Calculation of Propagated Errors in Airborne LiDAR Data

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Abstract

A process of determining the planimetric (or horizontal) and altimetric (altimetric) errors of airborne LiDAR data is demonstrated using law of error propagation for LiDAR observation equation. Standard values of the precisions are referred from relevant literature and specifications of the sensors. A set of complete and simplified expressions of propagated errors are also presented.

Introduction: This KanGAL report is prepared by referring Hoften et al. (2000), Glennie (2007), May (2008), and Schär (2010). Standard law of propagation of random errors is referred from Mekhail (1976).

The LiDAR observation equation is written as:

$$\mathbf{X}_p^m = \mathbf{X}_0^m + \mathbf{R}_b^m \left(\mathbf{l}^b + \mathbf{R}_s^b \, \mathbf{X}^s \right) \qquad \ldots (1)$$

For the purpose of analysis of random errors by statistical methods, epoch or duration of time is not participating as variable. Therefore, all variables are considered without time and denoted without subscript *t*, which indicates an instant of a time or epoch. Expanding the above equation with matrices and vectors gives:

$$\begin{bmatrix} X_p \\ Y_p \\ Z_p \end{bmatrix} = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} + \mathbf{R}_{(inc,\, ref)} \, \mathbf{R}_{(\omega,\, \varphi,\, \kappa)} \left[\begin{bmatrix} l_x \\ l_y \\ l_z \end{bmatrix} + \mathbf{R}_{(\alpha_0,\, \beta_0,\, \gamma_0)} \begin{bmatrix} 0 \\ -r \sin\eta \\ r\cos\eta \end{bmatrix} \right] \qquad \ldots (2)$$

Matrix $\mathbf{R}_{(\alpha_0,\beta_0,\gamma_0)}$ rotates the scanner coordinate system to body coordinate system with positive values of bore-sight angles (May, 2008). However, the positive signs of roll (ω), pitch (φ) and yaw (κ) angles rotates the mapping coordinate system to body coordinate system (May, 2008). Therefore, matrix $\mathbf{R}_{(\omega,\varphi,\kappa)}$ is constructed by roll, pitch and yaw angles with their negative values. Matrices are written with their full expressions as:

$$\mathbf{R}_{(inc,ref)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{aligned}\mathbf{R}_{(\omega,\varphi,\kappa)} &= \mathbf{R}_z(-\kappa)\,\mathbf{R}_y(-\varphi)\,\mathbf{R}_x(-\omega) \\ &= \begin{bmatrix} \cos\kappa & -\sin\kappa & 0 \\ \sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{bmatrix} \\ &= \begin{bmatrix} \cos\kappa\cos\varphi & -\sin\kappa\cos\omega+\cos\kappa\sin\varphi\sin\omega & \sin\kappa\sin\omega+\cos\kappa\sin\varphi\cos\omega \\ \sin\kappa\cos\varphi & \cos\kappa\cos\omega+\sin\kappa\sin\varphi\sin\omega & -\cos\kappa\sin\omega+\sin\kappa\sin\varphi\cos\omega \\ -\sin\varphi & \cos\varphi\sin\omega & \cos\varphi\cos\omega \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{R}_{(\alpha_0,\beta_0,\gamma_0)} &= \mathbf{R}_z(\gamma_0)\,\mathbf{R}_y(\beta_0)\,\mathbf{R}_x(\alpha_0) \\ &= \begin{bmatrix} \cos\gamma_0\cos\beta_0 & \sin\gamma_0\cos\alpha_0+\cos\gamma_0\sin\beta_0\sin\alpha_0 & \sin\gamma_0\sin\alpha_0-\cos\gamma_0\sin\beta_0\cos\alpha_0 \\ -\sin\gamma_0\cos\beta_0 & \cos\gamma_0\cos\alpha_0-\sin\gamma_0\sin\beta_0\sin\alpha_0 & \cos\gamma_0\sin\alpha_0+\sin\gamma_0\sin\beta_0\cos\alpha_0 \\ \sin\beta_0 & -\cos\beta_0\sin\alpha_0 & \cos\beta_0\cos\alpha_0 \end{bmatrix}\end{aligned}$$

Random errors propagated in the 3D coordinates of a LiDAR point can be calculated by standard law of propagation of error. 3D vector coordinates shown by equation (2) are functions of 14 variables: navigation sensor coordinates in mapping frame (X,Y,Z), attitude angles (ω,φ,κ), lever-arm offset components (l_x,l_y,l_z), bore-sight angles ($\alpha_0,\beta_0,\gamma_0$), scan angle ($\eta$) and range ($r$). Standard deviation and variance are denoted by σ and σ^2 with the subscript indicating the variable. Standard law of propagation of error (Mekhail, 1976) with all parameters considered independent gives the propagated error in the 3D vector coordinates of a LiDAR point as:

$$\mathbf{C} = \mathbf{A}\,\Sigma\,\mathbf{A}^T \qquad \ldots (3)$$

In equation (3), Matrix Σ is the covariance-variance matrix, which is a square and diagonal matrix having variances of parameter as non-zero diagonal elements. Matrix $\mathbf{A}$ is jacobian of $\mathbf{X}_p^m$ with respect to the 14 parameters. The diagonal elements of the matrix $\mathbf{C}$ represent the propagated variances ($\sigma_{X_p}^2, \sigma_{Y_p}^2, \sigma_{Z_p}^2$). Indicating the column vector of 14 parameters by matrix $\mathbf{p}$ and individual elements by $p_1, p_2, \cdots p_{14}$ gives:

