

MA 101 (Mathematics I)

Practice Problem Set - 1

1. State TRUE or FALSE giving proper justification for each of the following statements.
 - (a) If both (x_n) and (y_n) are unbounded sequences in $\mathbb{R}$, then the sequence $(x_n y_n)$ cannot be convergent.
 - (b) If both (x_n) and (y_n) are increasing sequences in $\mathbb{R}$, then the sequence $(x_n y_n)$ must be increasing.
 - (c) If $(x_n), (y_n)$ are sequences in $\mathbb{R}$ such that (x_n) is convergent and (y_n) is not convergent, then the sequence $(x_n + y_n)$ cannot be convergent.
 - (d) A monotonic sequence (x_n) in $\mathbb{R}$ is convergent iff the sequence (x_n^2) is convergent.
 - (e) If (x_n) is an unbounded sequence of nonzero real numbers, then the sequence $(\frac{1}{x_n})$ must converge to 0.
 - (f) If $x_n = (1 - \frac{1}{n}) \sin \frac{n\pi}{2}$ for all $n \in \mathbb{N}$, then the sequence (x_n) is not convergent although it has a convergent subsequence.
 - (g) If both the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ of real numbers are convergent, then the series $\sum_{n=1}^{\infty} x_n y_n$ must be convergent.
 - (h) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $f(x) > 0$ for all $x \in \mathbb{Q}$, then it is necessary that $f(x) > 0$ for all $x \in \mathbb{R}$.
 - (i) There exists a continuous function from $(0, 1)$ onto $(0, \infty)$.
 - (j) There exists a continuous function from $[0, 1]$ onto $(0, 1)$.
 - (k) There exists a continuous function from $(0, 1)$ onto $[0, 1]$.
 - (l) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and bounded, then there must exist $c \in \mathbb{R}$ such that $f(c) = c$.
 - (m) If both $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ are continuous at 0, then the composite function $g \circ f : \mathbb{R} \rightarrow \mathbb{R}$ must be continuous at 0.
 - (n) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is not differentiable at $x_0 \in \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ is not differentiable at $f(x_0)$, then $g \circ f : \mathbb{R} \rightarrow \mathbb{R}$ cannot be differentiable at x_0 .
 - (o) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is such that $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{h}$ exists (in $\mathbb{R}$) for every $x \in \mathbb{R}$, then f must be differentiable on $\mathbb{R}$.
2. Using the definition of convergence of sequence, examine whether the following sequences are convergent.
 - (a) $(n + \frac{3}{2})$
 - (b) $((-1)^n \frac{3}{n+2})$
 - (c) $((-1)^n (1 - \frac{1}{n}))$
 - (d) $(\frac{3n^2 + \sin n - 4}{2n^2 + 3})$
 - (e) $(\frac{2\sqrt{n+3n}}{2n+3})$
3. Examine whether the sequences (x_n) defined as below are convergent. Also, find their limits if they are convergent.
 - (a) $x_n = (a^n + b^n + c^n)^{\frac{1}{n}}$ for all $n \in \mathbb{N}$, where a, b, c are distinct positive real numbers.
 - (b) $x_n = \frac{1 - n + (-1)^n}{2n+1}$ for all $n \in \mathbb{N}$.
 - (c) $x_n = \frac{n^k}{\alpha^n}$, where $|\alpha| > 1$ and $k > 0$.
 - (d) $x_n = \frac{p(n)}{2^n}$ for all $n \in \mathbb{N}$, where $p(x)$ is a polynomial in the real variable x of degree 5.
 - (e) $x_n = \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$ for all $n \in \mathbb{N}$.
 - (f) $x_n = \frac{1}{n} \sin^2 n$ for all $n \in \mathbb{N}$.
 - (g) $x_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}$ for all $n \in \mathbb{N}$.
 - (h) $x_n = \frac{n}{n^3+1} + \frac{2n}{n^3+2} + \cdots + \frac{n^2}{n^3+n}$ for all $n \in \mathbb{N}$.

