad (101)$$

The quark mixing angles are found to be:

$$\begin{aligned} |V_{us}| &\simeq \left| \frac{h_{12}^d}{h_{22}^d} - \frac{h_{12}^u}{h_{22}^u} \right| \epsilon^2, \\ |V_{cb}| &\simeq \left| \frac{h_{23}^d}{h_{33}^d} - \frac{h_{23}^u}{h_{33}^u} \right| \epsilon^2, \\ |V_{ub}| &\simeq \left| \frac{h_{13}^d}{h_{33}^d} - \frac{h_{12}^u h_{23}^d}{h_{22}^u h_{33}^d} - \frac{h_{13}^u}{h_{33}^u} \right| \epsilon^4. \end{aligned} \quad (102)$$

With $\epsilon = 1/6.5$ and with all couplings $h_{ij}^{u,d}$ being of order one, excellent fits to the quark

masses and CKM mixing angles can be found. As an example, take the couplings to be

$$\begin{aligned}
\{|h_{33}^u|, |h_{22}^u|, |h_{11}^u - h_{12}^u h_{21}^u / h_{22}^u|\} &\simeq \{0.96, 0.14, 0.95\}, \\
\{|h_{33}^d|, |h_{22}^d|, |h_{11}^d|\} &\simeq \{0.68, 0.77, 1.65\}, \\
\{|h_{33}^\ell|, |h_{22}^\ell|, |h_{11}^\ell|\} &\simeq \{0.42, 1.06, 0.21\}.
\end{aligned} \tag{103}$$

The corresponding quark masses at $\mu = m_t(m_t)$ are:

$$\begin{aligned}
\{m_t, m_c, m_u\} &\simeq \{166, 0.60, 2.2 \times 10^{-3}\} \text{ GeV}, \\
\{m_b, m_s, m_d\} &\simeq \{2.78, 7.5 \times 10^{-2}, 3.8 \times 10^{-3}\} \text{ GeV}, \\
\{m_\tau, m_\mu, m_e\} &\simeq \{1.75, 0.104, 5.01 \times 10^{-4}\} \text{ GeV}.
\end{aligned} \tag{104}$$

All these are in agreement with values quoted in Table 1. Furthermore, the CKM mixing angles are also reproduced correctly with this choice of couplings.

In this scheme, the Yukawa coupling matrices of the physical quark fields are no longer proportional to the corresponding mass matrices. We obtain for the Yukawa couplings,

$$Y_u = \begin{pmatrix} 7h_{11}^u \epsilon^6 & 5h_{12}^u \epsilon^4 & 5h_{13}^u \epsilon^4 \\ 5h_{21}^u \epsilon^4 & 3h_{22}^u \epsilon^2 & 3h_{23}^u \epsilon^2 \\ 5h_{31}^u \epsilon^4 & 3h_{32}^u \epsilon^2 & h_{33}^u \end{pmatrix}, \quad Y_d = \begin{pmatrix} 7h_{11}^d \epsilon^6 & 7h_{12}^d \epsilon^6 & 7h_{13}^d \epsilon^6 \\ 7h_{21}^d \epsilon^6 & 5h_{22}^d \epsilon^4 & 5h_{23}^d \epsilon^4 \\ 7h_{31}^d \epsilon^6 & 5h_{32}^d \epsilon^4 & 3h_{33}^d \epsilon^2 \end{pmatrix}. \tag{105}$$

Take for example, the (3,3) entry in M_d . It arises from the operator $h_{33}^d Q_3 d_3^c \tilde{H} (H^\dagger H) / M^2$. The contribution to the mass matrix from this operator is $h_{33}^d v \epsilon^2$, while the contribution to the Yukawa coupling is $(h/\sqrt{2}) h_{33}^d (3\epsilon^2)$. The flavor factors (3 in this example) are not the same for various entries, and would result in flavor violation in Higgs interactions.

There is a tree-level contribution mediated by the Higgs boson for $K^0 - \bar{K}^0$ mass difference in this scheme. The new contribution, $\Delta m_K^{\text{Higgs}}$, is given by

$$\begin{aligned}
\Delta m_K^{\text{Higgs}} &\simeq \frac{4 f_K^2 m_K B_K}{3 m_{h^0}^2} \epsilon^{12} \left[\left\{ \frac{1}{6} \frac{m_K^2}{(m_d + m_s)^2} + \frac{1}{6} \right\} \text{Re} \left[\left(\frac{h_{12}^d + h_{21}^{d*}}{\sqrt{2}} \right)^2 \right] \right. \\
&\quad \left. - \left\{ \frac{11}{6} \frac{m_K^2}{(m_d + m_s)^2} + \frac{1}{6} \right\} \text{Re} \left[\left(\frac{h_{21}^d - h_{12}^{d*}}{\sqrt{2}} \right)^2 \right] \right].
\end{aligned} \tag{106}$$

Here B_K is the bag parameter. Using $B_K = 0.75$, $f_K \simeq 160$ MeV, $\epsilon \simeq 1/6.5 m_s(1 \text{ GeV}) = 175$ MeV, $m_d(1 \text{ GeV}) = 8.9$ MeV, and with $h_{12}^d = 1$, $h_{21}^d = 0.5$, we obtain $\Delta m_K^{\text{Higgs}} \simeq 3.1 \times 10^{-17}$ GeV, for $m_{h^0} = 100$ GeV. This is two orders of magnitude below the experimental value. We see broad consistency with data, primarily because of the appearance of high powers of ϵ in

processes involving the light generations. For heavy flavors, this suppression is not that strong. For example, the $\bar{t}ch^0$ vertex has a coefficient

$$\mathcal{L}_{\text{FCNC}}^{t-c} = \frac{2\epsilon^2 h^0}{\sqrt{2}}(h_{23}^u c t^c + h_{32}^u t c^c) + h.c. \quad (107)$$

This can lead to a branching ratio for $t \rightarrow ch^0$ at the level of $(0.1 - 1)\%$, depending on the actual value of the order one coupling h_{ij}^u . This decay may be observable at the LHC.

The most striking signature of this scenario is that the decay branching ratios of the Higgs boson will be modified considerably compared to the SM. Decays into light fermions are enhanced, while decay into W pair is not. For a specific set of flavor quantum numbers, the decay branching ratios are shown in Fig. 7, adopted from Ref. [43]. The solid lines correspond to branching ratios in the present model, while the dashed lines are the corresponding ones in the SM. Note that the branching ratio for $h \rightarrow b\bar{b}$ is enhanced. While the $h \rightarrow WW^*$ decay rate becomes comparable to $h \rightarrow b\bar{b}$ in the SM for a Higgs boson mass of 135 GeV, this crossover occurs at $m_h = 175$ GeV in the present case. Branching ratio for $h \rightarrow \mu^+\mu^-$ has increased, while the branching ratio for $h \rightarrow \gamma\gamma$ has diminished. These predictions are readily testable at the LHC once the Higgs boson is detected.

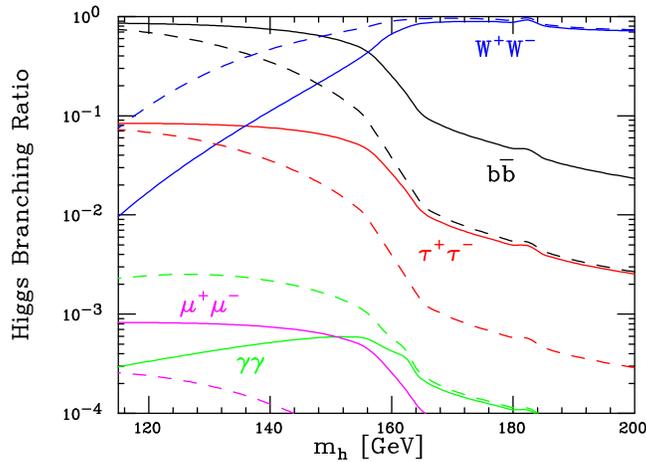


Figure 7: Higgs branching ratios with the SM Higgs as a flavon field [43]. The solid lines correspond to branching ratios with Higgs as a flavon, while the dashed lines are the corresponding SM branching fractions.

5 Grand Unification and the flavor puzzle

In this section we will develop ideas of Grand Unification which can provide significant insight into the flavor puzzle. When assisted by flavor symmetries, grand unified theories (GUTs) have great potential for addressing many of the puzzles.

Grand Unification is an ambitious program that attempts to unify the strong, weak and electromagnetic interactions [44, 45, 46]. It is strongly suggested by the unification of gauge couplings that happens in the minimal supersymmetric standard model. This is shown in Fig. 8, where the three gauge couplings of the standard model are extrapolated to high energies assuming weak scale supersymmetry. It is clear that data supports the merging of all three couplings to a common value. Besides its aesthetic appeal, in practical terms, grand unified theories reduce the number of parameters. For example, the three gauge couplings of the SM are unified into one at a very high energy scale $\Lambda_{\text{GUT}} \simeq 2 \times 10^{16}$ GeV. The apparent differences in the strengths of the various forces is attributed to the spontaneous breakdown of the GUT symmetry to the MSSM and the resulting renormalization flow of the gauge couplings. SUSY GUTs are perhaps the best motivated extensions of the SM. They explain the quantization of electric charge, as well as the quantum numbers of quarks and leptons. They provide ideal settings for understanding the flavor puzzle, which will be the focus of this discussion.

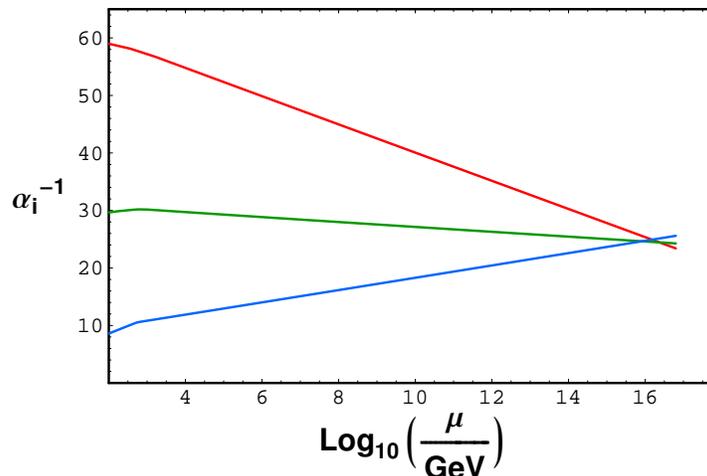


Figure 8: Evolution of the inverse gauge couplings (α_1^{-1} , α_2^{-1} , α_3^{-1}) (from top to bottom) in the MSSM as a function of momentum.

