CE 607: Random Vibration

Budhaditya Hazra

Room N-307

Department of Civil Engineering

Quadratic variation of Wiener Processes

Quadratic variation of Wiener Processes

Again, the considerations are based on an adequate partition of the interval $[0, t]$,

$$P_n([0,t]) : 0 = s_0 < s_1 < \ldots < s_n = t .$$

For a function g the **variation** over this partition is defined as :

$$V_n(g,t) = \sum_{i=1}^{n} |g(s_i) - g(s_{i-1})| .$$

If the limit exists independently of the decomposition for $n \to \infty$ one says that g is of finite variation and writes :

$$V_n(g,t) \to V(g,t), \quad n \to \infty .$$

Quadratic variation of Wiener Processes

The finite sum $V_n(g, t)$ measures for a certain partition the absolute increments of the function g on the interval $[0, t]$. If the function evolves sufficiently smooth, then $V(g, t)$ takes on a finite value for $(n \to \infty)$.

For very jagged functions

an increasing refinement $(n \to \infty)$ the increments of the graph of g become larger and larger even for fixed t, such that g is not of finite variation.

Quadratic variation of Wiener Processes

PROPOSITION-1 **(Variation of Continuously Differentiable Functions)** *Let g be a continuously differentiable function with derivative g' on $[0, t]$. Then g is of finite variation and it holds that*

$$V(g, t) = \int_0^t |g'(s)| \, ds.$$

Example: Let's see when this fails

(Sine Wave) Let us consider a sine cycle of the frequency k on the interval $[0, 2\pi]$:

$$g_k(s) = \sin(ks), \quad k = 1, 2, \ldots .$$

Quadratic variation of Wiener Processes

(Sine Wave) Let us consider a sine cycle of the frequency k on the interval $[0, 2\pi]$:

$$g_k(s) = \sin(ks), \quad k = 1, 2, \ldots .$$

$$g_k'(s) = k \cos(ks).$$

Accounting for the sign one obtains as the variation:

$$\begin{aligned} V(g_1, 2\pi) &= \int_0^{2\pi} |\cos(s)|\, ds = 4 \int_0^{\pi/2} \cos(s)\, ds \\ &= 4 \left(\sin\left(\frac{\pi}{2}\right) - \sin(0) \right) \\ &= 4\,, \end{aligned}$$

$$\begin{aligned} V(g_2, 2\pi) &= \int_0^{2\pi} 2\,|\cos(2s)|\, ds = 8 \int_0^{\pi/4} 2\cos(2s)\, ds \\ &= 8 \left(\sin\left(\frac{\pi}{2}\right) - \sin(0) \right) \\ &= 8\,, \end{aligned}$$

Quadratic variation of Wiener Processes

$$V(g_k, 2\pi) = \int_0^{2\pi} k\,|\cos(ks)|\; ds = 4k \int_0^{\pi/2k} k\cos(ks)\, ds$$
$$= 4k\left(\sin\left(\frac{\pi}{2}\right) - \sin(0)\right)$$
$$= 4k\,.$$

It can be observed, how the sum of (absolute) differences in amplitude grows with k growing. Accordingly, the absolute variation of $g_k(s) = \sin(ks)$ multiplies with k.

For $k \to \infty$, g_k' tends to infinity such that this derivative is not continuous anymore. Consequently, the absolute variation is not finite in the limiting case $k \to \infty$.

Quadratic variation of Wiener Processes

Quadratic Variation

In the same way as $V_n(g,t)$ a q-variation can be defined where we are only interested in the case $q = 2$, – the quadratic variation:

$$Q_n(g,t) = \sum_{i=1}^{n} |g(s_i) - g(s_{i-1})|^2 = \sum_{i=1}^{n} (g(s_i) - g(s_{i-1}))^2 \ .$$

As would seem natural, g is called of finite quadratic variation if it holds that

$$Q_n(g,t) \ \to \ Q(g,t) \, , \quad n \to \infty \, .$$

Quadratic variation of Wiener Processes

PROPOSITION-2 **(Absolute and Quadratic Variation)** *Let g be a continuous function on $[0, t]$. It then holds for $n \to \infty$:*

$$V_n(g, t) \to V(g, t) < \infty$$

implies

$$Q_n(g, t) \to 0 .$$

If g is a stochastic process, then "$\to$" is to be understood as convergence in mean square.

Quadratic variation of Wiener Processes

(Quadratic Variation of the WP) *For the Wiener process with* $n \to \infty$

$$Q_n(W,t) \xrightarrow{2} t = Q(W,t)\,.$$

The expression $Q(W,t) = t$ characterizes the level of jaggedness or irregularity of the Wiener process on the interval $[0,t]$. This non-vanishing quadratic variation causes the problems and specifics of the Ito integral.

Rephrasing it one can state

If the
Wiener process was continuously differentiable, then it would be of finite variation due to Proposition-**1** and it would have a vanishing quadratic variation due to Proposition-**2** . However, this is just not the case.

Quadratic variation of Wiener Processes

To prove: $Q_n(W,t) \to t = Q(W,t)$

1st stage: Show $E\{[Q_n(W,t) - t]^2\} \to 0$ as $\lim n \to \infty$

$$Q_n(W,t) = \sum_{i=1}^{n} (W(s_i) - W(s_{i-1}))^2$$

$$\mathrm{E}\,(Q_n(W,t)) = \sum_{i=1}^{n} \mathrm{Var}\,(W(s_i) - W(s_{i-1}))$$

$$= \sum_{i=1}^{n} (s_i - s_{i-1}) = s_n - s_0 = t$$

Quadratic variation of Wiener Processes

To prove: $Q_n(W,t) \to t$

2nd stage: Show $Var\,\{Q_n(W,t)\} \to 0$ as $\lim n \to \infty$

$$\mathrm{Var}\,(Q_n(W,t)) = \sum_{i=1}^{n} \mathrm{Var}\left[(W(s_i) - W(s_{i-1}))^2\right]$$

$$\begin{aligned}\mathrm{Var}\left[(W(s_i) - W(s_{i-1}))^2\right] &= \mathrm{E}\left[(W(s_i) - W(s_{i-1}))^4\right] - \left(\mathrm{E}\left[(W(s_i) - W(s_{i-1}))^2\right]\right)^2 \\ &= 3\;[\mathrm{Var}(W(s_i) - W(s_{i-1}))]^2 - (s_i - s_{i-1})^2 \\ &= 2\,(s_i - s_{i-1})^2\,.\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(Q_n(W,t)) \;&=\; 2\sum_{i=1}^{n}(s_i - s_{i-1})^2 \\ &\le\; 2\max_{1\le i\le n}(s_i - s_{i-1})\sum_{i=1}^{n}(s_i - s_{i-1}) \\ &=\; 2\max_{1\le i\le n}(s_i - s_{i-1})\,(s_n - s_0) \\ &\to 0\,, \quad n \to \infty\,,\end{aligned}$$