Indian Institute of Technology Guwahati

End-semester Examination

MA 101 (Mathematics I)

Maximum Marks : 50

Date : November 26, 2017 **Time :** 2 pm – 5 pm

No mark will be given for writing only TRUE or FALSE (without justification) in Questions 1 and 4.

For Questions 1 and 3, $M_{n,n}(\mathbb{R})$ denotes the vector space of all $n \times n$ matrices with entries from $\mathbb{R}$.

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1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) The set $V = \{B \in M_{4,4}(\mathbb{R}) : \det(B) < 100\}$ is a subspace of $M_{4,4}(\mathbb{R})$. [**1**]
   (b) If U is the subset of $M_{5,5}(\mathbb{R})$ consisting of all matrices of rank at most 3, then U is a subspace of $M_{5,5}(\mathbb{R})$. [**2**]

2. Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of all polynomials of degree at most 2 with coefficients in $\mathbb{R}$. Consider the map $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$, where

$$T(1 + x^2) = 1 - x^2,$$
$$T(-1 - 2x) = -1 + 2x,$$
$$T(x - 3x^2) = x + cx^2,$$
$$T(1 - 5x + 6x^2) = 1 + 3x - 6x^2.$$

   If T is a linear transformation, determine all possible values of $c \in \mathbb{R}$. [**2**]

3. Let W denote the vector subspace of $M_{3,3}(\mathbb{R})$ consisting of all skew-symmetric matrices.
   (a) Write down a basis for W (no justification is required). [**2**]
   (b) Consider the linear transformation $S : W \to \mathbb{R}^3$ given by

$$S(A) = A \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \qquad \forall\ A \in W.$$

   Determine a basis for range(S) (with proper justification). [**2**]
   (c) For S as above, determine the dimension of $\ker(S) = \{A \in W : S(A) = 0\}$ with proper justification. [**1**]

(*Continued on the next page*)

4. State TRUE or FALSE giving proper justification for each of the following statements. [**2 × 5**]

(a) If $f : \mathbb{R} \to \mathbb{R}$ is such that both $\lim_{x \to 1+} \frac{f(x) - f(1)}{x - 1}$ and $\lim_{x \to 1-} \frac{f(x) - f(1)}{x - 1}$ exist (in $\mathbb{R}$), then f must be continuous at 1.

(b) If (x_n) is a sequence in $\mathbb{R}$ such that for each $m \in \mathbb{N}$ with $m > 1$, the subsequence (x_{mn}) of (x_n) is convergent, then (x_n) must be convergent.

(c) There exists a power series $\sum_{n=0}^{\infty} a_n (x - 3)^n$ which is conditionally convergent for $x = -5$ and divergent for $x = 8$.

(d) There exists a continuous function $f : [1, 2] \to \mathbb{R}$ which is differentiable on $(1, 2)$ but not differentiable at 1 and 2.

(e) If $f : [1, 2] \to \mathbb{R}$ is Riemann integrable on $[1, 2]$ and if the function $F : [1, 2] \to \mathbb{R}$, defined by $F(x) = \int_1^x f(t)\, dt$ for all $x \in [1, 2]$, is differentiable on $[1, 2]$, then it is necessary that $F'(x) = f(x)$ for all $x \in [1, 2]$.

5. Let (x_n) be a sequence in $\mathbb{R}$ such that $\lim_{n \to \infty} \left| x_n + 3 \left(\frac{n}{n+1} \right)^n \right|^{\frac{1}{n}} = \frac{2}{3}$. Determine $\lim_{n \to \infty} x_n$. [**2**]

6. Determine all $p \in \mathbb{R}$ for which the series $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n^p}$ is convergent. [**3**]

7. If $f : [-1, 1] \to \mathbb{R}$ is a continuous function, then show that there exists $c \in [-1, 1]$ such that $|f(c)| = \frac{1}{4} \Big(|f(-1)| + 2|f(0)| + |f(1)| \Big)$. [**4**]

8. Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$ and $f(1) = 1$. Show that there exist $a, b \in (0, 1)$ with $a \neq b$ such that $\frac{1}{f'(a)} + \frac{1}{f'(b)} = 2$. [**4**]

9. Show that the Taylor series of $\log(1 + x)$ about $x = 0$ converges to $\log(1 + x)$ for each $x \in (-\frac{1}{2}, 1)$. [**4**]

10. Evaluate: $\lim_{x \to \infty} \left[(x + 1)^{\frac{x+2}{x+1}} - x^{\frac{x+1}{x}} \right]$ [**4**]

11. Examine whether the improper integral $\int_1^{\infty} \frac{\sqrt{x + 3}}{(x + 2)\sqrt{x^2 - 1}}\, dx$ is convergent. [**4**]

12. Find the area of the region that is inside the circle $r = 3 \sin \theta$ and outside the cardioid $r = 3(1 + \cos \theta)$. [**3**]

13. The region enclosed by the triangle with vertices $(1, 1)$, $(2, 3)$, and $(3, 2)$ in the xy-plane is revolved about the x-axis to generate a solid. Find the volume of the solid. [**2**]

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