The simplest GUT model is based on $SU(5)$ [45]. I will assume low energy supersymmetry, motivated by the gauge coupling unification and a solution to the hierarchy problem. For an understanding of quark–lepton masses and mixings SUSY is not crucial, but within the

context of SUSY there will be many interesting flavor violating processes. In $SU(5)$, the fifteen components of one family of quarks and leptons are organized into two multiplets: A $\mathbf{10}$ -plet and a $\bar{\mathbf{5}}$ -plet. The $\bar{\mathbf{5}}$ is of course the anti-fundamental representation of $SU(5)$, while the $\mathbf{10}$ is the anti-symmetric second rank tensor. These are represented by the following matrices:

$$\psi(\mathbf{10}) : \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_3^c & -u_2^c & u_1 & d_1 \\ -u_3^c & 0 & u_1^c & u_2 & d_2 \\ u_2^c & -u_1^c & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^c \\ -d_1 & -d_2 & -d_3 & -e^c & 0 \end{pmatrix}, \quad \chi(\bar{\mathbf{5}}) : \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e \\ -\nu_e \end{pmatrix}. \quad (108)$$

Each family of quarks and leptons is organized in a similar form. It is very nontrivial that this assignment of fermions under $SU(5)$ is anomaly free. The anomaly from the $\bar{\mathbf{5}}$ -plet is canceled by the anomaly from the $\mathbf{10}$ -plet. Note that quarks and leptons are unified into common multiplets. Furthermore, particles and antiparticles are also unified. These features imply that baryon number, which is a global symmetry of the SM, is violated, and that proton will decay. Because the unification scale is rather large, $\Lambda_{\text{GUT}} \approx 2 \times 10^{16}$ GeV, the decay rate of the proton is very slow, with a lifetime of order 10^{35} years. This is consistent with, but not very far from current experimental limits. Note that there is no ν^c field in the simplest version of $SU(5)$, but it can be added as a gauge singlet, as in the SM.

The symmetry breaking sector consists of two types of Higgs fields. One is an adjoint $\mathbf{24}_H$ -plet Σ , which acquires vacuum expectation value and breaks $SU(5)$ down to the SM gauge symmetry. The VEV of this traceless hermitian matrix is chosen as

$$\langle \Sigma \rangle = V \cdot \text{diag} \left\{ 1, 1, 1, -\frac{3}{2}, -\frac{3}{2} \right\}. \quad (109)$$

Under $SU(5)$ gauge transformation $\Sigma \rightarrow U \Sigma U^\dagger$. It is then clear that the VEV structure of Eq. (109) will leave invariant an $SU(3) \times SU(2) \times U(1)$ subgroup, identified as the SM gauge symmetry. 12 of the 24 gauge bosons of $SU(5)$ will acquire mass of order $V \sim \Lambda_{\text{GUT}} \approx 2 \times 10^{16}$ GeV, leaving the remaining 12 SM gauge bosons massless.

Σ cannot couple to the fermions. A pair of $\{\mathbf{5}_H + \bar{\mathbf{5}}_H\}$ Higgs fields, denoted as $(H + \bar{H})$, are used for generating fermion masses and for electroweak symmetry breaking. H contain the H_u field of MSSM, while \bar{H} contains the H_d field. These $(H + \bar{H})$ fields also contain color-triplet components, which must acquire GUT-scale masses, since they mediate proton decay. In minimal SUSY $SU(5)$ this splitting of color triplets and weak doublets is done by a special arrangement, by precisely tuning the mass term $M_H H \bar{H}$ and the coupling $\lambda H \bar{H} \Sigma$ of the

superpotential, so that the $SU(2)_L$ doublet components remain light, while their color-triplet partners acquire large masses. This is possible, since the VEV of Σ breaks the $SU(5)$ symmetry.

The Yukawa couplings of fermions and the (H, \overline{H}) fields are obtained from the superpotential

$$W_{\text{Yuk}} = \frac{(Y_u)_{ij}}{4} \psi_i^{\alpha\beta} \psi_j^{\gamma\delta} H^\rho \epsilon_{\alpha\beta\gamma\delta\rho} + \sqrt{2} (Y_d)_{ij} \psi_i^{\alpha\beta} \chi_{j\alpha} \overline{H}_\beta. \quad (110)$$

Here (i, j) are family indices, and $(\alpha, \beta \dots)$ are $SU(5)$ indices with ϵ being the completely antisymmetric Levi-Civita tensor. The \overline{H} field has components similar to χ of Eq. (108), so that its fifth component is neutral and acquires a VEV: $\langle \overline{H}_5 \rangle = v_d$. Similarly, the fifth component of H acquires a VEV: $\langle H_5 \rangle = v_u$. When these VEVs are inserted in Eq. (110), the following mass terms for quarks and leptons are generated:

$$\mathcal{L}_{\text{mass}} = \frac{1}{2} (Y_u)_{ij} v_u (u_i u_j^c + u_j u_i^c) + (Y_d)_{ij} v_d (d_i d_j^c + e_i^c e_j) + h.c. \quad (111)$$

This leads to the following fermion mass matrices:

$$M_u = Y_u v_u, \quad M_d = Y_d v_d, \quad M_\ell = Y_d^T v_d. \quad (112)$$

Note that M_u is a symmetric matrix in family space. Furthermore, there are only two Yukawa coupling matrices describing charged fermion masses, unlike the three matrices we have in the SM. The reason for this reduction of parameters is the higher symmetry and the unification of quarks with leptons. Specifically, we have the relation

$$M_d = M_\ell^T. \quad (113)$$

This identity leads to the asymptotic (valid at the GUT scale) relations for the mass eigenvalues

$$m_b^0 = m_\tau^0, \quad m_s^0 = m_\mu^0, \quad m_d^0 = m_e^0, \quad (114)$$

where the superscript ⁰ is used to indicate that the relation holds at the GUT scale.

In order to test the validity of the prediction of minimal SUSY $SU(5)$, we have to extrapolate the masses from GUT scale to low energy scale where the masses are measured. This is done by the renormalization group equations. The evolution of the b -quark and τ -lepton Yukawa couplings (λ_b and λ_τ), which are proportional to b -quark and τ -lepton masses, is shown in Fig. 9 for two different values of $\tan \beta = (1.7, 50)$. $\tan \beta = v_u/v_d$ is the ratio of the two Higgs VEVs in MSSM. Here we have extrapolated the Yukawa couplings derived from the observed masses from low scale to the GUT scale. It is remarkable that unification of masses occurs in this simple context. The main effect on the evolution comes from QCD enhancement of b quark mass as it evolves from high energy to low energy scale, which is absent for the τ lepton.

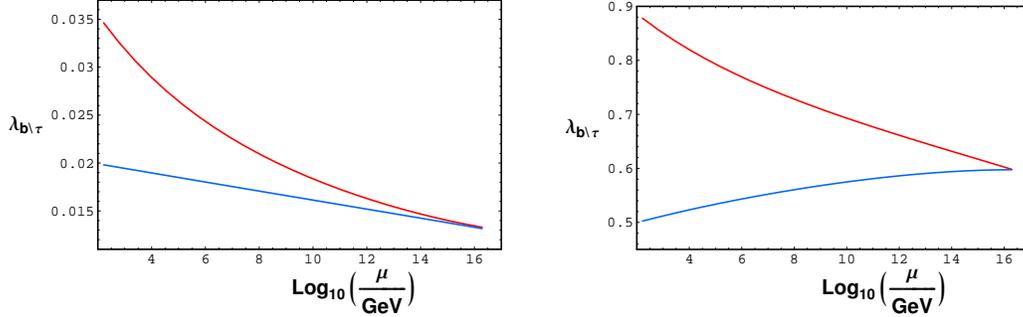


Figure 9: Evolution of the b -quark and τ -lepton Yukawa couplings in the MSSM for $\tan\beta = 1.7$ (left panel) and 50 (right panel). $m_b(m_b) = 4.65$ GeV has been used here.

Why these two specific values of $\tan\beta$? As it turns out, $b - \tau$ mass unification occurs only for specific values of $\tan\beta$, either for large values, or very small values. In Fig. 10 we plot the allowed values of $\tan\beta$ as a function of the strong coupling α_s [47]. From this figure, it is clear that intermediate values of $\tan\beta$ would lead to deviation from $m_b^0 = m_\tau^0$ by as much as 25%. For low and large values of $\tan\beta$, there is always good solution for $m_b(m_b)$, while for intermediate values there is no acceptable solution. It should be mentioned that there are significant finite corrections to the b -quark mass from loops involving the gluino, which is not included in the RGE analysis. These graphs, while loop suppressed, are enhanced by a factor of $\tan\beta$, and thus can be as large 30-40% for $m_b(m_b)$ [48]. So even intermediate values of $\tan\beta$ are not totally excluded.

The last two relations of Eq. (114) turn out to be not acceptable when compared to low energy values of the masses. One can see this without going through the RGE evolution. Eq. (114) implies $m_s^0/m_d^0 = m_\mu^0/m_e^0$. These mass ratios are RGE independent, so one can compare them directly with observations. We have seen that $m_s/m_d \simeq 20$, while $m_\mu/m_e \simeq 200$. So this relation is off by an order of magnitude.

There is an elegant way of fixing the light fermion masses in $SU(5)$. Consider modifying Eq. (114) to the following relations:

$$m_b^0 = m_\tau^0, \quad m_s^0 = \frac{1}{3} m_\mu^0, \quad m_d^0 = 3 m_e^0. \quad (115)$$

These relations were proposed by Georgi and Jarlskog and are known as the GJ relations [49]. The factors of 3 that appears in Eq. (115) have a simple group theoretic understanding in terms of $B - L$, under which lepton charges are (-3) times that of quark charges. The RGE independent quantity from Eq. (115) gives us

$$\frac{m_s}{m_d} = \frac{1}{9} \frac{m_\mu}{m_e}, \quad (116)$$

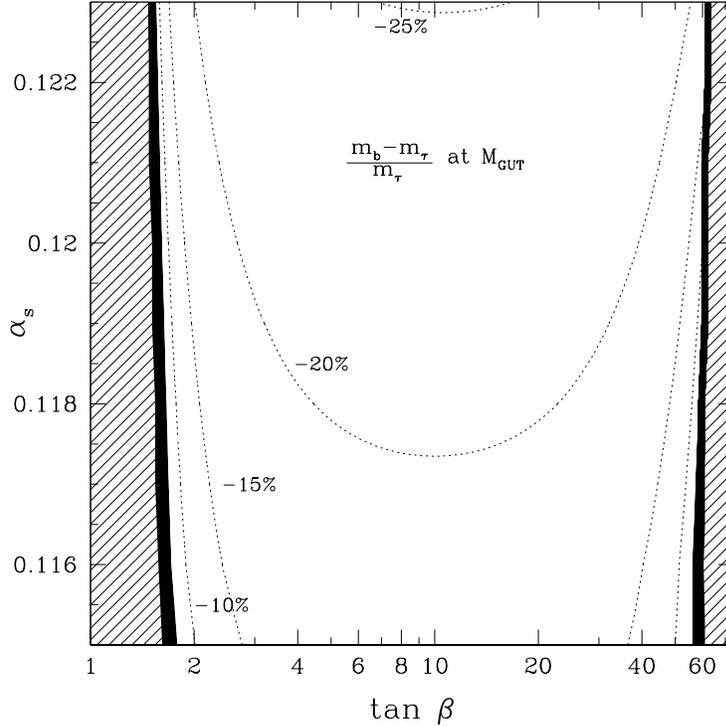


Figure 10: Deviation in the asymptotic relation $m_b^0 = m_\tau^0$ as a function of $\tan \beta$ and α_s [47].

which is in good agreement with observations. There is one other prediction, which can be taken to be the value of $m_d(1 \text{ GeV}) \simeq 8 \text{ MeV}$, which is also in good agreement with data, although recent lattice calculations prefer somewhat smaller values of m_d .