$$\begin{aligned}\mathbf{p} &= [X_0\ Y_0\ Z_0\ \omega\ \varphi\ \kappa\ l_x\ l_y\ l_z\ \alpha_0\ \beta_0\ \gamma_0\ r\ \phi]^T \\ &= [p_1\ p_2\ p_3\ p_4\ p_5\ p_6\ p_7\ p_8\ p_9\ p_{10}\ p_{11}\ p_{12}\ p_{13}\ p_{14}]^T \\ &= [p_j]^T;\ j = 1,2,...,14\end{aligned} \quad \ldots (4)$$

Writing the parameter p_j and corresponding precision by σ_j (where $j = 1,2,...,14$), the the diagonal elements of the matrix $\mathbf{C}$ (propagated variances) can be written as:

$$\begin{aligned}C_{1,1} &= \sigma_{X_p}^2 = \sum_{j=1}^{14} \left(A_{1,j}\right)^2 \left(\sigma_j^2\right) \\ C_{2,1} &= \sigma_{Y_p}^2 = \sum_{j=1}^{14} \left(A_{2,j}\right)^2 \left(\sigma_j^2\right) \\ C_{3,3} &= \sigma_{Z_p}^2 = \sum_{j=1}^{14} \left(A_{3,j}\right)^2 \left(\sigma_j^2\right)\end{aligned} \quad \ldots (5)$$

Matrix $\mathbf{A}$ contains 42 elements or partial derivatives of 3D vector coordinates with respect to variables in 3 rows and 14 columns. Elements of matrix $\mathbf{A}$ (i.e. $A_{i,j}$) can be calculated by evaluating the partial derivatives. In order to express matrix $\mathbf{A}$, elements of this matrix are grouped in five categories, namely, elements for 3D navigation sensor coordinates ($\mathbf{A}_{NSC}$), elements for attitude angles ($\mathbf{A}_{AA}$), elements for bore-sight angles ($\mathbf{A}_{BSA}$), elements for lever arm offsets ($\mathbf{A}_{LAO}$), and elements for scanner angle and range observations ($\mathbf{A}_{SAR}$). Matrix $\mathbf{A}$ is expressed in five partitions as:

$$\underset{(3\times14)}{\mathbf{A}} = \left[\underset{(3\times3)}{\mathbf{A}_{NSC}} \ \vdots\ \underset{(3\times3)}{\mathbf{A}_{AA}} \ \vdots\ \underset{(3\times3)}{\mathbf{A}_{LAO}} \ \vdots\ \underset{(3\times3)}{\mathbf{A}_{BSA}} \ \vdots\ \underset{(3\times2)}{\mathbf{A}_{SAR}} \right]$$

All variables are not interblended in equation (2). Therefore, variables, which are independent from the remaining variables, can be treated as constant in matrix calculations. As a result, additive variables can be ignored and multiplicative variables remain constant in matrix multiplication and thus not affecting the partial derivative calculations.

Elements of $\mathbf{A}_{NSC}$ matrix:

$$\underset{(3\times3)}{\mathbf{A}_{NSC}} = \begin{bmatrix} \frac{\partial X_p}{\partial X_0} & \frac{\partial X_p}{\partial Y_0} & \frac{\partial X_p}{\partial Z_0} \\ \frac{\partial Y_p}{\partial Y_0} & \frac{\partial Y_p}{\partial Y_0} & \frac{\partial Y_p}{\partial Y_0} \\ \frac{\partial Z_p}{\partial X_0} & \frac{\partial Z_p}{\partial Y_0} & \frac{\partial Z_p}{\partial Z_0} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Elements of $\mathbf{A}_{AA}$ matrix:

$$\underset{(3\times3)}{\mathbf{A}_{AA}} = \begin{bmatrix} \frac{\partial X_p}{\partial \omega} & \frac{\partial X_p}{\partial \varphi} & \frac{\partial X_p}{\partial \kappa} \\ \frac{\partial Y_p}{\partial \omega} & \frac{\partial Y_p}{\partial \varphi} & \frac{\partial Y_p}{\partial \kappa} \\ \frac{\partial Z_p}{\partial \omega} & \frac{\partial Z_p}{\partial \varphi} & \frac{\partial Z_p}{\partial \kappa} \end{bmatrix}$$

$$\begin{aligned} \frac{\partial X_p}{\partial \omega} = & -(\cos\omega\cos\kappa + \sin\varphi\sin\omega\sin\kappa)(l_z + r\cos\alpha_0\cos\beta_0\cos\eta + r\cos\beta_0\sin\alpha_0\sin\eta) \\ & -(\cos\kappa\sin\omega - \cos\omega\sin\varphi\sin\kappa)(l_y + r\cos\eta(\cos\gamma_0\sin\alpha_0 + \cos\alpha_0\sin\beta_0\sin\gamma_0) \\ & \qquad - r\sin\eta(\cos\alpha_0\cos\gamma_0 - \sin\alpha_0\sin\beta_0\sin\gamma_0)) \end{aligned}$$