- (i) $x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n+1}}$ for all $n \in \mathbb{N}$.
- (j) $x_n = \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{1+\sqrt{3}}} + \frac{1}{\sqrt{3+\sqrt{5}}} + \cdots + \frac{1}{\sqrt{2n-1+\sqrt{2n+1}}} \right)$ for all $n \in \mathbb{N}$.
- (k) $x_n = \left(\frac{\sin n + \cos n}{3} \right)^n$ for all $n \in \mathbb{N}$.
- (l) $x_n = \sqrt{4n^2 + n} - 2n$ for all $n \in \mathbb{N}$.
- (m) $x_n = \sqrt{n^2 + n} - \sqrt{n^2 + 1}$ for all $n \in \mathbb{N}$.
- (n) $x_1 = 1$ and $x_{n+1} = 1 + \sqrt{x_n}$ for all $n \in \mathbb{N}$.
- (o) $x_1 = 4$ and $x_{n+1} = 3 - \frac{2}{x_n}$ for all $n \in \mathbb{N}$.
- (p) $x_1 = 0$ and $x_{n+1} = \sqrt{6 + x_n}$ for all $n \in \mathbb{N}$.
- (q) $x_1 > 1$ and $x_{n+1} = \sqrt{x_n}$ for all $n \in \mathbb{N}$.
4. Let $(x_n), (y_n)$ be sequences in $\mathbb{R}$ such that $x_n \rightarrow x \in \mathbb{R}$ and $y_n \rightarrow y \in \mathbb{R}$. Show that $\lim_{n \rightarrow \infty} \max\{x_n, y_n\} = \max\{x, y\}$.
5. If a sequence (x_n) of positive real numbers converges to $\ell \in \mathbb{R}$, then show that $\lim_{n \rightarrow \infty} \sqrt{x_n} = \sqrt{\ell}$.
6. Let (x_n) be a convergent sequence in $\mathbb{R}$ with $\lim_{n \rightarrow \infty} x_n = \ell \neq 0$. Show that there exists $n_0 \in \mathbb{N}$ such that $x_n \neq 0$ for all $n \geq n_0$.
7. If $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$ for all $n \in \mathbb{N}$, then show that the sequence (x_n) is convergent.
8. Show that the sequences (x_n) in $\mathbb{R}$ defined as below are Cauchy (and hence convergent). Also, find their limits.
- (a) $x_1 = 1$ and $x_{n+1} = \frac{2+x_n}{1+x_n}$ for all $n \in \mathbb{N}$.
- (b) $x_1 > 0$ and $x_{n+1} = 2 + \frac{1}{x_n}$ for all $n \in \mathbb{N}$.
9. Examine whether the sequence (x_n) has a convergent subsequence, where for each $n \in \mathbb{N}$,
- (a) $x_n = (-1)^n n^2$ (b) $x_n = (-1)^n \frac{5n \sin^3 n}{3n-2}$.
10. If $a, b \in \mathbb{R}$, then show that the series $a + (a+b) + (a+2b) + \cdots$ is not convergent unless $a = b = 0$.
11. Examine whether the following series are convergent.
- (a) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$
- (b) $\sum_{n=1}^{\infty} \frac{(2n)!}{n^n}$
- (c) $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n}$
- (d) $\sum_{n=1}^{\infty} \sqrt{\frac{2n^2+3}{5n^3+1}}$
- (e) $\sum_{n=1}^{\infty} \frac{n^n}{2^{n^2}}$
- (f) $\sum_{n=1}^{\infty} ((n^3 + 1)^{\frac{1}{3}} - n)$
- (g) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n}$
- (h) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2}$
- (i) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n+1}}{n+1}$

