MA 101 (Mathematics I)

Solutions for Tutorial Problem Set

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1. Let (x_n) be a convergent sequence of positive real numbers such that $\lim\limits_{n\to\infty} x_n < 1$. Show that $\lim\limits_{n\to\infty} x_n^n = 0$.
   Solution: If $\ell = \lim\limits_{n\to\infty} x_n$, then $\frac{1}{2}(1-\ell) > 0$ and so there exists $n_0 \in \mathbb{N}$ such that $|x_n - \ell| < \frac{1}{2}(1-\ell)$ for all $n \geq n_0$. Hence $0 < x_n < \frac{1}{2}(1+\ell)$ for all $n \geq n_0 \Rightarrow 0 < x_n^n < (\frac{1+\ell}{2})^n$ for all $n \geq n_0$. Since $\frac{1}{2}(1+\ell) < 1$, $\lim\limits_{n\to\infty}(\frac{1+\ell}{2})^n = 0$. Therefore by sandwich theorem, $\lim\limits_{n\to\infty} x_n^n = 0$.

   Alternative solution: Since $\lim\limits_{n\to\infty}(x_n^n)^{\frac{1}{n}} = \lim\limits_{n\to\infty} x_n < 1$, by root test, the series $\sum\limits_{n=1}^{\infty} x_n^n$ converges and hence $\lim\limits_{n\to\infty} x_n^n = 0$.

2. Let (x_n) be a convergent sequence in $\mathbb{R}$ with limit $\ell \in \mathbb{R}$ and let $\alpha \in \mathbb{R}$.
   (a) If $x_n > \alpha$ for all $n \in \mathbb{N}$, then show that $\ell \geq \alpha$.
   (b) If $\ell > \alpha$, then show that there exists $n_0 \in \mathbb{N}$ such that $x_n > \alpha$ for all $n \geq n_0$.

   (Note that ℓ can be equal to α in (a).)
   Solution: (a) If possible, let $\ell < \alpha$. Then $\alpha - \ell > 0$ and since $x_n \to \ell$, there exists $n_0 \in \mathbb{N}$ such that $|x_n - \ell| < \alpha - \ell$ for all $n \geq n_0$. This implies that $x_n < \ell + \alpha - \ell = \alpha$ for all $n \geq n_0$, which is a contradiction. Hence $\ell \geq \alpha$.
   (b) Since $\ell - \alpha > 0$ and since $x_n \to \ell$, there exists $n_0 \in \mathbb{N}$ such that $|x_n - \ell| < \ell - \alpha$ for all $n \geq n_0$. This implies that $x_n > \ell - (\ell - \alpha) = \alpha$ for all $n \geq n_0$.
   (Note that although $\frac{1}{n} > 0$ for all $n \in \mathbb{N}$, $\lim\limits_{n\to\infty}\frac{1}{n} = 0$ and thus ℓ can be equal to α in (a).)

3. For $\alpha \in \mathbb{R}$, examine whether $\lim\limits_{n\to\infty}\frac{1}{n^2}([\alpha] + [2\alpha] + \cdots + [n\alpha])$ exists (in $\mathbb{R}$). Also, find the value if it exists.
   (For each $x \in \mathbb{R}$, $[x]$ denotes the greatest integer not exceeding x.)
   Solution: For each $x \in \mathbb{R}$, $[x] \leq x < [x] + 1 \Rightarrow x - 1 < [x] \leq x$. Hence, for all $n \in \mathbb{N}$, $\frac{1}{n^2}\{(\alpha - 1) + (2\alpha - 1) + \cdots + (n\alpha - 1)\} < x_n \leq \frac{1}{n^2}(\alpha + 2\alpha + \cdots + n\alpha) \Rightarrow \frac{\alpha}{2}(1 + \frac{1}{n}) - \frac{1}{n} < x_n \leq \frac{\alpha}{2}(1 + \frac{1}{n})$ for all $n \in \mathbb{N}$. Since $\frac{\alpha}{2}(1 + \frac{1}{n}) - \frac{1}{n} \to \frac{\alpha}{2}$ and $\frac{\alpha}{2}(1 + \frac{1}{n}) \to \frac{\alpha}{2}$, by sandwich theorem, (x_n) is convergent and $\lim\limits_{n\to\infty} x_n = \frac{\alpha}{2}$.