$$\begin{aligned} \frac{\partial X_p}{\partial \varphi} = & \cos\varphi\cos\omega\sin\kappa(l_z + r\cos\alpha_0\cos\beta_0\cos\eta + r\cos\beta_0\sin\alpha_0\sin\eta) \\ & -\sin\varphi\sin\kappa(l_x + r\cos\eta(\sin\alpha_0\sin\gamma_0 - \cos\alpha_0\cos\gamma_0\sin\beta_0) \\ & \qquad - r\sin\eta(\cos\alpha_0\sin\gamma_0 + \cos\gamma_0\sin\alpha_0\sin\beta_0)) \\ & +\cos\varphi\sin\omega\sin\kappa(l_y + r\cos\eta(\cos\gamma_0\sin\alpha_0 + \cos\alpha_0\sin\beta_0\sin\gamma_0) \\ & \qquad - r\sin\eta(\cos\alpha_0\cos\gamma_0 - \sin\alpha_0\sin\beta_0\sin\gamma_0)) \end{aligned}$$

$$
\begin{aligned}
\frac{\partial X_p}{\partial \kappa} = & (\sin\omega\sin\kappa+\cos\omega\cos\kappa\sin\varphi)(l_z + r\cos\alpha_0\cos\beta_0\cos\eta + r\cos\beta_0\sin\alpha_0\sin\eta) \\
& -(\cos\omega\sin\kappa-\cos\kappa\sin\varphi\sin\omega)(l_y + r\cos\eta(\cos\gamma_0\sin\alpha_0+\cos\alpha_0\sin\beta_0\sin\gamma_0) \\
& \qquad -r\sin\eta(\cos\alpha_0\cos\gamma_0-\sin\alpha_0\sin\beta_0\sin\gamma_0)) \\
& +\cos\varphi\cos\kappa(l_x + r\cos\eta(\sin\alpha_0\sin\gamma_0-\cos\alpha_0\cos\gamma_0\sin\beta_0) \\
& \qquad -r\sin\eta(\cos\alpha_0\sin\gamma_0+\cos\gamma_0\sin\alpha_0\sin\beta_0))
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial Y_p}{\partial \omega} = & ((\cos\omega\sin\kappa-\cos\kappa\sin\varphi\sin\omega)(l_z + r\cos\alpha_0\cos\beta_0\cos\eta + r\cos\beta_0\sin\alpha_0\sin\eta) \\
& +(\sin\omega\sin\kappa+\cos\omega\cos\kappa\sin\varphi)(l_y + r\cos\eta(\cos\gamma_0\sin\alpha_0+\cos\alpha_0\sin\beta_0\sin\gamma_0) \\
& \qquad -r\sin\eta(\cos\alpha_0\cos\gamma_0-\sin\alpha_0\sin\beta_0\sin\gamma_0))
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial Y_p}{\partial \varphi} = & \cos\varphi\cos\omega\cos\kappa(l_z + r\cos\alpha_0\cos\beta_0\cos\eta + r\cos\beta_0\sin\alpha_0\sin\eta) \\
& -\cos\kappa\sin\varphi(l_x + r\cos\eta(\sin\alpha_0\sin\gamma_0-\cos\alpha_0\cos\gamma_0\sin\beta_0) \\
& \qquad -r\sin\eta(\cos\alpha_0\sin\gamma_0+\cos\gamma_0\sin\alpha_0\sin\beta_0)) \\
& +\cos\varphi\cos\kappa\sin\omega(l_y + r\cos\eta(\cos\gamma_0\sin\alpha_0+\cos\alpha_0\sin\beta_0\sin\gamma_0) \\
& \qquad -r\sin\eta(\cos\alpha_0\cos\gamma_0-\sin\alpha_0\sin\beta_0\sin\gamma_0))
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial Y_p}{\partial \kappa} = & (\cos\kappa\sin\omega-\cos\omega\sin\varphi\sin\kappa)(l_z + r\cos\alpha_0\cos\beta_0\cos\eta \\
& \qquad + r\cos\beta_0\sin\alpha_0\sin\eta) \\
& -(\cos\omega\cos\kappa+\sin\varphi\sin\omega\sin\kappa)(l_y + r\cos\eta(\cos\gamma_0\sin\alpha_0+\cos\alpha_0\sin\beta_0\sin\gamma_0) \\
& \qquad -r\sin\eta(\cos\alpha_0\cos\gamma_0-\sin\alpha_0\sin\beta_0\sin\gamma_0)) \\
& -\cos\varphi\sin\kappa(l_x + r\cos\eta(\sin\alpha_0\sin\gamma_0-\cos\alpha_0\cos\gamma_0\sin\beta_0) \\
& \qquad -r\sin\eta(\cos\alpha_0\sin\gamma_0+\cos\gamma_0\sin\alpha_0\sin\beta_0))
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial Z_p}{\partial \omega} = & \cos\varphi\sin\omega(l_z + r\cos\alpha_0\cos\beta_0\cos\eta + r\cos\beta_0\sin\alpha_0\sin\eta) \\
& -\cos\varphi\cos\omega(l_y + r\cos\eta(\cos\gamma_0\sin\alpha_0+\cos\alpha_0\sin\beta_0\sin\gamma_0) \\
& \qquad -r\sin\eta(\cos\alpha_0\cos\gamma_0-\sin\alpha_0\sin\beta_0\sin\gamma_0))
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial Z_p}{\partial \varphi} = & +\cos\varphi(l_x + r\cos\eta(\sin\alpha_0\sin\gamma_0-\cos\alpha_0\cos\gamma_0\sin\beta_0) \\
& \qquad -r\sin\eta(\cos\alpha_0\sin\gamma_0+\cos\gamma_0\sin\alpha_0\sin\beta_0)) \\
& +\cos\omega\sin\varphi(l_z + r\cos\alpha_0\cos\beta_0\cos\eta + r\cos\beta_0\sin\alpha_0\sin\eta) \\
& +\sin\varphi\sin\omega(l_y + r\cos\eta(\cos\gamma_0\sin\alpha_0+\cos\alpha_0\sin\beta_0\sin\gamma_0) \\
& \qquad -r\sin\eta(\cos\alpha_0\cos\gamma_0-\sin\alpha_0\sin\beta_0\sin\gamma_0))
\end{aligned}
$$