12. Find all $x \in \mathbb{R}$ for which the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ is convergent.
13. Find all $x \in \mathbb{R}$ for which the series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{3^n \sqrt{2n+1}}$ is convergent.
14. Show that the series $\sum_{n=1}^{\infty} \frac{a^n}{a^n + n}$ is convergent if $0 < a < 1$ and is not convergent if $a > 1$.
15. If $0 < x_n < \frac{1}{2}$ for all $n \in \mathbb{N}$ and if the series $\sum_{n=1}^{\infty} x_n$ converges, then show that the series $\sum_{n=1}^{\infty} \frac{x_n}{1-x_n}$ converges.
16. Let $(x_n), (y_n)$ be sequences in $\mathbb{R}$ such that $|x_n| \leq |y_n|$ for all $n \in \mathbb{N}$. Find out (with justification) the true statement(s) from the following.
- If the series $\sum_{n=1}^{\infty} y_n$ converges, then the series $\sum_{n=1}^{\infty} x_n$ must converge.
 - If the series $\sum_{n=1}^{\infty} x_n$ converges, then the series $\sum_{n=1}^{\infty} y_n$ must converge.
 - If the series $\sum_{n=1}^{\infty} y_n$ converges absolutely, then the series $\sum_{n=1}^{\infty} x_n$ must converge absolutely.
 - If the series $\sum_{n=1}^{\infty} x_n$ converges absolutely, then the series $\sum_{n=1}^{\infty} y_n$ must converge absolutely.
17. If a series $\sum_{n=1}^{\infty} x_n$ is convergent but the series $\sum_{n=1}^{\infty} x_n^2$ is not convergent, then show that the series $\sum_{n=1}^{\infty} x_n$ is conditionally convergent.
18. Examine whether the following series are conditionally convergent.
- $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n^2 + 1} - n)$
 - $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 + (-1)^n}$
 - $\sum_{n=1}^{\infty} (-1)^n \frac{a^2 + n}{n^2}$, where $a \in \mathbb{R}$
19. Find all $x \in \mathbb{R}$ for which the series $\sum_{n=1}^{\infty} \frac{\log(n+1)}{\sqrt{n+1}} (x-5)^n$ is convergent.
20. Find all $x \in \mathbb{R}$ for which the series $\sum_{n=1}^{\infty} \frac{(-1)^n (x+3)^n}{n 5^n}$ is conditionally convergent.
21. Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be such that $|f(x)| \leq |g(x)|$ for all $x \in \mathbb{R}$. If g is continuous at 0 and $g(0) = 0$, then show that f is continuous at 0.
22. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = \begin{cases} \frac{1}{x} \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$
Examine whether f is continuous at 0.
23. Give an example (with justification) of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is discontinuous at every point of $\mathbb{R}$ but $|f| : \mathbb{R} \rightarrow \mathbb{R}$ is continuous.
24. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous such that $f(x) = x^2 + 5$ for all $x \in \mathbb{Q}$. Find $f(\sqrt{2})$.
25. Evaluate $\lim_{n \rightarrow \infty} \sin((2n\pi + \frac{1}{2n\pi}) \sin(2n\pi + \frac{1}{2n\pi}))$.

26. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous such that $f(0) > f(1) < f(2)$. Show that f is not one-one.
27. Let $f : [0, 1] \rightarrow [0, 1]$ be continuous. Show that there exists $c \in [0, 1]$ such that $f(c) + 2c^5 = 3c^7$.
28. Show that there exists $c \in \mathbb{R}$ such that $c^{179} + \frac{163}{1+c^2+\sin^2 c} = 119$.
29. Let $f, g : [-1, 1] \rightarrow \mathbb{R}$ be continuous such that $|f(x)| \leq 1$ for all $x \in [-1, 1]$ and $g(-1) = -1$, $g(1) = 1$. Show that there exists $c \in [-1, 1]$ such that $f(c) = g(c)$.
30. Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Show that
- if n is odd, then there exists unique $y \in \mathbb{R}$ such that $y^n = x$.
 - if n is even and $x > 0$, then there exists unique $y > 0$ such that $y^n = x$.
31. If $f : [0, 1] \rightarrow \mathbb{R}$ is continuous and $f(x) > 0$ for all $x \in [0, 1]$, then show that there exists $\alpha > 0$ such that $f(x) > \alpha$ for all $x \in [0, 1]$.
32. Give an example of each of the following.
- A function $f : [0, 1] \rightarrow \mathbb{R}$ which is not bounded.
 - A continuous and bounded function $f : \mathbb{R} \rightarrow \mathbb{R}$ which does not attain $\sup\{f(x) : x \in \mathbb{R}\}$ as well as $\inf\{f(x) : x \in \mathbb{R}\}$.
 - A continuous and bounded function $f : (0, 1) \rightarrow \mathbb{R}$ which attains both $\sup\{f(x) : x \in (0, 1)\}$ and $\inf\{f(x) : x \in (0, 1)\}$.
33. If $f(x) = x \sin x$ for all $x \in \mathbb{R}$, then show that $f : \mathbb{R} \rightarrow \mathbb{R}$ is neither bounded above nor bounded below.
34. Let p be an n th degree polynomial with real coefficients in one real variable such that $n(\neq 0)$ is even and $p(0) \cdot p^{(n)}(0) < 0$. Show that p has at least two real zeroes.
35. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous at 0 and let $g(x) = xf(x)$ for all $x \in \mathbb{R}$. Show that $g : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at 0.
36. Let $\alpha > 1$ and let $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfy $|f(x)| \leq |x|^\alpha$ for all $x \in \mathbb{R}$. Show that f is differentiable at 0.
37. Let $f(x) = x^2|x|$ for all $x \in \mathbb{R}$. Examine the existence of $f'(x)$, $f''(x)$ and $f'''(x)$, where $x \in \mathbb{R}$.
38. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2|\cos \frac{\pi}{x}| & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$
Examine whether f is differentiable (i) at 0 (ii) on $(0, 1)$.
39. Examine whether $f : \mathbb{R} \rightarrow \mathbb{R}$, defined as below, is differentiable at 0.
- $f(x) = \begin{cases} \frac{1}{2^{n+1}} & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$
 - $f(x) = \begin{cases} \frac{1}{4^n} & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$
40. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable at 0 and $f(0) = f'(0) = 0$. Show that $g : \mathbb{R} \rightarrow \mathbb{R}$, defined by $g(x) = \begin{cases} f(x) \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$ is differentiable at 0.