4. Let $x_1 = 6$ and $x_{n+1} = 5 - \frac{6}{x_n}$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find $\lim\limits_{n\to\infty} x_n$ if (x_n) is convergent.
   Solution: We have $x_1 > 3$ and if we assume that $x_k > 3$ for some $k \in \mathbb{N}$, then $x_{k+1} > 5 - 2 = 3$. Hence by the principle of mathematical induction, $x_n > 3$ for all $n \in \mathbb{N}$. Therefore (x_n) is bounded below. Again, $x_2 = 4 < x_1$ and if we assume that $x_{k+1} < x_k$ for some $k \in \mathbb{N}$, then $x_{k+2} - x_{k+1} = 6(\frac{1}{x_k} - \frac{1}{x_{k+1}}) < 0 \Rightarrow x_{k+2} < x_{k+1}$. Hence by the principle of mathematical induction, $x_{n+1} < x_n$ for all $n \in \mathbb{N}$. Therefore (x_n) is decreasing. Consequently (x_n) is convergent. Let $\ell = \lim\limits_{n\to\infty} x_n$. Then $\lim\limits_{n\to\infty} x_{n+1} = 5 - \frac{6}{\lim\limits_{n\to\infty} x_n} \Rightarrow \ell = 5 - \frac{6}{\ell}$ (since $x_n > 3$ for all $n \in \mathbb{N}$, $\ell \neq 0$) $\Rightarrow (\ell - 2)(\ell - 3) = 0 \Rightarrow \ell = 2$ or $\ell = 3$. But $x_n > 3$ for all $n \in \mathbb{N}$, so $\ell \geq 3$. Therefore $\ell = 3$.

   Alterbative solution: For all $n \in \mathbb{N}$, we have $|x_{n+2} - x_{n+1}| = \frac{6}{|x_{n+1}||x_n|}|x_{n+1} - x_n|$. Also, as shown in the above solution, $x_n > 3$ for all $n \in \mathbb{N}$. Hence $|x_{n+2} - x_{n+1}| \leq \frac{2}{3}|x_{n+1} - x_n|$ for all $n \in \mathbb{N}$. It follows that (x_n) is a Cauchy sequence in $\mathbb{R}$ and hence (x_n) is convergent. To show that $\lim\limits_{n\to\infty} x_n = 3$, we proceed as in the above solution.

5. Let (x_n) be a sequence of nonzero real numbers. If (x_n) does not have any convergent subsequence, then show that $\lim\limits_{n\to\infty}\frac{1}{x_n} = 0$.

Solution: If $\lim\limits_{n\to\infty} \frac{1}{x_n} \neq 0$, then there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$, there exists a positive integer $m > n$ satisfying $|\frac{1}{x_m}| \geq \varepsilon$, *i.e.* $|x_m| \leq \frac{1}{\varepsilon}$. Thus we get positive integers $n_1 < n_2 < \cdots$ such that $|x_{n_k}| \leq \frac{1}{\varepsilon}$ for each $k \in \mathbb{N}$. So (x_{n_k}) is a bounded subsequence of (x_n) and hence by Bolzano-Weierstrass theorem, (x_{n_k}) has a convergent subsequence, which is also a convergent subsequence of (x_n), which contradicts the hypothesis. Therefore $\lim\limits_{n\to\infty} \frac{1}{x_n} = 0$.

Alternative solution: Let $\varepsilon > 0$. We claim that there exist at most finitely many $n \in \mathbb{N}$ for which $x_n \in [-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]$. Because otherwise, we get a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \in [-\varepsilon, \varepsilon]$ for all $k \in \mathbb{N}$ and so (x_{n_k}) is bounded. By Bolzano-Weierstrass theorem, (x_{n_k}) has a convergent subsequence, which is also a subsequence of (x_n). This contradicts the given hypothesis. Hence our claim is proved and so there exists $n_0 \in \mathbb{N}$ such that $|x_n| > \frac{1}{\varepsilon}$ for all $n \geq n_0$. Thus $|\frac{1}{x_n}| < \varepsilon$ for all $n \geq n_0$ and therefore $\lim\limits_{n\to\infty} \frac{1}{x_n} = 0$.

6. Examine whether the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ is convergent.

   Solution: Let $x_n = \frac{1}{n^{1+\frac{1}{n}}}$ and let $y_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = 1 \neq 0$. Since $\sum\limits_{n=1}^{\infty} y_n$ is not convergent, by the limit comparison test, $\sum\limits_{n=1}^{\infty} x_n$ is also not convergent.

7. Let $x_n > 0$ for all $n \in \mathbb{N}$. Show that the series $\sum\limits_{n=1}^{\infty} x_n$ converges iff the series $\sum\limits_{n=1}^{\infty} \frac{x_n}{1+x_n}$ converges.

   Solution: We have $0 < \frac{x_n}{1+x_n} < x_n$ for all $n \in \mathbb{N}$. Hence by comparison test, $\sum\limits_{n=1}^{\infty} \frac{x_n}{1+x_n}$ converges if $\sum\limits_{n=1}^{\infty} x_n$ converges.

   Conversely, let $\sum\limits_{n=1}^{\infty} \frac{x_n}{1+x_n}$ converge. Then $\frac{x_n}{1+x_n} \to 0$ and so there exists $n_0 \in \mathbb{N}$ such that $\frac{x_n}{1+x_n} < \frac{1}{2}$ for all $n \geq n_0$. This implies that $x_n < 1$ for all $n \geq n_0$, *i.e.* $1 + x_n < 2$ for all $n \geq n_0$ and so $x_n < \frac{2x_n}{1+x_n}$ for all $n \geq n_0$. By comparison test, we conclude that $\sum\limits_{n=1}^{\infty} x_n$ converges.

