MA 101 (Mathematics I)

Continuity : Summary of Lectures

Definition: Let $D(\neq \emptyset) \subseteq \mathbb{R}$ and let $f : D \to \mathbb{R}$.
We say that f is continuous at $x_0 \in D$ if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in D$ satisfying $|x - x_0| < \delta$.

We say that $f : D \to \mathbb{R}$ is continuous if f is continuous at each $x_0 \in D$.

Definition: Let $D \subseteq \mathbb{R}$ and let $x_0 \in \mathbb{R}$ such that for some $h > 0$, $(x_0 - h, x_0 + h) \setminus \{x_0\} \subseteq D$.
If $f : D \to \mathbb{R}$, then $\ell \in \mathbb{R}$ is said to be the limit of f at x_0 if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ for all $x \in D$ satisfying $0 < |x - x_0| < \delta$.

We write: $\lim\limits_{x \to x_0} f(x) = \ell$.

Similarly we define: $\lim\limits_{x \to x_0+} f(x) = \ell$ and $\lim\limits_{x \to x_0-} f(x) = \ell$,
and also $\lim\limits_{x \to \infty} f(x) = \ell$, $\lim\limits_{x \to x_0} f(x) = -\infty$, etc.

Result: Let $D \subseteq \mathbb{R}$ and let $x_0 \in D$ such that for some $h > 0$, $(x_0 - h, x_0 + h) \subseteq D$. Then $f : D \to \mathbb{R}$ is continuous iff $\lim\limits_{x \to x_0} f(x) = f(x_0)$.
Similarly the other two cases.

Sequential criterion of continuity: $f : D \to \mathbb{R}$ is continuous at $x_0 \in D$ iff for every sequence (x_n) in D such that $x_n \to x_0$, we have $f(x_n) \to f(x_0)$.

Similar criterion for limit.

Example: $\lim\limits_{n \to \infty} \frac{\sin(\sqrt{n+1} - \sqrt{n})}{\sqrt{n+1} - \sqrt{n}} = 1$

Examples:

1. $f(x) = \begin{cases} 3x + 2 & \text{if } x < 1, \\ 4x^2 & \text{if } x \geq 1. \end{cases}$
2. $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$
3. $f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$
4. $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
5. $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ -x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Result: Let $f, g : D \to \mathbb{R}$ be continuous at $x_0 \in D$. Then

(a) $f + g$, fg and $|f|$ are continuous at x_0,
(b) f/g is continuous at x_0 if $g(x) \neq 0$ for all $x \in D$.

Result: Composition of two continuous functions is continuous.

Further examples of continuous functions:

Polynomial function, Rational function, sine function, cosine function, exponential function, etc.

Result: If $f : D \to \mathbb{R}$ is continuous at x_0 and $f(x_0) \neq 0$, then there exists $\delta > 0$ such that $f(x) \neq 0$ for all $x \in D$ satisfying $|x - x_0| < \delta$.

Result: If $f : [a, b] \to \mathbb{R}$ is continuous and if $f(a) \cdot f(b) < 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$.

Intermediate value theorem: Let I be an interval of $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be continuous. If $a, b \in I$ with $a < b$ and if $f(a) < k < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = k$.

Examples:

(a) The equation $x^2 = x \sin x + \cos x$ has at least two real roots.
(b) If $f : [0, 1] \to [0, 1]$ is continuous, then there exists $c \in [0, 1]$ such that $f(c) = c$.
(c) Let $f : [0, 2] \to \mathbb{R}$ be continuous such that $f(0) = f(2)$. Then there exist $x_1, x_2 \in [0, 2]$ such that $x_1 - x_2 = 1$ and $f(x_1) = f(x_2)$.

Result: If $f : [a, b] \to \mathbb{R}$ is continuous, then $f : [a, b] \to \mathbb{R}$ is bounded.

Example: There does not exist any continuous function from $[0, 1]$ onto $(0, \infty)$.

Result: If $f : [a, b] \to \mathbb{R}$ is continuous, then there exist $x_0, y_0 \in [a, b]$ such that $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in [a, b]$.