$$\frac{\partial Z_p}{\partial \kappa} = 0$$

Elements of $\mathbf{A}_{LAO}$ matrix:

$$\underset{(3\times3)}{\mathbf{A}_{LAO}} = \begin{bmatrix} \frac{\partial X_p}{\partial l_x} & \frac{\partial X_p}{\partial l_y} & \frac{\partial X_p}{\partial l_z} \\ \frac{\partial Y_p}{\partial l_x} & \frac{\partial Y_p}{\partial l_y} & \frac{\partial Y_p}{\partial l_z} \\ \frac{\partial Z_p}{\partial l_x} & \frac{\partial Z_p}{\partial l_y} & \frac{\partial Z_p}{\partial l_z} \end{bmatrix}$$

$$\frac{\partial X_p}{\partial l_x} = \cos\varphi \sin\kappa$$

$$\frac{\partial X_p}{\partial l_y} = \cos\omega \cos\kappa + \sin\varphi \sin\omega \sin\kappa$$

$$\frac{\partial X_p}{\partial l_z} = \cos\omega \sin\varphi \sin\kappa - \cos\kappa \sin\omega$$

$$\frac{\partial Y_p}{\partial l_x} = \cos\varphi \cos\kappa$$

$$\frac{\partial Y_p}{\partial l_y} = \cos\kappa \sin\varphi \sin\omega - \cos\omega \sin\kappa$$

$$\frac{\partial Y_p}{\partial l_z} = \sin\omega \sin\kappa + \cos\omega \cos\kappa \sin\varphi$$

$$\frac{\partial Z_p}{\partial l_x} = \sin\varphi$$

$$\frac{\partial Z_p}{\partial l_y} = -\cos\varphi \sin\omega$$

$$\frac{\partial Z_p}{\partial l_z} = -\cos\varphi \cos\omega$$

Elements of $\mathbf{A}_{BSA}$ matrix:

$$\underset{(3\times 3)}{\mathbf{A}_{BSA}} = \begin{bmatrix} \dfrac{\partial X_p}{\partial \alpha_0} & \dfrac{\partial X_p}{\partial \beta_0} & \dfrac{\partial X_p}{\partial \gamma_0} \\ \dfrac{\partial Y_p}{\partial \alpha_0} & \dfrac{\partial Y_p}{\partial \beta_0} & \dfrac{\partial Y_p}{\partial \gamma_0} \\ \dfrac{\partial Z_p}{\partial \alpha_0} & \dfrac{\partial Z_p}{\partial \beta_0} & \dfrac{\partial Z_p}{\partial \gamma_0} \end{bmatrix}$$

$$\begin{aligned}\frac{\partial X_p}{\partial \alpha_0} = &(r\cos\eta(\cos\alpha_0\cos\gamma_0 - \sin\alpha_0\sin\beta_0\sin\gamma_0) + r\sin\eta(\cos\gamma_0\sin\alpha_0 + \cos\alpha_0\sin\beta_0\sin\gamma_0)) \\ &(\cos\omega\cos\kappa + \sin\varphi\sin\omega\sin\kappa) \\ &-(\cos\kappa\sin\omega - \cos\omega\sin\varphi\sin\kappa)(r\cos\alpha_0\cos\beta_0\sin\eta - r\cos\beta_0\sin\alpha_0\cos\eta) \\ &+\cos\varphi\sin\kappa(r\cos\eta(\cos\alpha_0\sin\gamma_0 + \cos\gamma_0\sin\alpha_0\sin\beta_0) \\ &\qquad + r\sin\eta(\sin\alpha_0\sin\gamma_0 - \cos\alpha_0\cos\gamma_0\sin\beta_0))\end{aligned}$$

$$\begin{aligned}\frac{\partial X_p}{\partial \beta_0} = &(\cos\omega\cos\kappa + \sin\varphi\sin\omega\sin\kappa)(r\cos\alpha_0\cos\beta_0\cos\eta\sin\gamma_0 + r\cos\beta_0\sin\alpha_0\sin\gamma_0\sin\eta) \\ &+(r\cos\alpha_0\sin\beta_0\cos\eta + r\sin\alpha_0\sin\beta_0\sin\eta)(\cos\kappa\sin\omega - \cos\omega\sin\varphi\sin\kappa) \\ &-\cos\varphi\sin\kappa(r\cos\beta_0\cos\gamma_0\sin\alpha_0\sin\eta + r\cos\alpha_0\cos\beta_0\cos\gamma_0\cos\eta)\end{aligned}$$