   Alternative solution: If $\sum\limits_{n=1}^{\infty} x_n$ converges, then $\lim\limits_{n\to\infty} \frac{\frac{x_n}{1+x_n}}{x_n} = \lim\limits_{n\to\infty} \frac{1}{1+x_n} = 1$ (since $x_n \to 0$) and hence by limit comparison test, $\sum\limits_{n=1}^{\infty} \frac{x_n}{1+x_n}$ converges.

   Conversely, if $\sum\limits_{n=1}^{\infty} \frac{x_n}{1+x_n}$ converges, then $\lim\limits_{n\to\infty} \frac{\frac{x_n}{1+x_n}}{x_n} = \lim\limits_{n\to\infty} \frac{1}{1+x_n} = 1 \neq 0$ (since $\frac{x_n}{1+x_n} \to 0$ and so $\frac{1}{1+x_n} = 1 - \frac{x_n}{1+x_n} \to 1$) and hence by limit comparison test, $\sum\limits_{n=1}^{\infty} x_n$ converges.

8. Find all $x \in \mathbb{R}$ for which the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n(x-1)^n}{2^n n^2}$ converges.

   Solution: If $x = 1$, then the given series becomes $0 + 0 + \cdots$, which is clearly convergent. Let $x(\neq 1) \in \mathbb{R}$ and let $a_n = \frac{(-1)^n(x-1)^n}{2^n n^2}$ for all $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{1}{2}|x-1|$. Hence by ratio test, $\sum\limits_{n=1}^{\infty} a_n$ converges (absolutely) if $\frac{1}{2}|x-1| < 1$, *i.e.* if $x \in (-1, 3)$ and does not converge if $\frac{1}{2}|x-1| > 1$, *i.e.* if $x \in (-\infty, -1) \cup (3, \infty)$. If $\frac{1}{2}|x-1| = 1$, *i.e.* if $x \in \{-1, 3\}$, then $\sum\limits_{n=1}^{\infty} |a_n| = \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges and hence $\sum\limits_{n=1}^{\infty} a_n$ converges. Therefore the set of $x \in \mathbb{R}$ for which $\sum\limits_{n=1}^{\infty} a_n$ converges is $[-1, 3]$.

   Alternative solution: Instead of ratio test, one can find $\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{2}|x-1|$ and use root test.

The remaining part is same.

9. If $\alpha(\neq 0) \in \mathbb{R}$, then show that the series $\sum\limits_{n=1}^{\infty}(-1)^n \sin(\frac{\alpha}{n})$ is conditionally convergent.
Solution: We choose $n_0 \in \mathbb{N}$ such that $\frac{|\alpha|}{n_0} < \frac{\pi}{2}$. Then for all $n \geq n_0$, $\sin(\frac{\alpha}{n})$ has the same sign as that of α. Since the sine function is increasing in $(0, \frac{\pi}{2})$, it follows that the sequence $\left(\sin(\frac{|\alpha|}{n})\right)_{n=n_0}^{\infty}$ is decreasing. Also, $\lim\limits_{n\to\infty} \sin(\frac{|\alpha|}{n}) = 0$. Hence by Leibniz's test, $\sum\limits_{n=n_0}^{\infty}(-1)^n \sin(\frac{\alpha}{n})$ is convergent. Consequently $\sum\limits_{n=1}^{\infty}(-1)^n \sin(\frac{\alpha}{n})$ is convergent.
Again, $\sum\limits_{n=1}^{\infty} |(-1)^n \sin(\frac{\alpha}{n})| = \sum\limits_{n=1}^{\infty} |\sin(\frac{\alpha}{n})|$ is not convergent by limit comparison test, since (using $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$) $\lim\limits_{n\to\infty} \frac{|\sin(\alpha/n)|}{1/n} = |\alpha| \lim\limits_{n\to\infty} \left|\frac{\sin(\alpha/n)}{\alpha/n}\right| = |\alpha| \neq 0$ and $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is not convergent. Therefore the given series is conditionally convergent.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ [x] & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
Determine all the points of $\mathbb{R}$ where f is continuous.
Solution: Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Then there exists a sequence (r_n) in $\mathbb{Q}$ such that $r_n \to x$. So $f(r_n) = r_n \to x \neq [x] = f(x)$. Hence f is not continuous at x.
Again, let $y \in \mathbb{Q}$. Then there exists a sequence (t_n) in $\mathbb{R} \setminus \mathbb{Q}$ such that $t_n < y$ for all $n \in \mathbb{N}$ and $t_n \to y$. For each $n \in \mathbb{N}$, $f(t_n) = \begin{cases} [t_n] \leq y - 1 & \text{if } y \in \mathbb{Z}, \\ [t_n] \leq [y] < y & \text{if } y \notin \mathbb{Z}. \end{cases}$
In either case $f(t_n) \not\to f(y) = y$. Hence f is not continuous at y. Therefore f is not continuous at any point of $\mathbb{R}$.

11. Let $f : [0,1] \to \mathbb{R}$ be continuous such that $f(0) = f(1)$. Show that
(a) there exist $x_1, x_2 \in [0,1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{1}{2}$.
(b) there exist $x_1, x_2 \in [0,1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{1}{3}$.

