MA 101 (Mathematics I)

Tutorial Problem Set

- 1. Let (x_n) be a convergent sequence of positive real numbers such that $\lim_{n \to \infty} x_n < 1$. Show that $\lim_{n \to \infty} x_n^n = 0$.
- 2. Let (x_n) be a convergent sequence in R with limit l ∈ R and let α ∈ R.
 (a) If x_n > α for all n ∈ N, then show that l ≥ α.
 (b) If l > α, then show that there exists n₀ ∈ N such that x_n > α for all n ≥ n₀.
 (Note that l can be equal to α in (a).)
- 3. For α ∈ ℝ, examine whether lim_{n→∞} 1/n²([α] + [2α] + · · · + [nα]) exists (in ℝ). Also, find the value if it exists.
 (For each x ∈ ℝ, [x] denotes the greatest integer not exceeding x.)
- 4. Let $x_1 = 6$ and $x_{n+1} = 5 \frac{6}{x_n}$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find $\lim_{n \to \infty} x_n$ if (x_n) is convergent.
- 5. Let (x_n) be a sequence of nonzero real numbers. If (x_n) does not have any convergent subsequence, then show that $\lim_{n\to\infty} \frac{1}{x_n} = 0$.
- 6. Examine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ is convergent.
- 7. Let $x_n > 0$ for all $n \in \mathbb{N}$. Show that the series $\sum_{n=1}^{\infty} x_n$ converges iff the series $\sum_{n=1}^{\infty} \frac{x_n}{1+x_n}$ converges.
- 8. Find all $x \in \mathbb{R}$ for which the series $\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{2^n n^2}$ converges.
- 9. If $\alpha \neq 0 \in \mathbb{R}$, then show that the series $\sum_{n=1}^{\infty} (-1)^n \sin(\frac{\alpha}{n})$ is conditionally convergent.
- 10. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ [x] & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$ Determine all the points of \mathbb{R} where f is continuous.
- 11. Let $f:[0,1] \to \mathbb{R}$ be continuous such that f(0) = f(1). Show that (a) there exist $x_1, x_2 \in [0,1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{1}{2}$. (b) there exist $x_1, x_2 \in [0,1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{1}{3}$.

(In fact, if $n \in \mathbb{N}$, then there exist $x_1, x_2 \in [0, 1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{1}{n}$. However, it is not necessary that there exist $x_1, x_2 \in [0, 1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{2}{5}$.)

12. Let p be an odd degree polynomial with real coefficients in one real variable. If $g : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function, then show that there exists $x_0 \in \mathbb{R}$ such that $p(x_0) = g(x_0)$.

(In particular, this shows that

- (a) every odd degree polynomial with real coefficients in one real variable has at least one real zero.
- (b) the equation $x^9 4x^6 + x^5 + \frac{1}{1+x^2} = \sin 3x + 17$ has at least one real root.
- (c) the range of every odd degree polynomial with real coefficients in one real variable is \mathbb{R} .)

- 13. Does there exist a continuous function from (0, 1] onto \mathbb{R} ? Justify.
- 14. Let $f: \mathbb{R} \to \mathbb{R}$ be differentiable on $(-\delta, \delta)$ for some $\delta > 0$ and let f''(0) exist (in \mathbb{R}). If $f(\frac{1}{n}) = 0$ for all $n \in \mathbb{N}$, then find f'(0) and f''(0).
- 15. For $n \in \mathbb{N}$, show that the equation $1 x + \frac{x^2}{2} \frac{x^3}{3} + \dots + (-1)^n \frac{x^n}{n} = 0$ has exactly one real root if n is odd and has no real root if n is even.
- 16. Let $f: \mathbb{R} \to \mathbb{R}$ be differentiable such that f(0) = f(1) = 0 and f'(0) > 0, f'(1) > 0. Show that there exist $c_1, c_2 \in (0, 1)$ with $c_1 \neq c_2$ such that $f'(c_1) = f'(c_2) = 0$.
- 17. Let $f: \mathbb{R} \to \mathbb{R}$ be such that f''(c) exists (in \mathbb{R}), where $c \in \mathbb{R}$. Show that $\lim_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$ Give an example of an $f : \mathbb{R} \to \mathbb{R}$ and a point $c \in \mathbb{R}$ for which f''(c) does not exist (in \mathbb{R}) but the above limit exists (in \mathbb{R})
- 18. Let $f: [-1,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$

Show that f is Riemann integrable on [-1, 1] and that $\int_{-1}^{1} f(x) dx = 0$. If $F(x) = \int_{-1}^{x} f(t) dt$ for all $x \in [-1, 1]$, then show that $F : [-1, 1] \to \mathbb{R}$ is differentiable, and in particular, F'(0) = f(0), although f is not continuous at 0.

19. Let $f:[a,b] \to \mathbb{R}$ be continuous such that $f(x) \ge 0$ for all $x \in [a,b]$ and $\int_{a}^{b} f(x) dx = 0$. Show that f(x) = 0 for all $x \in [a, b]$.

(The above result need not be true if f is assumed to be only Riemann integrable on [a, b].)

- 20. If $f: [0,1] \to \mathbb{R}$ is continuous, then show that $\int_{0}^{x} (\int_{0}^{u} f(t) dt) du = \int_{0}^{x} (x-u)f(u) du$ for all $x \in [0,1]$.
- 21. Examine whether the integral $\int_{-\infty}^{\infty} \sin(x^2) dx$ is convergent.
- 22. Determine all real values of p for which the integral $\int_{0}^{\infty} \frac{x^{p-1}}{1+x} dx$ is convergent.
- 23. Find the area of the region that is inside the cardioid $r = a(1 + \cos \theta)$ and (a) inside the circle $r = \frac{3}{2}a$,
 - (b) outside the circle $r = \frac{3}{2}a$.
- 24. Find the length of the curve $y = \int_{0}^{x} \sqrt{\cos 2t} \, dt, \ 0 \le x \le \frac{\pi}{4}$.