$$\begin{aligned}\frac{\partial X_p}{\partial \gamma_0} = &\cos\varphi\sin\kappa(r\cos\eta(\cos\gamma_0\sin\alpha_0 + \cos\alpha_0\sin\beta_0\sin\gamma_0) \\ &\qquad - r\sin\eta(\cos\alpha_0\cos\gamma_0 - \sin\alpha_0\sin\beta_0\sin\gamma_0)) \\ &-(r\cos\eta(\sin\alpha_0\sin\gamma_0 - \cos\alpha_0\cos\gamma_0\sin\beta_0) \\ &\quad - r\sin\eta(\cos\alpha_0\sin\gamma_0 + \cos\gamma_0\sin\alpha_0\sin\beta_0)) \\ &(\cos\omega\cos\kappa + \sin\varphi\sin\omega\sin\kappa)\end{aligned}$$

$$\begin{aligned}\frac{\partial Y_p}{\partial \alpha_0} = &(\sin\omega\sin\kappa + \cos\omega\cos\kappa\sin\varphi)(r\cos\alpha_0\cos\beta_0\sin\eta - r\cos\beta_0\sin\alpha_0\cos\eta) \\ &-(r\cos\eta(\cos\alpha_0\cos\gamma_0 - \sin\alpha_0\sin\beta_0\cos\eta) + r\sin\eta(\cos\gamma_0\sin\alpha_0 + \cos\alpha_0\sin\beta_0\sin\gamma_0)) \\ &(\cos\omega\sin\kappa - \cos\kappa\sin\varphi\sin\omega) \\ &+\cos\varphi\cos\kappa(r\cos\eta(\cos\alpha_0\sin\gamma_0 + \cos\gamma_0\sin\alpha_0\sin\beta_0) \\ &\qquad + r\sin\eta(\sin\alpha_0\sin\gamma_0 - \cos\alpha_0\cos\gamma_0\sin\beta_0))\end{aligned}$$

$$\begin{aligned}\frac{\partial Y_p}{\partial \beta_0} = &(\cos\omega\sin\kappa - \cos\kappa\sin\varphi\sin\omega)(r\cos\alpha_0\cos\beta_0\cos\eta\sin\gamma_0 + r\cos\beta_0\sin\alpha_0\sin\gamma_0\sin\eta) \\ &-(r\cos\alpha_0\sin\beta_0\cos\eta + r\sin\alpha_0\sin\beta_0\sin\eta)(\sin\omega\sin\kappa + \cos\omega\cos\kappa\sin\varphi) \\ &-\cos\varphi\cos\kappa(r\cos\beta_0\cos\gamma_0\sin\alpha_0\sin\eta + r\cos\alpha_0\cos\beta_0\cos\gamma_0\cos\eta)\end{aligned}$$

$$\frac{\partial Y_p}{\partial \gamma_0} = (r\cos\eta\,(\sin\alpha_0\sin\gamma_0 - \cos\alpha_0\cos\gamma_0\sin\beta_0) - r\sin\eta\,(\cos\alpha_0\sin\gamma_0 + \cos\gamma_0\sin\alpha_0\sin\beta_0))$$
$$(\cos\omega\sin\kappa - \cos\kappa\sin\varphi\sin\omega)$$
$$+\cos\varphi\cos\kappa\,(r\cos\eta\,(\cos\gamma_0\sin\alpha_0 + \cos\alpha_0\sin\beta_0\sin\gamma_0)$$
$$-r\sin\eta\,(\cos\alpha_0\cos\gamma_0 - \sin\alpha_0\sin\beta_0\sin\gamma_0))$$

$$\frac{\partial Z_p}{\partial \alpha_0} = \sin\phi\,(r\cos\eta\,(\cos\alpha_0\sin\gamma_0 + \cos\gamma_0\sin\alpha_0\sin\beta_0)$$
$$+r\sin\eta\,(\sin\alpha_0\sin\gamma_0 - \cos\alpha_0\cos\gamma_0\sin\beta_0))$$
$$-\cos\varphi\sin\omega\,(r\cos\eta\,(\cos\alpha_0\cos\gamma_0 - \sin\alpha_0\sin\beta_0\sin\gamma_0)$$
$$+r\sin\eta\,(\cos\gamma_0\sin\alpha_0 + \cos\alpha_0\sin\beta_0\sin\gamma_0))$$
$$-\cos\varphi\cos\omega\,(r\cos\alpha_0\cos\beta_0\sin\eta\ - r\cos\beta_0\sin\alpha_0\cos\eta)$$

$$\frac{\partial Z_p}{\partial \beta_0} = \cos\varphi\cos\omega\,(r\cos\alpha_0\sin\beta_0\cos\eta + r\sin\alpha_0\sin\beta_0\sin\eta)$$
$$-\sin\varphi\,(r\cos\beta_0\cos\gamma_0\sin\alpha_0\sin\eta + r\cos\alpha_0\cos\beta_0\cos\gamma_0\cos\eta)$$
$$-\cos\varphi\sin\omega\,(r\cos\alpha_0\cos\beta_0\cos\eta\sin\gamma_0 + r\cos\beta_0\sin\alpha_0\sin\gamma_0\sin\eta)$$