(In fact, if $n \in \mathbb{N}$, then there exist $x_1, x_2 \in [0,1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{1}{n}$. However, it is not necessary that there exist $x_1, x_2 \in [0,1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{2}{5}$.)
Solution: (a) Let $g(x) = f(x + \frac{1}{2}) - f(x)$ for all $x \in [0, \frac{1}{2}]$. Since f is continuous, $g : [0, \frac{1}{2}] \to \mathbb{R}$ is continuous. Also $g(0) = f(\frac{1}{2}) - f(0)$ and $g(\frac{1}{2}) = f(1) - f(\frac{1}{2}) = -g(0)$, since $f(0) = f(1)$. If $g(0) = 0$, then we can take $x_1 = \frac{1}{2}$ and $x_2 = 0$. Otherwise, $g(\frac{1}{2})$ and $g(0)$ are of opposite signs and hence by the intermediate value property of continuous functions, there exists $c \in (0, \frac{1}{2})$ such that $g(c) = 0$, *i.e.* $f(c + \frac{1}{2}) = f(c)$. We take $x_1 = c + \frac{1}{2}$ and $x_2 = c$.

(b) Let $g(x) = f(x + \frac{1}{3}) - f(x)$ for all $x \in [0, \frac{2}{3}]$. Since f is continuous, $g : [0, \frac{2}{3}] \to \mathbb{R}$ is continuous. Also $g(0) + g(\frac{1}{3}) + g(\frac{2}{3}) = f(1) - f(0) = 0$. If at least one of $g(0)$, $g(\frac{1}{3})$ and $g(\frac{2}{3})$ is 0, then the result follows immediately. Otherwise, at least two of $g(0)$, $g(\frac{1}{3})$ and $g(\frac{2}{3})$ are of opposite signs and hence by the intermediate value property of continuous functions, there exists $c \in (0, \frac{2}{3})$ such that $g(c) = 0$, *i.e.* $f(c + \frac{1}{3}) = f(c)$. We take $x_1 = c + \frac{1}{3}$ and $x_2 = c$.

(Assuming $n > 1$, we define $g(x) = f(x + \frac{1}{n}) - f(x)$ for all $x \in [0, 1 - \frac{1}{n}]$. Since f is continuous, $g : [0, 1 - \frac{1}{n}] \to \mathbb{R}$ is continuous. Also $g(0) + g(\frac{1}{n}) + g(\frac{2}{n}) + \cdots + g(1 - \frac{1}{n}) = f(1) - f(0) = 0$. If at least one of $g(0)$, $g(\frac{1}{n})$, ..., $g(1 - \frac{1}{n})$ is 0, then the result follows immediately. Otherwise, at least two of $g(0)$, $g(\frac{1}{n})$, ..., $g(1 - \frac{1}{n})$ are of opposite signs and hence by the intermediate value property of continuous functions, there exists $c \in (0, 1 - \frac{1}{n})$ such that $g(c) = 0$, *i.e.* $f(c + \frac{1}{n}) = f(c)$. We take $x_1 = c + \frac{1}{n}$ and $x_2 = c$.
Again, if $f(x) = \sin^2(\frac{5}{2}\pi x) - x$ for all $x \in [0,1]$, then $f : [0,1] \to \mathbb{R}$ is continuous and $f(0) = 0 = f(1)$. However, $f(x) - f(x + \frac{2}{5}) = \frac{2}{5}$ for all $x \in [0, \frac{3}{5}]$ and so no points $x_1, x_2 \in [0,1]$

exist satisfying $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{2}{5}$.)

12. Let p be an odd degree polynomial with real coefficients in one real variable. If $g : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function, then show that there exists $x_0 \in \mathbb{R}$ such that $p(x_0) = g(x_0)$.

    (In particular, this shows that
    (a) every odd degree polynomial with real coefficients in one real variable has at least one real zero.
    (b) the equation $x^9 - 4x^6 + x^5 + \frac{1}{1+x^2} = \sin 3x + 17$ has at least one real root.
    (c) the range of every odd degree polynomial with real coefficients in one real variable is $\mathbb{R}$.)
    Solution: Let $f(x) = p(x) - g(x)$ for all $x \in \mathbb{R}$. Since both p and g are continuous, $f : \mathbb{R} \to \mathbb{R}$ is continuous.
    Since g is bounded, there exists $M > 0$ such that $|g(x)| \le M$ for all $x \in \mathbb{R}$.
    Let $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ for all $x \in \mathbb{R}$, where $a_i \in \mathbb{R}$ for $i = 0, 1, ..., n$, $n \in \mathbb{N}$ is odd and $a_0 \neq 0$. So $p(x) = a_0x^n(1 + \frac{a_1}{a_0} \cdot \frac{1}{x} + \cdots + \frac{a_{n-1}}{a_0} \cdot \frac{1}{x^{n-1}} + \frac{a_n}{a_0} \cdot \frac{1}{x^n})$ for all $x(\neq 0) \in \mathbb{R}$. We assume that $a_0 > 0$. (The case $a_0 < 0$ is almost similar.) Then $\lim\limits_{x\to\infty} p(x) = \infty$ and $\lim\limits_{x\to-\infty} p(x) = -\infty$ (since n is odd). So there exist $x_1 > 0$ and $x_2 < 0$ such that $p(x_1) > M$ and $p(x_2) < -M$. Hence $f(x_1) > 0$ and $f(x_2) < 0$. By the intermediate value property of continuous functions, there exists $x_0 \in (x_2, x_1)$ such that $f(x_0) = 0$, *i.e.* $p(x_0) = g(x_0)$.