41. Let $f(x) = x^3 + x$ and $g(x) = x^3 - x$ for all $x \in \mathbb{R}$. If f^{-1} denotes the inverse function of f and if $(g \circ f^{-1})(x) = g(f^{-1}(x))$ for all $x \in \mathbb{R}$, then find $(g \circ f^{-1})'(2)$.
42. If $a, b, c \in \mathbb{R}$, then show that the equation $4ax^3 + 3bx^2 + 2cx = a + b + c$ has at least one root in $(0, 1)$.
43. If $a_0, a_1, \dots, a_n \in \mathbb{R}$ satisfy $\frac{a_0}{1 \cdot 2} + \frac{a_1}{2 \cdot 3} + \dots + \frac{a_n}{(n+1)(n+2)} = 0$, then show that the equation $a_0 + a_1x + \dots + a_nx^n = 0$ has at least one root in $[0, 1]$.
44. Show that the equation $|x^{10} - 60x^9 - 290| = e^x$ has at least one real root.
45. Find the number of (distinct) real roots of the following equations.
 (a) $x^2 = \cos x$
 (b) $e^{2x} + \cos x + x = 0$
46. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable such that $f(0) = 0$, $f'(0) > 0$ and $f''(x) > 0$ for all $x \in \mathbb{R}$. Show that the equation $f(x) = 0$ has no positive real root.
47. Show that between any two (distinct) real roots of the equation $e^x \sin x = 1$, there exists at least one real root of the equation $e^x \cos x + 1 = 0$.
48. Let $f(x) = 3x^5 - 2x^3 + 12x - 8$ for all $x \in \mathbb{R}$. Show that $f : \mathbb{R} \rightarrow \mathbb{R}$ is one-one and onto.
49. Show that
 (a) $\frac{x-1}{x} < \log x < x - 1$ for all $x(\neq 1) > 0$.
 (b) $1 + x < e^x < 1 + xe^x$ for all $x(\neq 0) \in \mathbb{R}$.
 (c) $2 \sin x + \tan x > 3x$ for all $x \in (0, \frac{\pi}{2})$.
 (d) $(1+x)^\alpha \geq 1 + \alpha x$ for all $x \geq -1$ and for all $\alpha > 1$.
50. Determine all the differentiable functions $f : [0, 1] \rightarrow \mathbb{R}$ satisfying the conditions
 (a) $f(0) = 0$, $f(1) = 1$ and $|f'(x)| \leq \frac{1}{2}$ for all $x \in [0, 1]$.
 (b) $f(0) = 0$, $f(1) = 1$ and $|f'(x)| \leq 1$ for all $x \in [0, 1]$.
51. Let $f : [0, 2] \rightarrow \mathbb{R}$ be differentiable and $f(0) = f(1) = 0$, $f(2) = 3$. Show that there exist $a, b, c \in (0, 2)$ such that $f'(a) = 0$, $f'(b) = 3$ and $f'(c) = 1$.
52. Evaluate the following limits.
 (a) $\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$
 (b) $\lim_{x \rightarrow 0} \frac{e^{-\frac{1}{x^2}}}{x}$
 (c) $\lim_{x \rightarrow \infty} x(\log(1 + \frac{x}{2}) - \log \frac{x}{2})$
 (d) $\lim_{x \rightarrow 0} \frac{(1+x)^{\frac{1}{x}} - e}{x}$
 (e) $\lim_{x \rightarrow \infty} \frac{2x + \sin 2x + 1}{(2x + \sin 2x)(\sin x + 3)^2}$
53. If $f : (0, \infty) \rightarrow (0, \infty)$ is differentiable at $a \in (0, \infty)$, then evaluate $\lim_{x \rightarrow a} \left(\frac{f(x)}{f(a)} \right)^{\frac{1}{\log x - \log a}}$.