$$\frac{\partial Z_p}{\partial \gamma_0} = \sin\varphi\,(r\cos\eta\,(\cos\gamma_0\sin\alpha_0 + \cos\alpha_0\sin\beta_0\sin\gamma_0)$$
$$-r\sin\eta\,(\cos\alpha_0\cos\gamma_0 - \sin\alpha_0\sin\beta_0\sin\gamma_0))$$
$$+\cos\varphi\sin\omega\,(r\cos\eta\,(\sin\alpha_0\sin\gamma_0 - \cos\alpha_0\cos\gamma_0\sin\beta_0)$$
$$-r\sin\eta\,(\cos\alpha_0\sin\gamma_0 + \cos\gamma_0\sin\alpha_0\sin\beta_0))$$

Elements of $\mathbf{A}_{SAR}$ matrix:

$$\underset{(3\times 2)}{\mathbf{A}_{SAR}} = \begin{bmatrix} \frac{\partial X_p}{\partial \eta} & \frac{\partial X_p}{\partial r} \\ \frac{\partial Y_p}{\partial \eta} & \frac{\partial Y_p}{\partial r} \\ \frac{\partial Z_p}{\partial \eta} & \frac{\partial Z_p}{\partial r} \end{bmatrix}$$

$$\frac{\partial X_p}{\partial \eta}=(\cos\kappa\sin\omega-\cos\omega\sin\varphi\sin\kappa)(r\cos\alpha_0\cos\beta_0\sin\eta-r\cos\beta_0\sin\alpha_0\cos\eta)$$
$$-(r\cos\eta(\cos\alpha_0\cos\gamma_0-\sin\alpha_0\sin\beta_0\sin\gamma_0)+r\sin\eta(\cos\gamma_0\sin\alpha_0+\cos\alpha_0\sin\beta_0\sin\gamma_0))$$
$$(\cos\omega\cos\kappa+\sin\varphi\sin\omega\sin\kappa)$$
$$-\cos\varphi\sin\kappa(r\cos\eta(\cos\alpha_0\sin\gamma_0+\cos\gamma_0\sin\alpha_0\sin\beta_0)$$
$$+r\sin\eta(\sin\alpha_0\sin\gamma_0-\cos\alpha_0\cos\gamma_0\sin\beta_0))$$

$$\frac{\partial X_p}{\partial r}=(\cos\eta(\cos\gamma_0\sin\alpha_0+\cos\alpha_0\sin\beta_0\sin\gamma_0)-\sin\eta(\cos\alpha_0\cos\gamma_0-\sin\alpha_0\sin\beta_0\sin\gamma_0)$$
$$(\cos\omega\cos\kappa+\sin\varphi\sin\omega\sin\kappa)$$
$$-(\cos\alpha_0\cos\beta_0\cos\eta+\cos\beta_0\sin\alpha_0\sin\eta)(\cos\kappa\sin\omega-\cos\omega\sin\varphi\sin\kappa)$$
$$+\cos\varphi\sin\kappa(\cos\eta(\sin\alpha_0\sin\gamma_0-\cos\alpha_0\cos\gamma_0\sin\beta_0)$$
$$-\sin\eta(\cos\alpha_0\sin\gamma_0+\cos\gamma_0\sin\alpha_0\sin\beta_0))$$

$$\frac{\partial Y_p}{\partial \eta}=(r\cos\eta(\cos\alpha_0\cos\gamma_0-\sin\alpha_0\sin\beta_0\sin\gamma_0)+r\sin\eta(\cos\gamma_0\sin\alpha_0+\cos\alpha_0\sin\beta_0\sin\gamma_0))$$
$$(\cos\omega\sin\kappa-\cos\kappa\sin\varphi\sin\omega)$$
$$-(\sin\omega\sin\kappa+\cos\omega\cos\kappa\sin\varphi)(r\cos\alpha_0\cos\beta_0\sin\eta-r\cos\beta_0\sin\alpha_0\cos\eta)$$
$$-\cos\varphi\cos\kappa(r\cos\eta(\cos\alpha_0\sin\gamma_0+\cos\gamma_0\sin\alpha_0\sin\beta_0)$$
$$+r\sin\eta(\sin\alpha_0\sin\gamma_0-\cos\alpha_0\cos\gamma_0\sin\beta_0))$$

$$\frac{\partial Y_p}{\partial r}=(\cos\alpha_0\cos\beta_0\cos\eta+\cos\beta_0\sin\alpha_0\sin\eta)(\sin\omega\sin\kappa+\cos\omega\cos\kappa\sin\varphi)$$
$$-(\cos\eta(\cos\gamma_0\sin\alpha_0+\cos\alpha_0\sin\beta_0\sin\gamma_0)-\sin\eta(\cos\alpha_0\cos\gamma_0-\sin\alpha_0\sin\beta_0\sin\gamma_0))$$
$$(\cos\omega\sin\kappa-\cos\kappa\sin\varphi\sin\omega)$$
$$+\cos\varphi\cos\kappa(\cos\eta(\sin\alpha_0\sin\gamma_0-\cos\alpha_0\cos\gamma_0\sin\beta_0)$$
$$-\sin\eta(\cos\alpha_0\sin\gamma_0+\cos\gamma_0\sin\alpha_0\sin\beta_0))$$