5.1 A predictive GUT framework for fermion masses

How would one go about deriving the Georgi–Jarlskog mass relations? We invoke a flavor $U(1)$ symmetry as before. Consider the following mass matrices for up quarks, down quarks and charged leptons [49, 50, 51, 52].

$$M_u = \begin{pmatrix} 0 & a & 0 \\ a & 0 & b \\ 0 & b & c \end{pmatrix}, \quad M_d = \begin{pmatrix} 0 & de^{i\phi} & 0 \\ de^{-i\phi} & f & 0 \\ 0 & 0 & g \end{pmatrix}, \quad M_\ell = \begin{pmatrix} 0 & d & 0 \\ d & -3f & 0 \\ 0 & 0 & g \end{pmatrix}. \quad (117)$$

The factor (-3) in charged lepton versus down quark mass matrix is attributed to the $B - L$ quantum number, and the zeros are enforced by a flavor symmetry. In $SU(5)$ GUT, the $(1,2)$ and the $(2,1)$ entries of M_d (and M_ℓ) are unrelated, but in $SO(10)$ GUT discussed in the next

subsection, they can be related, as in Eq. (117). All parameters are complex to begin with, but after field redefinitions, only a single complex phase survives. There are 7 parameters in all to fit the 13 observables (9 masses, 3 mixing angles and one CP phase), thereby resulting in six predictions. Three of these predictions are the b , s and d -quark masses. We write them at the low energy scale by incorporating factors denoted as η which are the RGE evolution factors to go from the weak scale to the GUT scale. For light quark masses, there is a further evolution to go down from the weak scale to their respective mass (or hadron) scale. The predictions of the model for the quark masses are given by:

$$m_b = \eta_{b/\tau}^{-1} m_\tau; \quad \frac{m_d/m_s}{(1 - m_d/m_s)^2} = 9 \frac{m_e/m_\mu}{(1 - m_e/m_\mu)^2}; \quad (m_s - m_d) = \frac{1}{3} \eta_{s/\mu}^{-1} (m_\mu - m_e). \quad (118)$$

The other three predictions are for the quark mixing angles and the CP phase J . J is the rephasing invariant CP violation parameter (Jarlskog invariant) which can be defined as

$$J = \text{Im}(V_{us}V_{cb}V_{ub}^*V_{cs}^*) \quad (119)$$

and has a value of $J \simeq 2.8 \times 10^{-5}$. We have for the remaining three predictions [52]

$$|V_{cb}| = \eta_{KM}^{-1} \eta_{u/t}^{1/2} \sqrt{\frac{m_c}{m_t}}; \quad \frac{|V_{ub}|}{|V_{cb}|} = \sqrt{\frac{m_u}{m_c}}; \quad (120)$$

$$J = \eta_{KM}^{-2} \eta_{u/t} \sqrt{\frac{m_d}{m_s}} \sqrt{\frac{m_c}{m_t}} \sqrt{\frac{m_u}{m_t}} \left[1 - \frac{1}{4} \left(\sqrt{\frac{m_u}{m_c}} \sqrt{\frac{m_s}{m_d}} + \sqrt{\frac{m_c}{m_u}} \sqrt{\frac{m_d}{m_s}} - \sqrt{\frac{m_c}{m_u}} \sqrt{\frac{m_s}{m_d}} |V_{us}|^2 \right)^2 \right]^{\frac{1}{2}}.$$

Here $\eta_{ct} = [(m_c^0/m_t^0)/(m_c(m_t)/m_t(m_t))]$, $\eta_{KM} = |V_{cb}^0|/|V_{cb}|$, etc. One can write down semi-analytic results for the RGE factors, if the bottom-quark Yukawa coupling h_b is much smaller than the top Yukawa coupling h_t , (corresponding to $\tan\beta \lesssim 10$ or so). These RGE factors can be expressed then as

$$\begin{aligned} \eta_{KM} &= \eta_{d/b} = \left(1 - \frac{Y_t}{Y_f}\right)^{\frac{1}{12}}; \quad \eta_{u/t} = \left(1 - \frac{Y_t}{Y_f}\right)^{\frac{1}{4}}; \quad \eta_{s/\mu} = \left(\frac{\alpha_1}{\alpha_G}\right)^{-10/99} \left(\frac{\alpha_3}{\alpha_G}\right)^{-8/9} \\ \eta_{b/\tau} &= \left(\frac{\alpha_1}{\alpha_G}\right)^{-10/99} \left(\frac{\alpha_3}{\alpha_G}\right)^{-8/9} \left(1 - \frac{Y_t}{Y_f}\right)^{-1/12}. \end{aligned} \quad (121)$$

Here α_G is the unified gauge coupling strength, $Y_t = h_t^2$ at the weak scale and Y_f is the fixed point value of Y_t . That is, Y_f is the largest value Y_t can take consistent with perturbation theory being valid up to the GUT scale. Numerically, $Y_f \simeq 1.2$. Y_t is of course obtained from $Y_t = [m_t(m_t)/v_u]^2$, which for $M_t = 172.5$ GeV is $Y_t \simeq 0.876$. Note that the CKM mixing parameters and the mass ratios in the same charge sector evolve only due to Yukawa couplings.

The mass ratio m_s/m_μ does change with momentum proportional to the gauge interaction strength.

While five of the six predictions of this model agree well with experiments, the relation for $|V_{cb}|$ of Eq. (120) would imply that either the top quark mass is much higher than its observed value, or that the value of $|V_{cb}|$ is much larger than allowed. Indeed, if we use an acceptable value of $M_t = 172.5$ GeV, with $Y_f = 1.2$, Eq. (120) would lead to $|V_{cb}| \simeq 0.053$, which is more than 10 standard deviations away from its central value. If $|V_{cb}|$ is to be decreased down to any acceptable value, top quark mass will have to be very close to its perturbative upper limit, around 200 GeV, which is also excluded by experiments.

We conclude that, although very predictive and simple, the ansatz of Eq. (117) is excluded by data. It is interesting that while the original Fritszch ansatz of Eq. (79) was excluded since top quark mass was predicted to be too low, the present ansatz, which was very popular until a few years ago, is excluded for its prediction of top mass that is too large.

5.2 Fermion masses in a predictive $SO(10)$ model

Now let us turn to an even more interesting class of GUTs, those based on the gauge symmetry $SO(10)$ [53]. All members of a family are unified into a **16** dimensional spinor representation of $SO(10)$. This requires the existence of right-handed neutrino ν^c , leading naturally to the seesaw mechanism and small neutrino masses. $SU(5)$ has the option of having neutrino mass, but in that context there is no compelling argument for its existence. $SO(10)$ models are the canonical grand unified models, owing to the observed neutrino masses, and the fact that all members of a family are unified into a single **16**-dimensional spinor multiplet in $SO(10)$.

The spinor of $SO(10)$ breaks down under $SU(5)$ (which is one of its subgroups) as

$$\mathbf{16} = \mathbf{10} + \bar{\mathbf{5}} + \mathbf{1}, \quad (122)$$

where the **1** is the ν^c field. The **10** and the $\bar{\mathbf{5}}$ fields are identical to the case of $SU(5)$. We shall again assume low energy supersymmetry. Gauge symmetry breaking is accomplished in the SUSY limit by introducing Higgs fields in the adjoint $\mathbf{45}_{\mathbf{H}}$, spinor $\{\mathbf{16}_{\mathbf{H}} + \bar{\mathbf{16}}_{\mathbf{H}}\}$ and vector $\mathbf{10}_{\mathbf{H}}$ representations. Because there is more symmetry in $SO(10)$, more scalars are needed to achieve symmetry breaking down to the SM. The spinor Higgs fields break $SO(10)$ down to $SU(5)$ changing the rank of the gauge group, while the adjoint $\mathbf{45}_{\mathbf{H}}$ -plet breaks this symmetry down to the SM. The vector $\mathbf{10}_{\mathbf{H}}$ -plet is used for fermion mass generation and for electroweak symmetry breaking. The MSSM Higgs doublets $H_{u,d}$ are contained partially in the $\mathbf{10}_{\mathbf{H}}$ but can be partially also in the $\mathbf{16}_{\mathbf{H}}$.

Let me work out a specific flavor model based on $SO(10)$ supplemented by a $U(1)$ symmetry [54]. While this model will not be as predictive as the ansatz that generated the GJ relations in the previous subsection, there are still a number of predictions, and these predictions are consistent with data. Several variations of the theme can be found in the literature [55], but here I confine the discussions to the mass matrices of Ref. [54] and its slight generalization studied in Ref. [56].

The mass matrices for up and down quarks, and Dirac neutrino and charged leptons take the form:

$$\begin{aligned}
M_u &= \begin{bmatrix} 0 & \epsilon' & 0 \\ -\epsilon' & \zeta_{22}^u & \sigma + \epsilon \\ 0 & \sigma - \epsilon & 1 \end{bmatrix} \mathcal{M}_u^0, & M_d &= \begin{bmatrix} 0 & \eta' + \epsilon' & 0 \\ \eta' - \epsilon' & \zeta_{22}^d & \eta + \epsilon \\ 0 & \eta - \epsilon & 1 \end{bmatrix} \mathcal{M}_d^0 \\
M_\nu^D &= \begin{bmatrix} 0 & -3\epsilon' & 0 \\ 3\epsilon' & \zeta_{22}^u & \sigma - 3\epsilon \\ 0 & \sigma + 3\epsilon & 1 \end{bmatrix} \mathcal{M}_\nu^0, & M_\ell &= \begin{bmatrix} 0 & \eta' - 3\epsilon' & 0 \\ \eta' + 3\epsilon' & \zeta_{22}^d & \eta - 3\epsilon \\ 0 & \eta + 3\epsilon & 1 \end{bmatrix} \mathcal{M}_\ell^0.
\end{aligned} \tag{123}$$

Here M_ν^D is the Dirac neutrino mass matrix.

Notice the various correlations in these matrices. The overall scale associated with M_u and M_ν^D are identical, while those for M_d and M_ℓ are the same. The “1” entry in all matrices have a common origin, arising from the operator $\mathbf{16}_3 \mathbf{16}_3 \mathbf{10}_H$. The ϵ entry appears with coefficient 1 in the up and down quark matrices, and with coefficient -3 in the leptonic mass matrices. This factor (-3) is the ratio of the $B - L$ charge of leptons versus quarks. Specifically, the ϵ entry arises from an operator $\mathbf{16}_2 \mathbf{16}_3 (\mathbf{10}_H \times \mathbf{45}_H)/M$. Here the adjoint $\mathbf{45}_H$, which is a second rank antisymmetric tensor of $SO(10)$, acquires a VEV in a $B - L$ conserving direction:

$$\langle \mathbf{45}_H \rangle = i\tau_2 \times \text{diag.}(a, a, a, 0, 0) . \tag{124}$$

In the product $\mathbf{10}_H \times \mathbf{45}_H$, two fragments, an effective $\mathbf{10}_H$ and an effective $\mathbf{120}_H$, couple to the fermions. However, when the VEV of $\mathbf{45}_H$ from Eq. (124) is inserted, only the effective $\mathbf{120}_H$ is non-vanishing, leading to the relative factor of (-3) between leptons versus quarks. Note that the ϵ entry arises suppressed by $1/M$ so that $\epsilon \ll 1$, an idea familiar from the Froggatt–Nielsen mechanism. In an analogous fashion, the ϵ' entry arises from the operator $\mathbf{16}_1 \mathbf{16}_2 (\mathbf{10}_H \times \mathbf{45}_H) S/M^2$, where S is an $SO(10)$ singlet flavon field carrying a flavor $U(1)$ charge. This entry is then more suppressed compared to the ϵ entry. The σ entry originates from the operator $\mathbf{16}_2 \mathbf{16}_3 \mathbf{10}_H S/M$, and enters into all matrices with equal coefficient, just as the “1” entry. An operator $\mathbf{16}_2 \mathbf{16}_3 \mathbf{16}_H \mathbf{16}_H/M$ contributes equally to the down quark and

charged lepton mass matrices, but not to M_u and M_ν^D , since $\mathbf{16}_H$ contains only an H_d -type field, and not an H_u -type field. The η entry in M_d and M_ℓ is the sum of the last two operators. The entry η' originates from $\mathbf{16}_1\mathbf{16}_2\mathbf{16}_H\mathbf{16}_H S^2/M^3$ operator.