    (For (a), we take $g(x) = 0$ for all $x \in \mathbb{R}$. For (b), we take $p(x) = x^9 - 4x^6 + x^5 - 17$ and $g(x) = \sin 3x - \frac{1}{1+x^2}$ for all $x \in \mathbb{R}$ and we note that $|g(x)| \le 2$ for all $x \in \mathbb{R}$. For (c), given $y \in \mathbb{R}$, we take $g(x) = y$ for all $x \in \mathbb{R}$.)

13. Does there exist a continuous function from $(0, 1]$ onto $\mathbb{R}$? Justify.
    Solution: If $f(x) = \frac{1}{x} \sin \frac{1}{x}$ for all $x \in (0, 1]$, then $f : (0, 1] \to \mathbb{R}$ is continuous and $f(\frac{2}{(4n+1)\pi}) = 2n\pi + \frac{\pi}{2}$, $f(\frac{2}{(4n+3)\pi}) = -2n\pi - \frac{3\pi}{2}$ for all $n \in \mathbb{N}$. For each $y \in \mathbb{R}$, we can find $n \in \mathbb{N}$ such that $-2n\pi - \frac{3\pi}{2} < y < 2n\pi + \frac{\pi}{2}$ and hence by the intermediate value property of continuous functions, there exists $x \in \mathbb{R}$ such that $f(x) = y$. Thus $f : (0, 1] \to \mathbb{R}$ is onto. Therefore there exists such a function.

14. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on $(-\delta, \delta)$ for some $\delta > 0$ and let $f''(0)$ exist (in $\mathbb{R}$). If $f(\frac{1}{n}) = 0$ for all $n \in \mathbb{N}$, then find $f'(0)$ and $f''(0)$.
    Solution: Since f is continuous at 0 and since $\frac{1}{n} \to 0$, we have $f(0) = \lim\limits_{n\to\infty} f(\frac{1}{n}) = 0$. Also, since $f'(0)$ exists (in $\mathbb{R}$) and since $\frac{1}{n} \to 0$, we have $f'(0) = \lim\limits_{n\to\infty} \frac{f(\frac{1}{n})-f(0)}{1/n} = 0$. Again, we can choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \delta$. By Rolle's theorem, for each $n \ge n_0$, there exists $x_n \in (\frac{1}{n+1}, \frac{1}{n})$ such that $f'(x_n) = 0$. By sandwich theorem, $x_n \to 0$. Since $f''(0)$ exists, we have $f''(0) = \lim\limits_{n\to\infty} \frac{f'(x_n)-f'(0)}{x_n} = 0$.

15. For $n \in \mathbb{N}$, show that the equation $1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots + (-1)^n \frac{x^n}{n} = 0$ has exactly one real root if n is odd and has no real root if n is even.
    Solution: Let $p(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots + (-1)^n \frac{x^n}{n}$ for all $x \in \mathbb{R}$. Then $p'(x) = -1 + x - x^2 + \cdots + (-1)^n x^{n-1}$ for all $x \in \mathbb{R}$.
    We first assume that n is odd. By Ex.12 of Tutorial Problem Set, the equation $p(x) = 0$ has at least one real root. Also, $p'(-1) = -n \neq 0$ and $p'(x) = -(\frac{1+x^n}{1+x}) \neq 0$ for all $x \in \mathbb{R} \setminus \{-1\}$. As a consequence of Rolle's theorem, the equation $p(x) = 0$ can have at most one real root. Therefore the equation $p(x) = 0$ has exactly one real root.
    We now assume that n is even. Then $p'(-1) = -n < 0$ and $p'(x) = -(\frac{1-x^n}{1+x})$ for all $x \in \mathbb{R}\setminus\{-1\}$. So $p'(x) > 0$ for all $x > 1$ and $p'(x) < 0$ for all $x < 1$. Hence p is strictly increasing in $[1, \infty)$ and p is strictly decreasing in $(-\infty, 1]$. So $p(x) > p(1)$ for all $x > 1$ and also $p(x) > p(1)$ for all $x < 1$, *i.e.* $p(x) > p(1)$ for all $x(\neq 1) \in \mathbb{R}$. Since $p(1) = (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{4} - \frac{1}{5}) + \cdots + (\frac{1}{n-2} - \frac{1}{n-1}) + \frac{1}{n} > 0$,

we get $p(x) > 0$ for all $x \in \mathbb{R}$. Therefore the equation $p(x) = 0$ has no real root.

16. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that $f(0) = f(1) = 0$ and $f'(0) > 0$, $f'(1) > 0$. Show that there exist $c_1, c_2 \in (0, 1)$ with $c_1 \neq c_2$ such that $f'(c_1) = f'(c_2) = 0$.
Solution: Since $f'(0) > 0$, there exists $\delta_1 \in (0, \frac{1}{2})$ such that $f(x) > f(0) = 0$ for all $x \in (0, \delta_1)$. Also, since $f'(1) > 0$, there exists $\delta_2 \in (0, \frac{1}{2})$ such that $f(x) < f(1) = 0$ for all $x \in (1 - \delta_2, 1)$. By the intermediate value property of continuous functions, there exists $c \in (\frac{\delta_1}{2}, 1 - \frac{\delta_2}{2})$ such that $f(c) = 0$. Now, by Rolle's theorem, there exists $c_1 \in (0, c)$ and $c_2 \in (c, 1)$ such that $f'(c_1) = f'(c_2) = 0$.

Alternative solution: If possible, let $f'(x) \geq 0$ for all $x \in (0, 1)$. Then f is an increasing function on $[0, 1]$. So $0 = f(0) \leq f(x) \leq f(1) = 0$ for all $x \in [0, 1]$, *i.e.* $f(x) = 0$ for all $x \in [0, 1]$. This gives $f'(0) = 0$, which is a contradiction. Therefore there exists $c \in (0, 1)$ such that $f'(c) < 0$. Then by the intermediate value property of derivatives, there exist $c_1 \in (0, c)$ and $c_2 \in (c, 1)$ such that $f'(c_1) = f'(c_2) = 0$.

17. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f''(c)$ exists (in $\mathbb{R}$), where $c \in \mathbb{R}$. Show that $\lim\limits_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c)$.
Give an example of an $f : \mathbb{R} \to \mathbb{R}$ and a point $c \in \mathbb{R}$ for which $f''(c)$ does not exist (in $\mathbb{R}$) but the above limit exists (in $\mathbb{R}$).
Solution: Since $f''(c)$ exists (in $\mathbb{R}$), there exists $\delta > 0$ such that $f'(x)$ exists (in $\mathbb{R}$) for each $x \in (c - \delta, c + \delta)$. Hence by L'Hôpital's rule, $\lim\limits_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = \lim\limits_{h \to 0} \frac{f'(c+h) - f'(c-h)}{2h}$, provided the second limit exists (in $\mathbb{R}$).
Now $\lim\limits_{h \to 0} \frac{f'(c+h) - f'(c-h)}{2h} = \frac{1}{2}[\lim\limits_{h \to 0} \frac{f'(c+h) - f'(c)}{h} + \lim\limits_{h \to 0} \frac{f'(c-h) - f'(c)}{-h}] = \frac{1}{2}[f''(c) + f''(c)] = f''(c)$. Hence $\lim\limits_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c)$.
If $f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0, \end{cases}$ then $f : \mathbb{R} \to \mathbb{R}$ is not continuous at 0 and hence $f''(0)$ does not exist (in $\mathbb{R}$), but $\lim\limits_{h \to 0} \frac{f(0+h) - 2f(0) + f(0-h)}{h^2} = 0$, since $f(h) + f(-h) = 0$ for all $h (\neq 0) \in \mathbb{R}$.

18. Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$
Show that f is Riemann integrable on $[-1, 1]$ and that $\int\limits_{-1}^{1} f(x)\, dx = 0$.
If $F(x) = \int\limits_{-1}^{x} f(t)\, dt$ for all $x \in [-1, 1]$, then show that $F : [-1, 1] \to \mathbb{R}$ is differentiable, and in particular, $F'(0) = f(0)$, although f is not continuous at 0.
Solution: If $P = \{x_0, x_1, ..., x_n\}$ is any partition of $[-1, 1]$, then clearly $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\} = 0$ and $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\} \geq 0$ for $i = 1, 2, ..., n$ and so $L(f, P) = 0$ and $U(f, P) \geq 0$. Hence $\underline{\int\limits_{-1}^{1}} f(x)\, dx = 0$ and $\overline{\int\limits_{-1}^{1}} f(x)\, dx \geq 0$. Let $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{\varepsilon}{2}$. We choose u, v and s_k, t_k for $k = 2, 3, ..., n_0$ such that $\frac{1}{n_0+1} < u < s_{n_0} < \frac{1}{n_0} < t_{n_0} < \cdots < s_2 < \frac{1}{2} < t_2 < v < 1$ and also $1 - v < \frac{\varepsilon}{2n_0}$ and $t_k - s_k < \frac{\varepsilon}{2n_0}$ for $k = 2, 3, ..., n_0$. Then the partition $P_0 = \{-1, 0, u, s_{n_0}, t_{n_0}, ..., s_2, t_2, v, 1\}$ of $[-1, 1]$ is such that $U(f, P_0) < \varepsilon$. It follows that $0 \leq \overline{\int\limits_{-1}^{1}} f(x)\, dx \leq U(f, P_0) < \varepsilon$ and so $\overline{\int\limits_{-1}^{1}} f(x)\, dx = 0$. Thus $\underline{\int\limits_{-1}^{1}} f(x)\, dx = \overline{\int\limits_{-1}^{1}} f(x)\, dx = 0$. Therefore f is Riemann integrable on $[-1, 1]$ and $\int\limits_{-1}^{1} f(x)\, dx = 0$.
As above we can see that $F(x) = 0$ for all $x \in [-1, 1]$. Hence F is differentiable and

