CE 513: Statistical Methods in Civil Engineering

INTRODUCTION TO STOCHASTIC CALCULUS

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Convergence of Random Process

Definitions:

Sequence of RVs: $\{X_n, \text{n} \geq 1\}$; $n \in N$

$E\,[X_n^2] < \infty$

Mean squared Error: MSE $(X_n, X) \coloneqq E\,[(X_n - X)^2]$

Limit in mean square: $\underset{n\to\infty}{\text{l.i.m.}}\ X_n = X$

Is the same way of stating $\lim_{n\to\infty} \text{MSE}\,(X_n, X) \to 0$

Convergence of Random Process

Let $\{X_n\}$ *converge in mean square to* X*. Then it holds for* $n \to \infty$

(a) $E(X_n) \to E(X)$;

(b) $E(X_n^2) \to E(X^2)$;

(c) *if* $\{X_n\}$ *is Gaussian, then* X *follows a Gaussian distribution as well*

Mean Square Calculus

A random process X(t) is said to be continuous in mean square (m.s.) if

$$\lim_{\varepsilon \to 0} E\,\{\,[X(t+\varepsilon) - X(t)]^2\} = 0$$

Theorem:

A random process X(t) is m.s. continuous, then it is also continuous in mean

$$\lim_{\varepsilon \to 0} \mu_X(t+\varepsilon) = \mu_X(t)$$

Mean Square Calculus

Theorem:
A random process X(t) is said to be m.s. continuous if and only if its autocorrelation function $R_X(t,s)$ is continuous

Theorem:
A WSS stationary random process X(t) is said to be m.s. continuous if and only if its autocorrelation function $R_X(\tau)$ is continuous at $\tau = 0$

Lemma: Wiener process X(t) is m.s. continuous

Mean Square Calculus

A random process X(t) is said to have a m.s. derivative X'(t) if

$$\underset{\varepsilon\to 0}{\text{l.i.m.}}\frac{X(t+\varepsilon)-X(t)}{\varepsilon}=X'(t)$$

$\underset{\varepsilon\to 0}{\text{l.i.m.}}$ implies limit in mean square

X(t) has the m.s. derivative X'(t), then its mean and autocorrelation-function are given by

$$E[X'(t)]=\frac{d}{dt}E[X(t)]=\mu'_X(t)$$

$$R_{X'}(t,s)=\frac{\partial^2 R_X(t,s)}{\partial t\,\partial s}$$

Mean Square Calculus

Theorem:

A random process X(t) has a m.s. derivative $X'(t)$ if $\frac{\partial^2 R_X(t,s)}{\partial t\ \partial s}$ exists at s = t

Mean Square Calculus

The autocorrelation function of X'; $R_{X'}(t,s)$ is $\frac{\partial^2 R_X(t,s)}{\partial t \, \partial s}$

If X(t) is WSS then $R_{X'}(\tau) = -\frac{d^2 R_X(\tau)}{d\tau^2}$

Although Wiener process X(t) is m.s. continuous, it does not possess m.s. derivative.

Mean Square Calculus

- Wiener process X(t) does not possess m.s. derivative

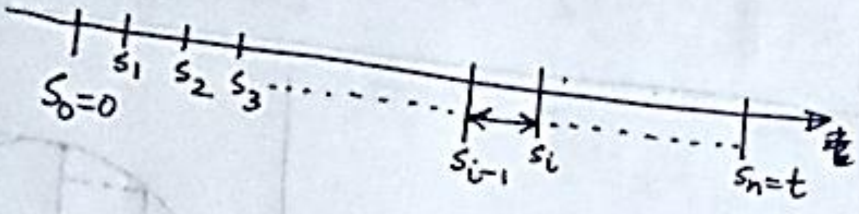
$$\frac{\partial R_X(t,\ s)}{\partial s} = \sigma^2\, \mathcal{U}(t-s)$$

where $\mathcal{U}$ is the unit step function

- However, a generalized derivative can be defined by using the relationship between step function and the Dirac Delta function

$$R_{X'}(t,s) = \frac{\partial^2 R_X(t,\ s)}{\partial s \partial t} = \sigma^2\, \delta(t-s)$$