These are precisely the operators one would obtain when the three families of fermions and the Higgs fields are assigned the following $U(1)$ charges:

$$\begin{array}{cccccccc} \mathbf{16}_3 & \mathbf{16}_2 & \mathbf{16}_1 & \mathbf{10}_H & \mathbf{16}_H & \overline{\mathbf{16}}_H & \mathbf{45}_H & \mathbf{S} \\ a & a+1 & a+2 & -2a & -a-1/2 & -a & 0 & -1 \end{array}. \quad (125)$$

In Ref. [54], where for simplicity, CP violation was ignored, the diagonal (2,2) entries were not introduced. In subsequent work these (2,2) entries, especially ζ_{22}^d , were used to accommodate CP violation. Here we present the predictions of the model as given in Ref. [54]. An acceptable fit to all mass and mixing parameters is obtained by the following choice of parameters at the GUT scale:

$$\begin{aligned} \sigma &= -0.1096, \quad \eta = -0.1507, \quad \epsilon = 0.0954, \\ \epsilon' &= 1.76 \times 10^{-4}, \quad \eta' = 4.14 \times 10^{-3}. \end{aligned} \quad (126)$$

With these input, one obtains the following predictions:

$$\begin{aligned} m_b(m_b) &= 4.9 \text{ GeV}, \quad m_s(1 \text{ GeV}) = 116 \text{ MeV}, \quad m_d(1 \text{ GeV}) = 8 \text{ MeV} \\ \theta_C &\simeq \left| \sqrt{\frac{m_d}{m_s}} - e^{i\phi} \sqrt{\frac{m_u}{m_c}} \right|, \quad \frac{|V_{ub}|}{|V_{cb}|} \simeq \sqrt{\frac{m_u}{m_c}} \simeq 0.07. \end{aligned} \quad (127)$$

These predictions are in general agreement with data. When the (2,2) entries are included in the mass matrices, realistic CP violation phenomenology also follows [56].

Light neutrino masses are generated in this scheme via the seesaw mechanism. Note that the Dirac neutrino mass matrix elements are completely fixed, because of $SO(10)$ symmetry, from the charged fermion sectors. The mechanism that generates heavy Majorana neutrino masses for the ν^c fields should be specified. The model already contains operators that do this, as given by

$$W_{\text{Maj}} = \mathbf{16}_i \mathbf{16}_j (\overline{\mathbf{16}}_H \overline{\mathbf{16}}_H) / M. \quad (128)$$

The natural scale of the cut-off M is $M = M_{\text{Planck}} = 2 \times 10^{18} \text{ GeV}$. Then with order one couplings in Eq. (128), one would obtain, for the (third family) right-handed Majorana mass, $M_{\nu_3^c} \sim \Lambda_{\text{GUT}}^2 / M_{\text{Planck}} \sim 10^{14} \text{ GeV}$. This in turn leads to the light neutrino mass $m_\nu \sim m_t^2 / M_{\nu_3^c} \simeq 0.05 \text{ eV}$, nicely consistent with the value desired for atmospheric neutrino oscillation data.

In Ref. [54], it was shown, with a specific choice of the flavor structure of M_{ν^c} , that large neutrino oscillation angles arise naturally, while preserving the smallness of quark mixing angles. Specifically, while $|V_{cb}| \simeq 0.041$, $\sin^2 2\theta_{23} \simeq (0.9 - 0.99)$ was obtained, as a function of the light neutrino mass ratio m_2/m_3 .

5.2.1 Flavor violation in SUSY GUTs

How do we go about testing ideas of grand unification in the flavor sector? Since the GUT scale is below the Planck scale, even though the flavor symmetry is broken near the GUT scale, soft SUSY breaking parameters can remember flavor violating interaction due to their running between the Planck scale and the GUT scale. Such running is expected in supergravity models, where the messengers of SUSY breaking have masses at the Planck scale. The most significant flavor violation in the model of Ref. [54] arises due to the splitting of the third family sfermions from those of the first two families. This is seen by the solution to the RGE equations for these masses [57].

$$\Delta\hat{m}_{\tilde{b}_L}^2 = \Delta\hat{m}_{\tilde{b}_R}^2 = \Delta\hat{m}_{\tilde{\tau}_L}^2 = \Delta\hat{m}_{\tilde{\tau}_R}^2 \equiv \Delta \approx -\left(\frac{30m_0^2}{16\pi^2}\right)h_t^2 \log(M^*/M_{GUT}). \quad (129)$$

Here M^* is the fundamental scale where SUSY breaking messengers reside, with $M^* > M_{GUT}$. h_t is the top quark Yukawa coupling. Note that leptons also feel the effect of top Yukawa, because leptons and quarks are unified. In Eq. (129) m_0 is the universal SUSY breaking scalar mass parameter. One sees that, because of the GUT threshold, universality is not preserved in this type of models. In going from gauge basis to the mass eigenbasis for the fermions, Eq. (129) would imply that there will be flavor changing scalar interactions. Because SUSY particles have masses of order TeV, these flavor violation can manifest in the MSSM sector via SUSY loops.

The most constraining FCNC process in the present model turns out to be $\mu \rightarrow e\gamma$. The diagrams inducing such processes in SUSY GUT models are shown in Fig. 11. In the present case it turns out that the decay $\tau \rightarrow \mu\gamma$ is not very significant, while the new contributions to $b \rightarrow s\gamma$ is not negligible. Predictions for the branching ratio for the decay $\mu \rightarrow e\gamma$ are depicted in Fig. 12 as a function of slepton mass [57]. Part of the parameter space is already ruled out, so there is a good chance that this process will be discovered at the MEG experiment at PSI.

There are other sources of flavor violation in SUSY GUTs. A widely discussed process is the $\ell_i \rightarrow \ell_j\gamma$ decay arising from neutrino mass physics [58]. The heavy right-handed neutrino mass is expected to be in the range $(10^{10} - 10^{14})$ GeV in SUSY GUTs. Even when the supergravity boundary conditions on the soft SUSY breaking parameters are valid at the GUT scale (and not the Planck scale), there is a momentum regime $\mu, M_{\nu^c} \leq \mu \leq \Lambda_{GUT}$, where the ν^c fields are active. In this momentum regime the neutrino Dirac Yukawa couplings will affect the RGE evolution of the soft slepton mass parameters and generate lepton flavor violation. The FCNC effect in the slepton soft squared mass is given by

$$(\Delta m_{\tilde{L}}^2)_{ij} \simeq -\frac{\log(\Lambda_{GUT}/M_{\nu^c})}{8\pi^2} \{3m_0^2(Y_\nu^\dagger Y_\nu)_{ij} + (A_\nu^\dagger A_\nu)_{ij}\}. \quad (130)$$

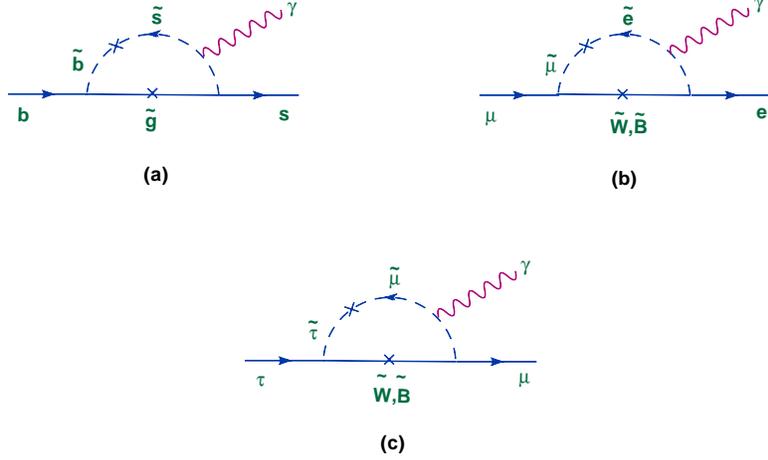


Figure 11: Rare decays induced by penguin diagrams via the exchange of SUSY particles. The flavor mixing occurs during the RGE flow between M_{GUT} and M_* .

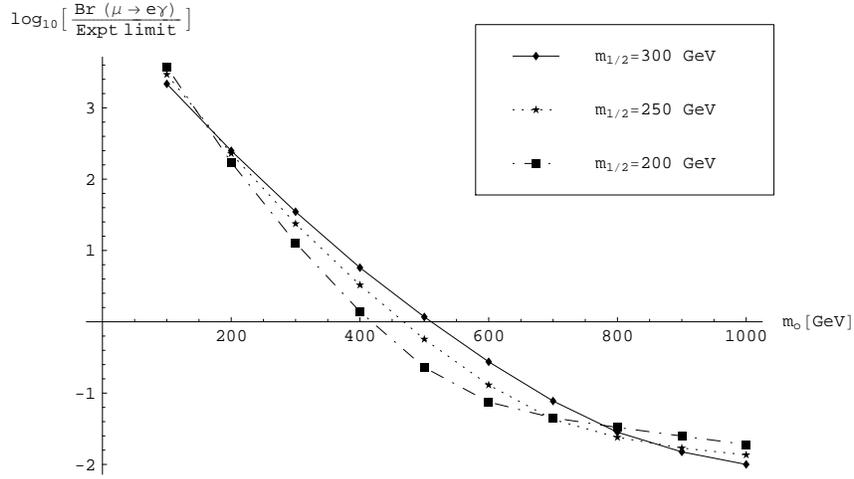


Figure 12: Prediction for the branching ratio for $\mu \rightarrow e\gamma$ in the SUSY $SO(10)$ model as a function of slepton mass. The horizontal line indicates current experimental limit [57].

Here Y_ν is the neutrino Dirac Yukawa coupling, while A_ν is the corresponding soft trilinear A -term.

In the MSSM, or in the SUSY $SU(5)$ model, the Yukawa coupling Y_ν cannot be determined from neutrino oscillation data. This is because the seesaw formula for light neutrinos goes as $m_\nu \sim Y_\nu^2 v^2 / M_{\nu c}$, and knowing m_ν does not determine Y_ν uniquely. However, if some of the entries of Y_ν are of order $(10^{-2} - 1)$, then the decay rate $\mu \rightarrow e\gamma$ will be within reach of ongoing

experiments.

In SUSY $SO(10)$ there is a more crisp prediction for $\mu \rightarrow e\gamma$ arising from the neutrino sector. This happens because $SO(10)$ symmetry relates Y_ν with the up-quark Yukawa couplings. Specifically, for the third family, we have $(Y_\nu)_{33} = Y_t$, the top quark Yukawa coupling. Since Y_t is of order one, the FCNC effects from the neutrino sector in SUSY $SO(10)$ are *predicted* to be significant. That is, they cannot be tuned to disappear, unlike in the SUSY $SU(5)$ model.