$F'(0) = 0 = f(0)$. However, f is not continuous at 0, because $\frac{1}{n} \to 0$ but $f(\frac{1}{n}) \to 1$ (since $f(\frac{1}{n}) = 1$ for all $n \in \mathbb{N}$).

(*Alternative method of showing* $F(x) = 0$ *for all* $x \in [-1,1]$: Since $f(t) \geq 0$ for all $t \in [-1,1]$, we have $0 \leq F(x) \leq F(x) + \int_x^1 f(t)\,dt = \int_{-1}^1 f(t)\,dt = 0$ for all $x \in [-1,1]$. Hence $F(x) = 0$ for all $x \in [-1,1]$.)

19. Let $f : [a,b] \to \mathbb{R}$ be continuous such that $f(x) \geq 0$ for all $x \in [a,b]$ and $\int_a^b f(x)\,dx = 0$. Show that $f(x) = 0$ for all $x \in [a,b]$.

(The above result need not be true if f is assumed to be only Riemann integrable on $[a,b]$.)
Solution: If possible, let $f(c) \neq 0$ for some $c \in (a,b)$, so that $f(c) > 0$. Since f is continuous at c, there exists $\delta > 0$ such that $|f(x) - f(c)| < \frac{1}{2}f(c)$ for all $x \in (c-\delta, c+\delta)$. (We may choose δ such that $(c-\delta, c+\delta) \subset [a,b]$.) This implies that $f(x) > \frac{1}{2}f(c)$ for all $x \in (c-\delta, c+\delta)$. So $\int_a^b f(x)\,dx = \int_a^{c-\delta/2} f(x)\,dx + \int_{c-\delta/2}^{c+\delta/2} f(x)\,dx + \int_{c+\delta/2}^b f(x)\,dx \geq \frac{1}{2}\delta f(c) > 0$, a contradiction. Hence $f(x) = 0$ for all $x \in (a,b)$. Almost similar arguments work if $c = a$ or $c = b$.

(Taking $f(0) = 1$ and $f(x) = 0$ for all $x \in (0,1]$, we find that $f : [0,1] \to \mathbb{R}$ is Riemann integrable on $[0,1]$ with $f(x) \geq 0$ for all $x \in [0,1]$ and $\int_0^1 f(x)\,dx = 0$ but $f(0) \neq 0$.)

20. If $f : [0,1] \to \mathbb{R}$ is continuous, then show that $\int_0^x (\int_0^u f(t)\,dt)\,du = \int_0^x (x-u)f(u)\,du$ for all $x \in [0,1]$.

Solution: Let $F(u) = \int_0^u f(t)\,dt$ for all $u \in [0,1]$. Then for all $x \in [0,1]$,
$\int_0^x (\int_0^u f(t)\,dt)\,du = \int_0^x F(u) \cdot 1\,du = F(u)u|_0^x - \int_0^x f(u)u\,du$ (integrating by parts and using the fact that $F'(u) = f(u)$ for all $u \in [0,1]$, since f is continuous on $[0,1]$) $= xF(x) - \int_0^x uf(u)\,du = x\int_0^x f(u)\,du - \int_0^x uf(u)\,du = \int_0^x (x-u)f(u)\,du$.

Alternative solution: Let $F(x) = \int_0^x (\int_0^u f(t)\,dt)\,du$ and $G(x) = \int_0^x (x-u)f(u)\,du = x\int_0^x f(u)\,du - \int_0^x uf(u)\,du$ for all $x \in [0,1]$. Since f is continuous on $[0,1]$, both $F : [0,1] \to \mathbb{R}$ and $G : [0,1] \to \mathbb{R}$ are differentiable and $F'(x) = \int_0^x f(t)\,dt$ and $G'(x) = xf(x) + \int_0^x f(u)\,du - xf(x) = \int_0^x f(u)\,du$ for all $x \in [0,1]$. Thus $(F-G)'(x) = F'(x) - G'(x) = 0$ for all $x \in [0,1]$ and hence $F - G$ is a constant function on $[0,1]$. Since $(F-G)(0) = F(0) - G(0) = 0 - 0 = 0$, we get $(F-G)(x) = 0$ for all $x \in [0,1] \Rightarrow F(x) = G(x)$ for all $x \in [0,1]$.