$$\frac{\partial Z_p}{\partial \eta}=\cos\varphi\sin\omega(r\cos\eta(\cos\alpha_0\cos\gamma_0-\sin\alpha_0\sin\beta_0\sin\gamma_0)$$
$$+r\sin\eta(\cos\gamma_0\sin\alpha_0+\cos\alpha_0\sin\beta_0\sin\gamma_0)$$
$$-\sin\varphi(r\cos\eta(\cos\alpha_0\sin\gamma_0+\cos\gamma_0\sin\alpha_0\sin\beta_0)$$
$$+r\sin\eta(\sin\alpha_0\sin\gamma_0-\cos\alpha_0\cos\gamma_0\sin\beta_0))$$
$$+\cos\varphi\cos\omega(r\cos\alpha_0\cos\beta_0\sin\eta-r\cos\beta_0\sin\alpha_0\cos\eta)$$

$$\frac{\partial Z_p}{\partial r}=\sin\varphi(\cos\eta(\sin\alpha_0\sin\gamma_0-\cos\alpha_0\cos\gamma_0\sin\beta_0)$$
$$-\sin\eta(\cos\alpha_0\sin\gamma_0+\cos\gamma_0\sin\alpha_0\sin\beta_0))$$
$$-\cos\varphi\cos\omega(\cos\alpha_0\cos\beta_0\cos\eta+\cos\beta_0\sin\alpha_0\sin\eta)$$
$$-\cos\varphi\sin\omega(\cos\eta(\cos\gamma_0\sin\alpha_0+\cos\alpha_0\sin\beta_0\sin\gamma_0)$$
$$-\sin\eta(\cos\alpha_0\cos\gamma_0-\sin\alpha_0\sin\beta_0\sin\gamma_0))$$

Typical examples of parameters and their precision values are presented in following table (1).

Table 1. Typical Values of Precisions in Airborne LiDAR Data Acquisition

S.N.	Variable	Precision Values	Values and Remarks
1	3D Coordinates of navigation sensor	$\sigma_{X_0} = \sigma_{Y_0} = 5\,cm$ $\sigma_{Z_0} = 1.5(\sigma_{X_0}) = 7.5\,cm$	$\sigma_{X_0} = \sigma_{Y_0} = 2\,cm + 1\,ppm;$ $\sigma_{Z_0} = 2cm + 2\,ppm$ (Baseline length = $30\,km$)
2	Orientation angles	**IMU model: Applanix 510** $\sigma_{\omega} = \sigma_{\varphi} = 0.005^{\circ};$ $\sigma_{k} = 0.008^{\circ}$	$\omega = \varphi = \kappa = 0^{\circ}$
3	Lever-arm components	$\sigma_{l_x} = \sigma_{l_y} = \sigma_{l_z} = \pm 0.02\,m$	$l_x = l_y = l_z = -0.5\,m$
4	Bore-Sight Angles	Values by least squares : $\sigma_{\alpha_0} = \sigma_{\beta_0} = 0.001^{\circ};$ $\sigma_{\gamma_0} = 0.004^{\circ}$	$\alpha_0 = \beta_0 = \gamma_0 = 2^{\circ}$
5	Range and scan angle	$\sigma_r = 0.02\,m;$ $\sigma_{\phi} = 0.0044^{\circ}$	$r = H\sec\phi$

Using values of various variables ($\omega, \varphi, \kappa, l_x, l_y, l_z, \alpha_0, \beta_0, \gamma_0$) given in table (1), the elements of the matrix $\mathbf{A}$ are written below in terms of range and scan angle:

$$\underset{(3\times3)}{\mathbf{A}_{NSC}} = \begin{bmatrix} \frac{\partial X_p}{\partial X_0} & \frac{\partial X_p}{\partial Y_0} & \frac{\partial X_p}{\partial Z_0} \\ \frac{\partial Y_p}{\partial Y_0} & \frac{\partial Y_p}{\partial Y_0} & \frac{\partial Y_p}{\partial Y_0} \\ \frac{\partial Z_p}{\partial X_0} & \frac{\partial Z_p}{\partial Y_0} & \frac{\partial Z_p}{\partial Z_0} \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{(3\times3)}{\mathbf{A}_{AA}} = \begin{bmatrix} \frac{\partial X_p}{\partial \omega} & \frac{\partial X_p}{\partial \varphi} & \frac{\partial X_p}{\partial \kappa} \\ \frac{\partial Y_p}{\partial \omega} & \frac{\partial Y_p}{\partial \varphi} & \frac{\partial Y_p}{\partial \kappa} \\ \frac{\partial Z_p}{\partial \omega} & \frac{\partial Z_p}{\partial \varphi} & \frac{\partial Z_p}{\partial \kappa} \end{bmatrix} = \begin{bmatrix} A_{1,4} & A_{1,5} & A_{1,6} \\ A_{2,4} & A_{2,5} & A_{2,6} \\ A_{3,4} & A_{3,5} & A_{3,6} \end{bmatrix}$$