6 Radiative fermion mass generation

The hierarchical structure of the quark and lepton masses and the quark mixing angles can be elegantly understood by the mechanism of radiative mass generation. This is an alternative to the Froggatt–Nielsen mechanism. Here the idea is that only the heaviest fermions (eg. the third family quarks) acquire tree level masses. The next heaviest fermions (second family quarks) acquire masses as one loop radiative corrections, which are suppressed by a typical loop factor $\sim 1/(16\pi^2) \sim 10^{-2}$ relative to the heaviest fermions. The lightest fermions (u and d quarks) acquire masses as two loop radiative corrections, which are then a factor $\sim [1/(16\pi^2)]^2 \sim 10^{-4}$ suppressed relative to the heaviest fermions. Thus, even without putting in small Yukawa couplings one understands the hierarchy in the mass spectrum of the fermions.

There is another appeal to this idea. If the electron mass is radiatively generated from the muon mass, then there must be no counter-term needed in the Lagrangian to absorb infinity associated with the electron mass. In other words, electron mass is “calculable”, in terms of other parameters of the model. This idea was originally suggested by 'tHooft in his classic paper on the renormalizability of non-Abelian gauge theories [59]. This also implies that there must be some symmetry reason for the light fermions not to have tree level masses, otherwise the idea cannot be implemented consistently. Early attempts along this line were presented in Ref. [60]. More realistic models came along somewhat later [61, 62, 63, 64, 65].

There is a resurgence of interest in this idea as the LHC turns on, since new particles with specific properties which may be seen at the LHC are predicted. There exist rather nice models of this type by Mohapatra and collaborators [61] from the late 80's. Recently Dobrescu and Fox have written a nice paper on the subject [64], which I recommend to you. As in past examples, I will try to convey the main idea, with the understanding that implementation can vary considerably. I will discuss an implementation which I worked out with Mohapatra based on the permutation symmetry [62].

Let us focus on the quark sector of the SM first. We wish to have a scenario where only the top quark and the bottom quark have tree level masses. In the same limit, there should

be no CKM mixing induced. This can be realized if one has the following “democratic” mass matrices for up and down quarks.

$$M_{u,d} = \frac{m_{t,b}}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (131)$$

Of course, these matrices have rank 1, implying that only the top and the bottom acquire masses from here. A common unitary matrix will diagonalize M_u and M_d , so there is no CKM mixing induced at this stage.

How do we obtain democratic mass matrices of Eq. (131)? It turns out that the symmetry of these matrices is $S_{3L} \times S_{3R}$, where S_3 is the group of permutation of three letters. The Lagrangian that would generate Eq. (131) for M_u is of the form

$$\mathcal{L}_{\text{Yukawa}} = h_u (\bar{Q}_{1L} + \bar{Q}_{2L} + \bar{Q}_{3L}) \tilde{H} (u_{1R} + u_{2R} + u_{3R}) \quad (132)$$

which is manifestly symmetric under separate permutations of the left-handed and the right-handed quark fields. So it is tempting to start with this symmetry group $S_{3L} \times S_{3R}$, but it is not necessary to have the S_{3R} group, since right-handed rotations are un-physical in the SM. So consider the following Lagrangian which only has the S_{3L} symmetry.

$$\begin{aligned} \mathcal{L}_{\text{Yukawa}} &= (\bar{Q}_{1L} + \bar{Q}_{2L} + \bar{Q}_{3L}) \tilde{H} (h_1^u u_{1R} + h_2^u u_{2R} + h_3^u u_{3R}) \\ &+ (\bar{Q}_{1L} + \bar{Q}_{2L} + \bar{Q}_{3L}) H (h_1^d d_{1R} + h_2^d d_{2R} + h_3^d d_{3R}). \end{aligned} \quad (133)$$

By right-handed rotations on u_R and d_R fields, we can bring Eq. (133) into the form of Eq. (132). Two combinations of the Q_{iL} and (u_{iR}, d_{iR}) fields orthogonal to Eq. (133) will be massless.

These massless Q_{iL} modes actually form the **2** dimensional representations of S_3 . It is convenient to directly go to the irreducible representations of S_3 . They are a true singlet **1**, an odd singlet **1'** and a doublet **2** = (x_1, x_2) . The product of two **1'** gives a **1**, while the product of two **2** gives **1** + **1'** + **2**. The Clebsch–Gordon coefficients for this product (in a certain basis) are [66]:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathbf{1} : (x_1 y_1 + x_2 y_2); \quad \mathbf{1}' : (x_1 y_2 - x_2 y_1); \quad \mathbf{2} : \begin{pmatrix} x_1 y_2 + x_2 y_1 \\ x_1 y_1 - x_2 y_2 \end{pmatrix}. \quad (134)$$

Now, consider the following assignment of quarks and scalars under S_3 :

$$\begin{aligned} \begin{pmatrix} Q_{1L} \\ Q_{2L} \end{pmatrix} &: \mathbf{2}; \quad Q_{3L} : \mathbf{1}; \quad u_{iR} : \mathbf{1}, \\ H &: \mathbf{1}, \quad \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} : \mathbf{2}, \quad \omega_3 : \mathbf{1}. \end{aligned} \quad (135)$$

Here the gauge structure is simply that of SM with H being the SM Higgs doublet. In order to radiatively generate light fermion masses, new ingredients are needed. The simplest possibility is to introduce scalar fields which have Yukawa couplings connecting the heavy (3rd generation) and the light fermions. We have assumed existence of $\omega_i(3, 1, -1/3)$ fields, which can have such Yukawa couplings, without inducing direct mass terms for the light fermions. Note that these ω_i fields are colored and charged, so they do not acquire vacuum expectation values.

The most general Yukawa couplings allowed in this $\text{SM} \times S_3$ model is given by

$$\begin{aligned} \mathcal{L}_{\text{Yukawa}} &= h_t \bar{Q}_{3L} t_R \tilde{H} + h_b \bar{Q}_{3L} b_R H + h_1 (Q_{1L}^T C Q_{3L} \omega_1 + Q_{2L}^T C Q_{3L} \omega_2) \\ &+ h_2 (Q_{1L}^T C Q_{1L} + Q_{2L}^T C Q_{2L}) \omega_3 + h_3 Q_{3L}^T C Q_{3L} \omega_3 \\ &+ h_4 \{ Q_{1L}^T C Q_{2L} + Q_{2L}^T C Q_{1L} \} \omega_1 + (Q_{1L}^T C Q_{1L} - Q_{2L}^T C Q_{2L}) \omega_2 + h.c. \end{aligned} \quad (136)$$

Here we have redefined the combination of u_R that couples to Q_{3L} as simply t_R (and similarly for b_R).

Clearly, from Eq. (136), only the top and bottom quarks acquire tree-level masses. There is no tree-level CKM mixing angle. So by symmetry reason, we have achieved the first stage of the program. Now, if S_3 is unbroken, none of the light fermions will acquire masses, even though they have Yukawa couplings via the ω_i fields. We can break S_3 spontaneously, or by soft bilinear terms in the Higgs potential:

$$V = \sum_{i,j=1}^3 \mu_{ij}^2 \omega_i^* \omega_j + h.c. \quad (137)$$

With these soft breaking terms, light fermion masses will be induced. In Fig. 13 we have the one-loop and the two-loop mass generation diagrams.

The one-loop diagram of Fig. 13 only generates charm quark mass, and not the up quark mass. This can be understood as follows. At tree-level, among the down quarks, only b has a mass. There is a single linear combination of up quarks which couples to the b quark via the ω_i fields. It is this combination that picks up mass at one-loop. The orthogonal combination remains massless at this order. Now, the two-loop diagram connects up quarks to both b and

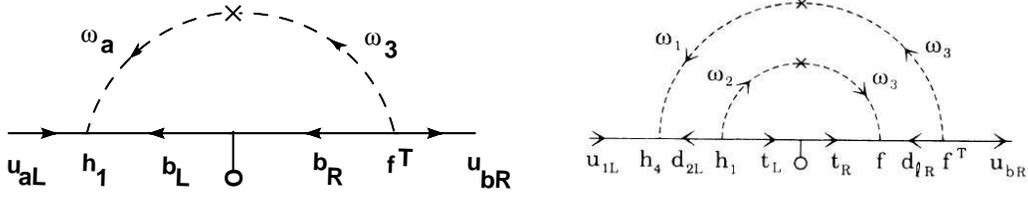


Figure 13: One-loop diagram inducing charm quark mass (left) and two-loop diagram inducing up quark mass (right).

s quarks. The inner loop of the two-loop diagram is the one-loop diagram that generates the s quark mass. As a result, u quark will acquire a mass proportional to the s quark mass at two-loop.

Including the one-loop diagram, the mass matrix for the (c, t) sector has the form

$$M_u^{1\text{-loop}} = \begin{pmatrix} \epsilon & a\epsilon \\ 0 & m_t^0 \end{pmatrix} \quad (138)$$

where a is of order one and the small parameter ϵ is found to be

$$\epsilon \simeq \left(\frac{h_1 f}{8\pi^2} \right) m_b \left(\frac{\mu_{a3}^2}{M_\omega^2} \right) \log \left(\frac{M_\omega^2}{m_b^2} \right). \quad (139)$$

With the Yukawa couplings being order one, we can explain why the charm is much lighter than the top. The mixing angle V_{cb} is of order m_s/m_b , in agreement with observations. The two-loop diagrams which induce the up and down quark masses also induce the mixings of the first family. There is a natural hierarchy of mixing angles where $|V_{us}| \gg |V_{cb}| \gg |V_{ub}|$.

It is straightforward to extend the S_3 model to the leptonic sector. Consider the following assignment of leptons and ω_ℓ fields under S_3 , where ω_ℓ are $(3^*, 1, -1/3)$ scalar fields. (These are not the conjugates of the ω_i fields from the quark sector, or else, there will be proton decay mediated by these scalars. We assume separate baryon number conservation, so the proton is stable.)

$$\begin{pmatrix} \psi_{1L} \\ \psi_{2L} \end{pmatrix} : \mathbf{2}, \quad \psi_{3L} : \mathbf{1}; \quad e_{iR} : \mathbf{1}' \\ \omega_\ell : \mathbf{1}, \quad \omega'_\ell : \mathbf{1}' \quad (140)$$

The general Yukawa coupling of leptons is given by

$$\begin{aligned} \mathcal{L}'_{\text{Yukawa}} &= h'_1 Q_{3L}^T C \psi_{3L} \omega_\ell + h'_2 (Q_{1L}^T C \psi_{1L} + Q_{2L}^T C \psi_{2L}) \omega_\ell \\ &+ h'_3 (Q_{1L}^T C \psi_{2L} - Q_{2L}^T C \psi_{1L}) \omega'_\ell + f'_{ab} u_{aR}^T C e_{bR} \omega'_\ell + h.c. \end{aligned} \quad (141)$$

Note that all leptons are massless at the tree level. The one-loop diagram shown in Fig. 14 will induce the τ lepton mass, and is proportional to the top quark mass with a loop suppression. Only τ acquires a one-loop mass. The muon mass arises from the two-loop diagram of Fig. 14. The electron remains massless at this order, and acquires a mass only via a three-loop diagram.

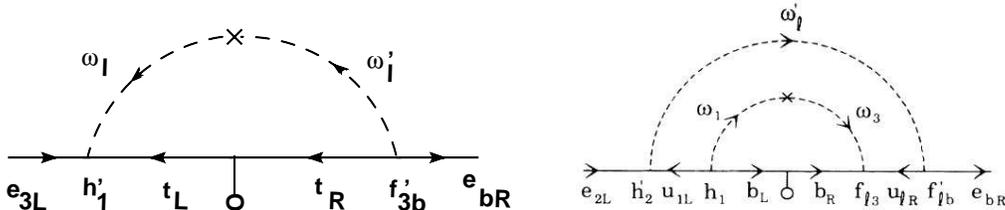


Figure 14: One loop diagram inducing τ lepton mass (left) and two-loop diagram inducing the muon mass (right).