21. Examine whether the integral $\int_0^\infty \sin(x^2)\,dx$ is convergent.

Solution: Since the Riemann integral $\int_0^1 \sin(x^2)\,dx$ exists (in $\mathbb{R}$), $\int_0^\infty \sin(x^2)\,dx$ is convergent if $\int_1^\infty \sin(x^2)\,dx$ is convergent. Let $f(x) = \frac{1}{2x}$ and $g(x) = 2x\sin(x^2)$ for all $x \in [1,\infty)$. Then f is decreasing on $[1,\infty)$ and $\lim\limits_{x\to\infty} f(x) = 0$. Also $\left|\int_1^x g(t)\,dt\right| = |\cos 1 - \cos(x^2)| \leq 2$ for all

$x \in [1,\infty)$. Hence by Dirichlet's test, $\int\limits_1^\infty f(x)g(x)\,dx$, *i.e.* $\int\limits_1^\infty \sin(x^2)\,dx$ is convergent. Consequently $\int\limits_0^\infty \sin(x^2)\,dx$ is convergent.

22. Determine all real values of p for which the integral $\int\limits_0^\infty \frac{x^{p-1}}{1+x}\,dx$ is convergent.

Solution: The given integral is convergent iff both the integrals $\int\limits_0^1 \frac{x^{p-1}}{1+x}\,dx$ and $\int\limits_1^\infty \frac{x^{p-1}}{1+x}\,dx$ are convergent. If $p \geq 1$, then $\int\limits_0^1 \frac{x^{p-1}}{1+x}\,dx$ exists (in $\mathbb{R}$) as a Riemann integral. For $p < 1$, since $\lim\limits_{x\to 0+} \frac{x^{p-1}}{1+x} \cdot x^{1-p} = 1 \neq 0$, by the limit comparison test, $\int\limits_0^1 \frac{x^{p-1}}{1+x}\,dx$ converges iff $\int\limits_0^1 \frac{1}{x^{1-p}}\,dx$ converges. We know that $\int\limits_0^1 \frac{1}{x^{1-p}}\,dx$ converges iff $1-p<1$, *i.e.* iff $p>0$. Hence $\int\limits_0^1 \frac{x^{p-1}}{1+x}\,dx$ converges iff $p>0$. Again, since $\lim\limits_{x\to\infty} \frac{x^{p-1}}{1+x} \cdot x^{2-p} = \lim\limits_{x\to\infty} \frac{x}{1+x} = 1 \neq 0$, by the limit comparison test, $\int\limits_1^\infty \frac{x^{p-1}}{1+x}\,dx$ converges iff $\int\limits_1^\infty \frac{1}{x^{2-p}}\,dx$ converges. We know that $\int\limits_1^\infty \frac{1}{x^{2-p}}\,dx$ converges iff $2-p>1$, *i.e.* iff $p<1$. Hence $\int\limits_1^\infty \frac{x^{p-1}}{1+x}\,dx$ converges iff $p<1$. Therefore the given integral is convergent iff $0<p<1$.

23. Find the area of the region that is inside the cardioid $r = a(1+\cos\theta)$ and
(a) inside the circle $r = \frac{3}{2}a$,
(b) outside the circle $r = \frac{3}{2}a$.
Solution: At a point of intersection of the cardioid $r = a(1+\cos\theta)$ and the circle $r = \frac{3}{2}a$, we have $a(1+\cos\theta) = \frac{3}{2}a$. So $\theta = \frac{\pi}{3}$ corresponds to a point of intersection. Hence the area of the region that is inside the cardioid $r = a(1+\cos\theta)$ and inside the circle $r = \frac{3}{2}a$ is
$2\left[\frac{1}{2}\int\limits_0^{\pi/3} (\frac{3}{2}a)^2\,d\theta + \frac{1}{2}\int\limits_{\pi/3}^{\pi} a^2(1+\cos\theta)^2\,d\theta\right] = (\frac{7\pi}{4} - \frac{9\sqrt{3}}{8})a^2$. Also, the area of the region that is inside the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = \frac{3}{2}a$ is
$2\left[\frac{1}{2}\int\limits_0^{\pi/3} a^2(1+\cos\theta)^2\,d\theta - \frac{1}{2}\int\limits_0^{\pi/3} (\frac{3}{2}a)^2\,d\theta\right] = (\frac{9\sqrt{3}}{8} - \frac{\pi}{4})a^2$.

24. Find the length of the curve $y = \int\limits_0^x \sqrt{\cos 2t}\,dt$, $0 \leq x \leq \frac{\pi}{4}$.

Solution: Let $y = f(x) = \int\limits_0^x \sqrt{\cos 2t}\,dt$ for all $x \in [0,\frac{\pi}{4}]$. Then $f'(x) = \sqrt{\cos 2x}$ for all $x \in [0,\frac{\pi}{4}]$ (by the first fundamental theorem of calculus). Hence the length of the given curve is $\int\limits_0^{\frac{\pi}{4}} \sqrt{1+(f'(x))^2}\,dx = \int\limits_0^{\frac{\pi}{4}} \sqrt{1+\cos 2x}\,dx = \sqrt{2}\int\limits_0^{\frac{\pi}{4}} \cos x\,dx = 1$.