Mean Square Riemann Integral



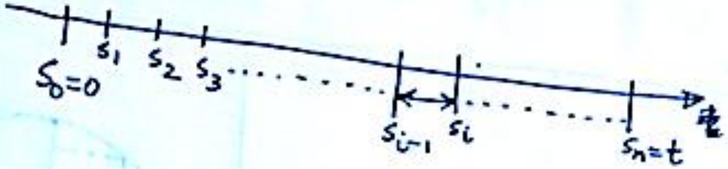
$$\max_{1 \le i \le n} (s_i - s_{i-1}) \to 0 \quad \text{as } n \to \infty$$

$$s_i^* \in [s_{i-1}, s_i] \quad i = 1, 2, \ldots\ldots, n$$

It holds that $\sum_{i=1}^{n} (s_i - s_{i-1}) = s_n - s_0 = t$

In general for a function $\sum_{i=1}^{n} \{ \phi(s_i) - \phi(s_{i-1}) \} = \phi(t) - \phi(0)$

Mean Square Riemann Integral



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The Riemann sum R_n

$$R_n = \sum_{i=1}^{n} f(s_i^*)\, X(s_i^*)(s_i - s_{i-1})$$

If the limit of the sum exits as $n\to\infty$ uniquely and independent of the choice of s_i^* and $(s_i - s_{i-1})$ then we define the Riemann sum as

$$\sum f(s_i^*)\, X(s_i^*)\,(s_i - s_{i-1}) \xrightarrow{2} \int_0^t f(s) X(s)\, ds$$

(2 → in mean square)

Mean Square Riemann Integral

Existence : For the integral $\int_0^t f(s)\,X(s)\,ds$ to exist

$$\int_0^t \int_0^t f(s)\, f(r)\, \mathbb{E}\{X(s)\, X(r)\}\, dr\, ds$$

must exist in a mean square sense

Fubini's Theorem

$$E\left(\int_0^t X(s)\, ds\right) = \int_0^t E(X(s))\, ds$$

Mean Square Riemann Integral

Variance of Riemann Integral $\int_0^t f(s)W(s)ds$

$$\begin{aligned}\operatorname{Var}(Y(t)) &= \mathrm{E}\left[Y^2(t)\right] \\ &= \mathrm{E}\left[\int_0^t f(s)\,W(s)ds \int_0^t f(r)\,W(r)dr\right] \\ &= \mathrm{E}\left[\int_0^t \left(\int_0^t f(r)\,W(r)dr\right) f(s)\,W(s)ds\right] \\ &= \mathrm{E}\left[\int_0^t \left(\int_0^t f(r)f(s)\,W(r)W(s)dr\right) ds\right]\end{aligned}$$

Now Apply Fubini's Theorem twice and derive the final integral form of

$$E[Y^2(t)] = ??$$

Mean Square Riemann Integral

Example-1: Variance of the Integrated WP

$$\mathrm{Var}\left(\int_0^t W(s)ds\right) = \int_0^t \int_0^t \min(r,s)\, drds$$

The integral with respect to *r* is decomposed into the sum of two integrals with *s* as the integration limit such that the minimum function can be specified explicitly

$$\int_0^t \int_0^t \min(r,s)\, drds = \int_0^t \left[\int_0^s \min(r,s)\, dr + \int_s^t \min(r,s)\, dr\right] ds$$

$$= \int_0^t \left[\int_0^s r\, dr + \int_s^t s\, dr\right] ds\,.$$

$$\mathrm{Var}\left(\int_0^t W(s)ds\right) = \int_0^t \left[\int_0^s r\, dr + \int_s^t s\, dr\right] ds$$

$$= \int_0^t \left[\frac{s^2}{2} + s\,(t-s)\right] ds = \left[\frac{s^2\, t}{2} - \frac{s^3}{6}\right]_0^t = \frac{t^3}{3}$$