Note that we cannot constrain the masses of the ω fields from this process, since by taking the masses of the ω fields and the soft breaking μ^2 term to large values, the light fermion masses will remain unchanged.

However, in the supersymmetric version of the radiative mass generation mechanism, the new scalars should remain light, to about 1 TeV, since the superpotential is un-renormalized. That is to say that in a SUSY context, in the exact SUSY limit, the fermionic and bosonic loop diagrams add up to give zero. Once SUSY breaking terms are turned on, these diagrams will no longer cancel, and will generate finite quark and lepton masses. Thus, there is a prediction in this scenario. In addition to SUSY particles, LHC should discover these ω_i particles and their superpartners.

7 The strong CP problem and its resolution

There is no indication of CP violation in strong interactions. Yet, the QCD Lagrangian admits a term

$$\mathcal{L}_{QCD} = \frac{\theta g^2}{32\pi^2} G_{\mu\nu}^a \tilde{G}^{a\mu\nu} \quad (142)$$

which is P and T violating, and thus, owing to CPT invariance CP violating as well. In Eq. (142), $\tilde{G}^{a\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}^a$ is the dual field strength for the gluon. The Lagrangian in Eq. (142) is a total divergence, since $G_{\mu\nu}^a \tilde{G}^{a\mu\nu} = \partial_\mu K^\mu = \partial_\mu [\epsilon^{\mu\nu\rho\sigma} A_\nu^a (F_{\rho\sigma}^a - \frac{2}{3}\epsilon^{abc} A_\rho^b A_\sigma^c)]$. In a $U(1)$ gauge theory, the resulting surface term in the action would vanish for finite energy configurations.

Thus a term analogous to Eq. (142) does not lead to P or T violation in QED. However, in QCD, the surface term gives rise to non-zero contributions, owing to finite energy “instanton” configurations, causing P and T violation.

It is not the parameter θ in Eq. (142) that is physical. Recall that the QCD Lagrangian also contains quark mass matrices M_u and M_d , which are generated after electroweak symmetry breaking. These matrices are complex, and generate the KM phase for CP violation in weak interactions. As discussed in Sec. 2, one makes bi-unitary transformations to bring these matrices into diagonal form: $U_L^{u\dagger} M_u U_R^u = \text{diag}(m_u, m_c, m_t)$, and similarly for M_d . If U_L and U_R belong to the global $SU(N_f)_L \times SU(N_f)_R$ chiral symmetry (N_f is the number of quark flavors), which has no QCD anomaly, the diagonal quark masses cannot be made real. Specifically, $\text{Det}(M_u) \rightarrow \text{Det}(M_u)$ under such a special bi-unitary transformation. If the phases of the quark masses are denoted as $\theta_{u,c,t}$ and $\theta_{d,s,b}$, the combination

$$\theta_{\text{QFD}} = \theta_u + \theta_c + \theta_t + \theta_d + \theta_s + \theta_b = \text{Arg}[\text{Det}(M_q)] \quad (143)$$

cannot be removed by anomaly-free rotations. A chiral rotation on the quark fields is necessary in order to remove this phase. This however will generate an anomaly term in the Lagrangian, of the same form as in Eq. (142). The physical parameter is then

$$\bar{\theta} = \theta + \text{Arg}[\text{Det}M_q]. \quad (144)$$

With $\bar{\theta}$ physical, there will be CP violation in strong interactions. However, there are stringent constraints on the value of $\bar{\theta}$ from experimental limits on the electric dipole moment (EDM) of the neutron: $\bar{\theta} < 10^{-10}$. This arises since in the presence of $\bar{\theta}$ neutron EDM can be shown to have a non-zero value given by

$$d_n \simeq [10^{-16} \times \bar{\theta}] \text{ e - cm}. \quad (145)$$

From the experimental limit on neutron EDM, $d_n < 10^{-26}$ e-cm, one obtains the limit $\bar{\theta} < 10^{-10}$. Why is it that a fundamental dimensionless parameter of the Lagrangian, which should naturally be of order one, so small is the strong CP problem. If CP were a good symmetry of the entire Lagrangian, small $\bar{\theta}$ would have been quite natural. However, weak interactions do break CP invariance, which makes the strong CP problem acute.

There are various proposed solutions to the problem. At some point in time it was thought that the up quark mass may be zero. If true, that would solve the strong CP problem, since θ_u is then un-physical, and therefore $\bar{\theta}$ can be removed from the theory. But now we know, especially from lattice gauge theory results, that $m_u = 0$ is not an acceptable solution.

7.1 Peccei–Quinn symmetry and the axion solution

The most widely studied solution of the strong CP problem is the Peccei–Quinn (PQ) mechanism [67], which yields a light pseudo–Goldstone boson, the axion [68]. Here the parameter $\bar{\theta}$ is promoted to a dynamical field. This field acquires a non–perturbative potential induced by the QCD anomaly. Minimization of the potential yields the desired solution $\bar{\theta} = 0$, solving the strong CP problem.

In the presence of the $\bar{\theta}$ term in the Lagrangian, non–perturbative QCD effects will induce a vacuum energy given by

$$E_{\text{vac}} = \mu^4 \cos \bar{\theta}, \quad (146)$$

where $\mu \sim \Lambda_{\text{QCD}} \sim 100$ MeV. This observation is crucially used in the PQ mechanism. What if $\bar{\theta}$ is a dynamical field? Then this non–perturbative potential will have to be minimized to locate the ground state (unlike the case where $\bar{\theta}$ is a constant in the Lagrangian). Minimization of this potential will yield $\bar{\theta} = 0$, as desired.

The essence of the PQ mechanism can be explained with a simple toy model [69]. Consider QCD with one quark flavor (q) and no weak interactions. Suppose there is a global $U(1)$ symmetry under which $q \rightarrow e^{-i\alpha\gamma_5/2}q$. Such a symmetry has a QCD anomaly, and can only be imposed at the classical level. A bare mass for q is then forbidden. Introduce now a complex color singlet scalar field ϕ which transforms under this $U(1)$ as $\phi \rightarrow e^{i\alpha}\phi$. The following Yukawa interaction is then allowed.

$$\mathcal{L}_{\text{Yuk}} = Y\bar{q}_L\phi q_R + Y^*\bar{q}_R\phi^* q_L. \quad (147)$$

The potential for ϕ also respects the $U(1)$ symmetry, and is given by

$$V(\phi) = -m_\phi^2|\phi|^2 + \lambda|\phi|^4 \quad (148)$$

With a negative sign for m_ϕ^2 , the ϕ field will acquire a non-zero VEV, spontaneously breaking the $U(1)$. In this broken symmetric phase, we can parametrize ϕ as

$$\phi = \left[f_a + \tilde{\phi}(x^\mu) \right] e^{ia(x)/f_a}. \quad (149)$$

Here f_a is a real constant, while $\tilde{\phi}(x^\mu)$ and $a(x^\mu)$ are dynamical (real) fields. The quark q now acquires a mass, given by $M_q = Yf_a e^{ia(x)/f_a}$. Making the quark mass real by a field redefinition will induce a $\bar{\theta}$ given by

$$\bar{\theta}_{\text{eff}} = \theta + \text{Arg}[\text{Det } Y] + \frac{1}{f_a}a(x^\mu). \quad (150)$$

The crucial point is that now $\bar{\theta}$ is a dynamical field, because of the presence of the a field, the axion. Without non–perturbative QCD effects, a will be massless, since it is the Goldstone

boson associated with the spontaneous breaking of the global $U(1)$. The vacuum energy analog of Eq. (146) is now

$$E_{\text{vac}} = -\mu^4 \cos \bar{\theta}_{\text{eff}}. \quad (151)$$

Minimizing this potential with respect the dynamical a field would yield $\bar{\theta}_{\text{eff}} = 0$.

The field-dependent redefinition on q , $q(x^\mu) \rightarrow q(x^\mu) e^{-i(a(x^\mu)/f_a)(\gamma_5/2)}$ would remove the axion field from quark interactions except via derivatives, originating from the kinetic terms. The axion also will have couplings to the gluon field strength. These couplings are given by

$$\mathcal{L}_a = - \left(\frac{\partial_\mu a}{f_a} \right) \bar{q} \gamma_\mu \gamma_5 q + \frac{g^2}{32\pi^2} \left(\frac{a}{f_a} \right) G\tilde{G}. \quad (152)$$

It is the second term of Eq. (152) that actually induces the potential for the axion. Because of this potential, axion will have a mass of order $m_a \sim \Lambda_{\text{QCD}}^2/f_a$.

The essentials of realistic axion model are already present in this toy model. We need to turn on weak interactions, and we need to add three families of quarks. The straightforward implementation would involve the SM extended to have two Higgs doublets, one coupling to the up-type quarks, and the other coupling to the down-type quarks [68]. A global $U(1)$ can then be defined classically, which has a QCD anomaly. The axion will now be part of the Higgs doublet, with the axion decay constant $f_a \sim v \sim 10^2$ GeV. The couplings of the axion to quarks, Eq. (152), will now be rather strong. The decay $K^+ \rightarrow \pi^+ a$ will occur at an observable strength. This process has been searched for, but has not been observed. Negative results in searches for this and other such processes have excluded the weak scale axion model.

Acceptable axion models of the “invisible” type [70, 71] involving high scale PQ symmetry breaking are fully consistent. In the model of Ref. [70], in addition to the two Higgs doublets, a complex singlet Higgs scalar S is also introduced. The axion decay constant f_a is now the VEV of S , which can be much above the weak scale. The axion is primarily in S , with very weak couplings to the SM fermions. There are non-trivial constraints from astrophysics and cosmology on such a weakly interacting light particle. For example, axion can be produced inside supernovae. Once produced, they will escape freely, draining the supernova of its energy. Consistency with supernova observations requires that $f_a > 10^9$ GeV. Cosmological abundance of the axion requires that $f_a < 10^{12}$ GeV.

In the invisible axion model of Ref. [71], there is only a single Higgs doublet of the SM. A Higgs singlet R and a heavy quark Q , which has vectorial properties under the SM, are introduced. The PQ $U(1)$ symmetry acts on Q and the scalar R . Q acquires its mass only via its Yukawa coupling with R . (This example is essentially the same as the toy model described above.) The phase of R is the axion in this case, with phenomenology similar to, but somewhat different from, the axion model of Ref. [70].

It should be noted that axion is a leading candidate for the cosmological dark matter. For reviews of axion physics, astrophysics, cosmology, and detection techniques, see Ref. [72].

7.2 Solving strong CP problem with Parity symmetry

There is another class of solution to the strong CP problem. One can assume Parity [73, 74] to set $\theta = 0$. If the fermion mass matrices have real determinant, then $\bar{\theta}$ can be zero at the tree level. Loop induced $\bar{\theta}$ needs to be small, but this is not difficult to realize.

Let me illustrate this idea with the left–right symmetric model which has Parity invariance. The Yukawa couplings are hermitian in this setup. To make the mass matrices also hermitian, we must ensure that the VEVs of scalars are real. This is easily done in the SUSY version, which is what I will describe [74]. In SUSY models, one should also take into account the contributions from the gluino to $\bar{\theta}$.