$$A_{1,4} = (0.5 - 0.0349\, r \sin\phi - 0.9988\, r \cos\phi)$$

$$A_{1,5} = 0$$

$$A_{1,6} = -(0.5 + 0.0336\, r \cos\phi + 0.0361\, r \sin\phi)$$

$$A_{2,4} = 0$$

$$A_{2,5} = (-0.5 + 0.9988\, r \cos\phi + 0.0349\, r \sin\phi)$$

$$A_{2,6} = (0.5 - 0.0361\, r \cos\phi + 0.9987\, r \sin\phi)$$

$$A_{3,4} = (0.5 - 0.0361\, r \cos\phi + 0.9987\, r \sin\phi)$$

$$A_{3,5} = (-0.5 - 0.0336\, r \cos\phi - 0.0361\, r \sin\phi)$$

$$A_{3,6} = 0$$

$$\underset{(3\times3)}{\mathbf{A}_{LAO}} = \begin{bmatrix} \frac{\partial X_p}{\partial l_x} & \frac{\partial X_p}{\partial l_y} & \frac{\partial X_p}{\partial l_z} \\ \frac{\partial Y_p}{\partial l_x} & \frac{\partial Y_p}{\partial l_y} & \frac{\partial Y_p}{\partial l_z} \\ \frac{\partial Z_p}{\partial l_x} & \frac{\partial Z_p}{\partial l_y} & \frac{\partial Z_p}{\partial l_z} \end{bmatrix} = \begin{bmatrix} A_{1,7} & A_{1,8} & A_{1,9} \\ A_{2,7} & A_{2,8} & A_{2,9} \\ A_{3,7} & A_{3,8} & A_{3,9} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A_{1,7} = 0$$

$$A_{1,8} = 1$$

$$A_{1,9} = 0$$

$$A_{2,7} = 1$$

$$A_{2,8} = 0$$

$$A_{2,9} = 0$$

$A_{3,7} = 0$

$A_{3,8} = 0$

$A_{3,9} = -1$

$$\underset{(3\times 3)}{\mathbf{A}_{BSA}} = \begin{bmatrix} \dfrac{\partial X_p}{\partial \alpha_0} & \dfrac{\partial X_p}{\partial \beta_0} & \dfrac{\partial X_p}{\partial \gamma_0} \\ \dfrac{\partial Y_p}{\partial \alpha_0} & \dfrac{\partial Y_p}{\partial \beta_0} & \dfrac{\partial Y_p}{\partial \gamma_0} \\ \dfrac{\partial Z_p}{\partial \alpha_0} & \dfrac{\partial Z_p}{\partial \beta_0} & \dfrac{\partial Z_p}{\partial \gamma_0} \end{bmatrix} = \begin{bmatrix} A_{1,10} & A_{1,11} & A_{1,12} \\ A_{2,10} & A_{2,11} & A_{2,12} \\ A_{3,10} & A_{3,11} & A_{3,12} \end{bmatrix}$$

$A_{1,10} = (0.9987\, r\cos\phi + 0.0361\, r\sin\phi)$

$A_{1,11} = (0.0349\, r\cos\phi + 0.0012\, r\sin\phi)$

$A_{1,12} = (0.0336\, r\cos\phi + 0.0361\, r\sin\phi)$

$A_{2,10} = (0.0361\, r\cos\phi - 0.0336\, r\sin\phi)$

$A_{2,11} = (-0.9982\, r\cos\phi - 0.0349\, r\sin\phi)$

$A_{2,12} = (0.0361\, r\cos\phi - 0.9987\, r\sin\phi)$

$A_{3,10} = (0.0349\, r\cos\phi - 0.9988\, r\sin\phi)$

$A_{3,11} = (0.0349\, r\cos\phi + 0.0012\, r\sin\phi)$

$A_{3,12} = 0$

$$\underset{(3\times 2)}{\mathbf{A}_{SAR}} = \begin{bmatrix} \dfrac{\partial X_p}{\partial \eta} & \dfrac{\partial X_p}{\partial r} \\ \dfrac{\partial Y_p}{\partial \eta} & \dfrac{\partial Y_p}{\partial r} \\ \dfrac{\partial Z_p}{\partial \eta} & \dfrac{\partial Z_p}{\partial r} \end{bmatrix} = \begin{bmatrix} A_{1,13} & A_{1,14} \\ A_{2,13} & A_{2,14} \\ A_{3,13} & A_{3,14} \end{bmatrix}$$

$A_{1,13} = (-0.9987\, r\cos\phi - 0.0361\, r\sin\phi)$

$A_{1,14} = (0.0361\cos\phi - 0.9987\sin\phi)$

$A_{2,13} = (-0.0361\, r\cos\phi + 0.0336\, r\sin\phi)$

$$A_{2,14} = (-0.0336\cos\phi - 0.0361\sin\phi)$$

$$A_{3,13} = (0.9988\, r\sin\phi - 0.0349\, r\cos\phi)$$

$$A_{3,14} = (-0.9988\cos\phi - 0.0349\sin\phi)$$

Conclusion

This report demonstrates the calculation of propagated error in 3D coordinates obtained by airborne LiDAR by law of error propagation. The standard equation of physical model that describes that how range and half scan angle data are transformed into error expressions using GPS, IMU and spatial arrangements between these sensor units, is used for determining the propagated error. For choosen precision values of the specific GPS, IMU, and LiDAR sensors, simplified expressions of the propagated errors are presented.

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