54. Let $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x(\neq 0) \in \mathbb{R}, \\ 1 & \text{if } x = 0. \end{cases}$
Examine whether $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable.
55. Using Taylor's theorem, show that
 (a) $|\sqrt{1+x} - (1 + \frac{x}{2} - \frac{x^2}{8})| \leq \frac{1}{2}|x|^3$ for all $x \in (-\frac{1}{2}, \frac{1}{2})$.
 (b) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} > \cos x > 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$ for all $x \in (0, \pi)$.
 (c) $x - \frac{x^3}{3!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}$ for all $x \in (0, \pi)$.
56. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} n!x^n$.
57. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$.
58. Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function. If there is a partition P of $[a, b]$ such that $L(f, P) = U(f, P)$, then show that f is a constant function.
59. Evaluate the following limits.
 (a) $\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n \sqrt{n^2 - k^2}$
 (b) $\lim_{n \rightarrow \infty} \frac{1}{n} [(n+1)(n+2) \cdots (n+n)]^{\frac{1}{n}}$
 (c) $\lim_{x \rightarrow 0} \frac{x}{1-e^{x^2}} \int_0^x e^{t^2} dt$
 (d) $\lim_{n \rightarrow \infty} \left(\frac{1^8 + 3^8 + \cdots + (2n-1)^8}{n^9} \right)$
60. If $f : [-1, 1] \rightarrow \mathbb{R}$ is continuously differentiable, then evaluate $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f'(\frac{k}{3n})$.
61. Show that
 (a) $\frac{\pi^2}{9} \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{x}{\sin x} dx \leq \frac{2\pi^2}{9}$.
 (b) $\frac{\sqrt{3}}{8} \leq \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin x}{x} dx \leq \frac{\sqrt{2}}{6}$.
62. If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then show that there exists $c \in [a, b]$ such that $\int_a^b f(x) dx = (b-a)f(c)$.
(This result is called the mean value theorem of Riemann integrals.)
63. Let $f : [a, b] \rightarrow \mathbb{R}$ and $g : [a, b] \rightarrow \mathbb{R}$ be continuous and let $g(x) \geq 0$ for all $x \in [a, b]$. Show that there exists $c \in [a, b]$ such that $\int_a^b f(x)g(x) dx = f(c) \int_a^b g(x) dx$.
(This result is called the generalized mean value theorem of Riemann integrals.)
64. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and let $g(x) = \int_0^x (x-t)f(t) dt$ for all $x \in \mathbb{R}$. Show that $g''(x) = f(x)$ for all $x \in \mathbb{R}$.

65. Let $f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } 1 < x \leq 2, \end{cases}$ and let $F(x) = \int_0^x f(t) dt$ for all $x \in [0, 2]$. Is $F : [0, 2] \rightarrow \mathbb{R}$ differentiable? Justify.
66. If $f : [0, 1] \rightarrow [0, 1]$ is continuous, then show that the equation $2x - \int_0^x f(t) dt = 1$ has exactly one root in $[0, 1]$.
67. Examine whether the following improper integrals are convergent.
- $\int_0^{\infty} e^{-t^2} dt$
 - $\int_{-\infty}^{\infty} te^{-t^2} dt$
 - $\int_0^1 \frac{dt}{\sqrt{t-t^2}}$
68. Determine all real values of p for which the integral $\int_1^{\infty} t^p e^{-t} dt$ converges.
69. Find the area of the region enclosed by the curve $y = \sqrt{|x+1|}$ and the line $5y = x + 7$.
70. The region bounded by the parabola $y = x^2 + 1$ and the line $y = x + 3$ is revolved about the x -axis to generate a solid. Find the volume of the solid.
71. The region bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ (where $a > 0$) is revolved about the x -axis to generate a solid. Find the volume of the solid.
72. Find the area of the region that is inside the circle $r = 2\cos\theta$ and outside the cardioid $r = 2(1 - \cos\theta)$.
73. Find the area of the region which is inside both the cardioids $r = a(1 + \cos\theta)$ and $r = a(1 - \cos\theta)$, where $a > 0$.
74. Consider the funnel formed by revolving the curve $y = \frac{1}{x}$ about the x -axis, between $x = 1$ and $x = a$, where $a > 1$. If V_a and S_a denote respectively the volume and the surface area of the funnel, then show that $\lim_{a \rightarrow \infty} V_a = \pi$ and $\lim_{a \rightarrow \infty} S_a = \infty$.