The model is the SUSY version of left–right symmetric model based on the gauge symmetry $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ discussed in Sec. 3. Two bi-doublet scalars $\Phi_i(1, 2, 2, 0)$ ($i = 1, 2$) are used to generate quark and lepton masses as well as CKM mixings. The relevant superpotential is given as

$$W = Y_u Q Q^c \Phi_u + Y_d Q Q^c \Phi_d. \quad (153)$$

The Yukawa coupling matrices Y_u and Y_d will be hermitian, owing to Parity invariance. Parity also implies that the QCD Lagrangian parameter $\theta = 0$ and that the gluino mass is real. The soft SUSY breaking A -terms, analogous to W in Eq. (153) will also be hermitian. We shall consider the case where these A terms are proportional to the respective Yukawa matrices. Furthermore, we assume universal masses for the squarks, as in minimal supergravity, or in gauge mediated SUSY breaking models.

The quark mass matrices $M_{u,d}$ are hermitian at tree level since the VEVs of the bi-doublet scalars turn out to be real. Therefore $\bar{\theta} = 0$ at tree level. We wish to demonstrate that loop induced contributions to $\bar{\theta}$ are not excessive. Note that this setup has two hermitian matrices Y_u and Y_d which are complex, with all other (flavor singlet) parameters being real.

Since parity is broken at a high scale (denoted as v_R), a nonzero value of $\bar{\theta}$ will be induced at the weak scale through renormalization group extrapolation below v_R . This is because the SM gauge symmetry does not permit the Yukawa couplings to remain hermitian. The induced $\bar{\theta}$ will have the general structure given by

$$\delta\bar{\theta} = \text{ImTr}[\Delta M_u M_u^{-1} + \Delta M_d M_d^{-1}] - 3 \text{Im}(\Delta M_{\tilde{g}} M_{\tilde{g}}^{-1}) \quad (154)$$

where $M_{u,d,\tilde{g}}$ denote the tree level contribution to the up–quark matrix, down–quark matrix and the gluino mass respectively, and $\Delta M_{u,d,\tilde{g}}$ are the loop corrections. To estimate the corrections

from ΔM_u and ΔM_d , we note that the beta function for the evolution of Y_u below v_R is given by $\beta_{Y_u} = Y_u/(16\pi^2)(3Y_u^\dagger Y_u + Y_d^\dagger Y_d + G_u)$ with the corresponding one for Y_d obtained by the interchange $Y_u \leftrightarrow Y_d$ and $G_u \rightarrow G_d$. Here G_u is a family-independent contribution arising from gauge bosons and the $\text{Tr}(Y_u^\dagger Y_u)$ term. The $3Y_u^\dagger Y_u$ term and the G_u term cannot induce non-hermiticity in Y_u , given that Y_u is hermitian at v_R . The interplay of Y_d with Y_u will however induce deviations from hermiticity. Repeated iteration of the solution with $Y_u \propto Y_u Y_d^\dagger Y_d$ and $Y_d \propto Y_d Y_u^\dagger Y_u$ in these equations will generate the following structure:

$$\delta\bar{\theta} \simeq \left(\frac{\ln(M_U/M_W)}{16\pi^2} \right)^4 [c_1 \text{ImTr}(Y_u^2 Y_d^4 Y_u^4 Y_d^2) + c_2 \text{ImTr}(Y_d^2 Y_u^4 Y_d^4 Y_u^2)], \quad (155)$$

where M_U is the unification scale. Here c_1 and c_2 are order one coefficients which are not equal. To estimate the induced $\bar{\theta}$, we choose a basis where Y_u is diagonal, $Y_u = D$ and $Y_d = V D' V^\dagger$ where $D_u v_u = \text{diag}(m_u, m_c, m_t)$, $D_d v_d = \text{diag}(m_d, m_s, m_b)$ with V being the CKM matrix. The Trace of the first term in Eq. (155) is then $\text{Im}(D_i^2 D_k^4 D_j^4 D_l^2 V_{ij} V_{kl} V_{il}^* V_{kj}^*)$. The leading contribution in this sum is $(m_t^4 m_c^2 m_b^4 m_s^2)/(v_u^6 v_d^6) \text{Im}(V_{cb} V_{ts} V_{cs}^* V_{tb}^*)$. The second Trace in Eq.(155) is identical, except that it has an opposite sign. Numerically we find

$$\delta\bar{\theta} \sim 3 \times 10^{-27} (\tan \beta)^6 (c_1 - c_2), \quad (156)$$

which is well below the experimental limit of 10^{-10} from neutron EDM.

There are also finite corrections to the quark and gluino masses, which are not contained in the RG equations. Consider first the finite one loop corrections to the quark mass matrices. A typical diagram involving the exchange of squarks and gluino is shown in Fig. 15, where the crosses on the \tilde{Q} and \tilde{Q}^c lines represent (LL) and (RR) mass insertions that will be induced in the process of RGE evolution. From this figure we can estimate the form for $\Delta M_u = \frac{2\alpha_s}{3\pi} m_{\tilde{Q}}^2 A_u m_{\tilde{u}^c}^2$ where \tilde{Q} is the squark doublet and \tilde{u}^c is the right-handed singlet up squark. Without RGE effects, the trace of this term will be real, and will not contribute to $\bar{\theta}$. Looking at the RGE for $m_{\tilde{u}^c}^2$ upto two loop order, we see that for the case of proportionality of A_u and Y_u , $m_{\tilde{u}^c}^2$ gets corrections having the form $m_0^2 Y_u^2$ or $m_0^2 Y_u^4$ or $m_0^2 Y_u Y_d^2 Y_u$. Therefore in $\Delta M_u M_u^{-1}$, the M_u^{-1} always cancels and we are left with a product of matrices of the form $Y_u^n Y_d^m Y_u^p Y_d^q \dots$. A similar comment applies when we look at the RGE corrections for $m_{\tilde{Q}}^2$ or A_u . If the product is hermitian, then its trace is real. So to get a nonvanishing contribution to theta, we have to find the lowest order product of Y_u^2 and Y_d^2 that is non-hermitian and we get

$$\delta\bar{\theta} = \frac{2\alpha_s}{3\pi} \left(\frac{\ln(M_U/M_W)}{16\pi^2} \right)^4 (k_1 \text{ImTr}[Y_u^2 Y_d^4 Y_u^4 Y_d^2] + k_2 \text{ImTr}[Y_d^2 Y_u^4 Y_d^4 Y_u^2]) \quad (157)$$

where $k_{1,2}$ are calculable constants. The numerical estimate of this contribution parallels that of the previous discussions, $\delta\bar{\theta} \sim (k_1 - k_2) \times 10^{-28} (\tan \beta)^6$. The contributions from the up-

quark and down quark matrices tend to cancel, but since the \tilde{d}^c and the \tilde{u}^c squarks are not degenerate, $k_1 \neq k_2$ and the cancellation is incomplete.

In Fig. 15 we have also displayed the one-loop contribution to the gluino mass arising from the quark mass matrix. Here again one encounters the imaginary trace of two hermitian matrices Y_u and Y_d , in the case of universality and proportionality of SUSY breaking parameters. Our estimate for $\delta\bar{\theta}$ is similar to that of the quark mass matrix of Eq. (157).

This exercise shows that the strong CP problem can be consistently resolved with the imposition of parity symmetry.

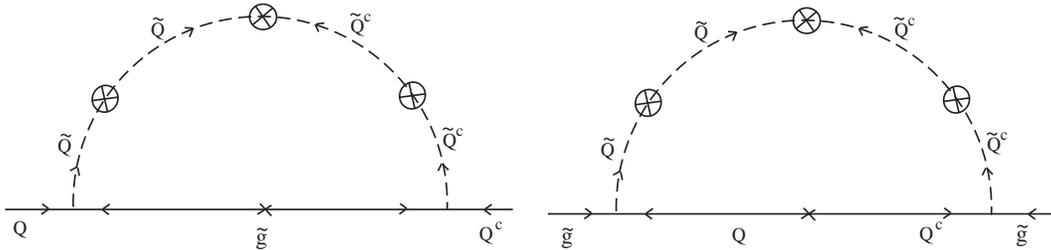


Figure 15: One loop diagram inducing complex correction to the quark mass (left) and to the gluino mass (right).

7.3 Solving the strong CP problem by CP symmetry

The idea of Ref. [75] is to use CP as a spontaneously broken symmetry. The QCD θ is then zero. In order to generate KM CP violation in weak interactions, the mass matrices of the up and down quarks will have to be complex. This can be realized consistently, while keeping the determinant of the quark mass matrix real by breaking CP spontaneously. Then at tree-level $\bar{\theta}$ will be zero.

A model of this type can be readily constructed. Consider the addition of three vector-like $D + D^c$ quarks to the SM. These are $SU(2)$ singlets with $Y = \mp 2/3$, so that they can mix with the down-type quarks (d, d^c) of the SM. Suppose there is a discrete symmetry Z_2 under which the d^c quarks reflect sign. Several SM singlet Higgs scalar fields S_i with $i \geq 2$ are also needed. Under Z_2 these S_i fields are odd.

The Yukawa Lagrangian of this theory is given by

$$\mathcal{L}_{\text{Yuk}} = Y_u Q u^c H + Y_D Q D^c \tilde{H} + M_D D d^c + F_i D D^c S_i + h.c. \quad (158)$$

CP invariance implies that all the coupling matrices (Y_u, Y_D, M_D, F_i) are real. Complex phases appear only in the VEVs of the S_i fields, which break CP spontaneously. The down-type quark

mass matrix arising from Eq. (158) is given by

$$M_{d-D} = \begin{pmatrix} 0 & Y_D v \\ M_D & \sum_i F_i \langle S_i \rangle \end{pmatrix}. \quad (159)$$

When the heavy D states are integrated out, the light 3×3 quark mass matrix for the down quarks will have a complex form, yielding weak CP violation. The determinant of M_{d-D} is real, owing to its structure (with all complex phases residing in the lower right-hand block). So $\bar{\theta} = 0$ in this model at tree level.

Loop corrections will induce non-zero $\bar{\theta}$ at the one loop level, which has a magnitude of order $\bar{\theta} \sim F^2/(16\pi^2)$. For $F \sim 10^{-4}$, this induced $\bar{\theta}$ will be within experimental limits.

8 Rare B meson decay and new physics

In this section we turn to specific processes where new physics may show up at colliders. It is quite likely that such processes will show up first in the heaviest fermion (t, b, τ) systems. Specifically, LHCb will be sensitive to such effects occurring in the B meson system. We focus on this system here.

New physics may show up at the LHC in decays of the B meson that are rare or forbidden in the SM. Low energy supersymmetry can provide such possibilities. Specifically, in the framework of SUSY with minimal flavor violation [76], that is, flavor violation arising only via the MSSM Yukawa couplings, there are processes that are enhanced at large $\tan\beta$ which can be in the observable range.

One such example is the rare decay $B_{s,d} \rightarrow \mu^+ \mu^-$ that has not been observed so far. In the SM, this process occurs via penguin and box diagrams. The branching ratio has been calculated to be [77]

$$\begin{aligned} Br(B_s \rightarrow \mu^+ \mu^-) &= (3.35 \pm 0.32) \times 10^{-9}, \\ Br(B_d \rightarrow \mu^+ \mu^-) &= (1.03 \pm 0.09) \times 10^{-10}. \end{aligned} \quad (160)$$

This prediction is to be compared with the current experimental limits from CDF and D0 [5]

$$\begin{aligned} Br(B_s \rightarrow \mu^+ \mu^-) &< (5.8 \pm 0.32) \times 10^{-8}, \\ Br(B_d \rightarrow \mu^+ \mu^-) &< (1.8 \pm 0.09) \times 10^{-8}. \end{aligned} \quad (161)$$

There is a lot of room for new physics in these processes. At the LHC, sensitivity of the experiments will be better than the SM prediction.

8.1 $B_s \rightarrow \mu^+ \mu^-$ in MSSM at large $\tan \beta$

Minimal supersymmetry at large $\tan \beta$ can significantly enhance the decay rate $B_s \rightarrow \mu^+ \mu^-$. This occurs via exchange of Higgs bosons of MSSM [78]. MSSM Yukawa couplings do preserve flavor at the tree level, see Eq. (41). That is, in the quark sector only H_u couples to the up-quarks, while only H_d couples to the down-quarks. There is no tree-level FCNC mediated by the Higgs boson. However, this situation changes once loop corrections to the Yukawa couplings are included.

To see this, let us begin by writing the effective Lagrangian for the interactions of the two Higgs doublets with the quarks in an arbitrary basis:

$$-\mathcal{L}_{eff} = \bar{D}_R \mathbf{Y}_D Q_L H_d + \bar{D}_R \mathbf{Y}_D \left[\epsilon_g + \epsilon_u \mathbf{Y}_U^\dagger \mathbf{Y}_U \right] Q_L H_u^* + h.c. \quad (162)$$

Here \mathbf{Y}_D and \mathbf{Y}_U are the 3×3 Yukawa matrices of the microscopic theory, while the $\epsilon_{g,u}$ are the finite, loop-generated non-holomorphic Yukawa coupling coefficients. The leading contributions to ϵ_g and ϵ_u are generated by the two diagrams in Fig. 16.

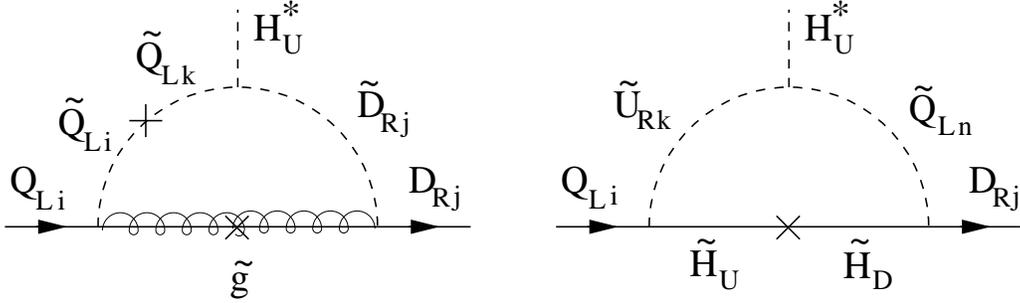


Figure 16: One loop diagram inducing τ lepton mass (left) and two-loop diagram inducing the muon mass (right).

Consider the first diagram in Fig. 16. If all \tilde{Q}_i masses are assumed degenerate at some scale M_{unif} then, at lowest order, $i = k$ and the diagram contributes only to ϵ_g :

$$\epsilon_g \simeq \frac{2\alpha_3}{3\pi} \mu^* M_3 f(M_3^2, m_{\tilde{Q}_L}^2, m_{\tilde{d}_R}^2), \quad (163)$$

where

$$f(x, y, z) = -\frac{xy \log(x/y) + yz \log(y/z) + zx \log(z/x)}{(x-y)(y-z)(z-x)}. \quad (164)$$

Meanwhile, the second diagram of Fig. 16 contributes to ϵ_u :

$$\epsilon_u \simeq \frac{1}{16\pi^2} \mu^* A_U f(\mu^2, m_{\tilde{Q}_L}^2, m_{\tilde{u}_R}^2). \quad (165)$$

(We assume that the trilinear A -terms can be written as some flavor-independent mass times \mathbf{Y}_U .) For typical inputs, one usually finds $|\epsilon_g|$ is about 4 times larger than $|\epsilon_u|$.

Owing to these loop corrections, the CKM mixing angles receive finite corrections. In particular,

$$V_{ub} \simeq V_{ub}^0 \left[\frac{1 + \epsilon_g \tan \beta}{1 + (\epsilon_g + \epsilon_u y_t^2) \tan \beta} \right]. \quad (166)$$

The same form also holds for the corrected V_{cb} , V_{td} and V_{ts} .

For $\epsilon_u \neq 0$, however, the rotation that diagonalized the mass matrix does not diagonalize the Yukawa couplings of the Higgs fields, leading to FCNC Higgs couplings given by

$$\mathcal{L}_{FCNC} = \frac{\bar{y}_b V_{tb}^*}{\sin \beta} \chi_{FC} [V_{td} \bar{b}_R d_L + V_{ts} \bar{b}_R s_L] (\cos \beta H_u^{0*} - \sin \beta H_d^0) + h.c. \quad (167)$$

with the quark fields in the physical/mass eigenbasis, and defining

$$\chi_{FC} = \frac{-\epsilon_u y_t^2 \tan \beta}{(1 + \epsilon_g \tan \beta)[1 + (\epsilon_g + \epsilon_u y_t^2) \tan \beta]} \quad (168)$$

to parameterize the amount of flavor-changing induced.

We now consider the rare decay $B^0 \rightarrow \mu^+ \mu^-$. This occurs via emission off the quark current of a single virtual Higgs boson which then decays leptonically. The amplitude for the process $B_{(d,s)}^0 \rightarrow \mu^+ \mu^-$ is given by:

$$\mathcal{A} = \eta_{QCD} \frac{\bar{y}_b y_\mu V_{t(d,s)} V_{tb}^*}{2 \sin \beta} \chi_{FC} \langle 0 | \bar{b}_R d_L | B_{(d,s)}^0 \rangle [\bar{\mu} (a_1 + a_2 \gamma^5) \mu] \quad (169)$$

where

$$\begin{aligned} a_1 &= \frac{\sin(\beta - \alpha) \cos \alpha}{m_H^2} - \frac{\cos(\beta - \alpha) \sin \alpha}{m_h^2}, \\ a_2 &= -\frac{\sin \beta}{m_A^2}. \end{aligned} \quad (170)$$

The partial width is then

$$\Gamma(B_{(d,s)}^0 \rightarrow \mu^+ \mu^-) = \frac{\eta_{QCD}^2}{128\pi} m_B^3 f_B^2 \bar{y}_b^2 y_\mu^2 |V_{t(d,s)}^* V_{tb}|^2 \chi_{FC}^2 (a_1^2 + a_2^2). \quad (171)$$

For SUSY scalar masses of order 500 GeV, we can estimate the branching ratio to be near current experimental limit for $\tan \beta$ larger than about 30. The reason for this enhancement has to do with the dependence of this rate on $\tan \beta$. For large values of $\tan \beta$, the rate scales as $(\tan \beta)^6$. Two powers of $\tan \beta$ arise each from \bar{y}_b^2 and y_μ^2 , while the remaining two powers arise from χ_{FC}^2 .

New physics contributions in B meson system can arise in SUSY GUTs [79, 80]. Generically, these models predict large $\tilde{b}_R - \tilde{s}_R$ mixing, especially when large neutrino mixing angles are induced. As a result, there is a SUSY box diagram that contributes to $B_s - \overline{B}_s$ mixing, shown in Fig. 17. This contribution can be at the level of 30% of SM box diagram. Now, in the SM, CP violation arising from mixing in B_s is very small, but the new diagrams can significantly alter this scenario. There is also new contribution to direct B decays, which can also be comparable to the SM contribution. These ideas will therefore be tested in the near future at the LHC.

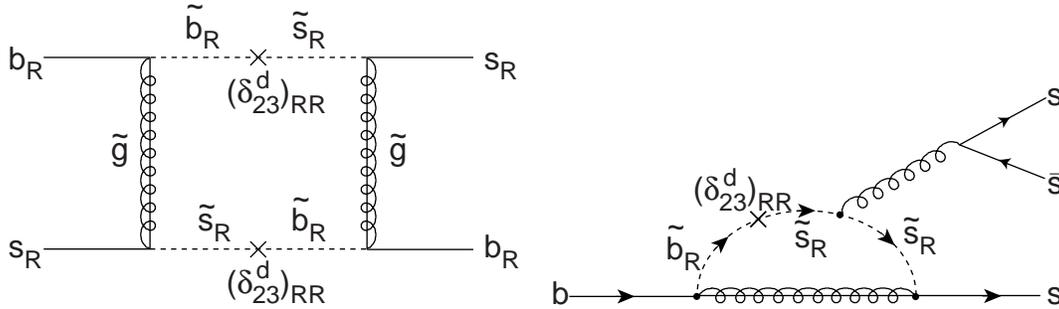


Figure 17: New physics contributions to $B_s - \overline{B}_s$ mixing and $b \rightarrow s\overline{s}s$ in SUSY GUTs.

9 Conclusion and Outlook

Flavor physics is quite rich, in these lecture notes I have only scratched the surface of a subset of the various issues.

It is a great triumph for experiment and theory, that we know so much about the fundamental parameters of the flavor sector. Even a few years ago, it looked unlikely that so much would be learned with such high precision. On the experimental side, the two B factories, BABAR and BELLE, have contributed tremendously to the improved understanding. We have seen substantial progress on the theoretical understanding, especially from lattice gauge theory and Heavy Quark Effective Theory in the last decade. Both have played crucial roles in the precise determination of the fundamental parameters of the quark flavor sector, viz., quark masses, CKM mixing angles and CP violation. While we have learned a great deal about the fundamental parameters of the neutrino sector, in these lectures we focused primarily on the quark sector.

Knowing the fundamental parameters precisely is only the start. It is imperative that we seek explanations to these observations. Any such attempt will take us beyond the realms of

the standard model. There is great hope that the LHC will actually test some of the new ideas introduced to explain some the puzzles in the flavor sector. It should, however, be cautioned that flavor dynamics could very well happen near the Planck scale, which would mask its direct effects. If there is low energy supersymmetry, there is a good chance that flavor physics, even if it occurs at a very high scale, transmits information to the SUSY breaking sector, which may be observed. The prime candidate for these effects are rare decays of the type $\ell_i \rightarrow \ell_j \gamma$. Observing such decays will show the existence of new flavor physics, but it would be impossible, from these processes alone, to distinguish between various possibilities. We have seen that neutrino mass physics, GUT physics, and flavor physics related to anomalous $U(1)$, all lead to the prediction that $\mu \rightarrow e \gamma$ is in the observable range.

I have discussed at some length some, but not all, of the popular ideas that address the puzzles from the flavor sector. The mixing–mass sum rules in the quark sector appeared quite promising, but with more precise data, many of the models in this class have already been excluded. It has become increasingly difficult to find precise patterns in the masses and mixings that fit observations. Perhaps the best setting to address these issues is supersymmetric grand unification, supplemented by flavor symmetries. SUSY GUTs are well motivated on independent grounds, they have the power to shed light on the flavor puzzle. Some recent ideas along this line are discussed in Sec. 5. I have also emphasized the close connection between the strong CP problem and the flavor puzzle. Axion solution to this problem is the most popular, but using P or CP symmetries seem to work equally well. These ideas may have collider signals, such as the discovery of right–handed W_R^\pm gauge bosons.

With some luck, the path chosen by Nature may be revealed at the LHC in the coming years. Let us wait with hope.

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