On Spectrum and Arithmetic

C. S. Rajan

School of Mathematics,
Tata Institute of Fundamental Research, Mumbai
rajan@math.tifr.res.in

11 August 2010

Arithmetic

Let X be an algebraic variety defined over a number field K. Our aim is to study the arithmetic of X and the relationship with the geometry of the space $X(\mathbf{C})$.

There have been two themes that have dominated so far the relationship between the geometry of $X(\mathbf{C})$ and the arithmetic of X.

Topology and arithmetic: Weil conjectures

Suppose X is a smooth projective variety defined over K. Let v be a finite place of K where X has good reduction, and let $\bar{X}_v$ be the reduction modulo v of X. As a part of the Weil conjectures, the zeta function,

$$Z(\bar{X}_v, t) = \exp\left(\sum_{n\geq 1} N_n(\bar{X}_v(\mathbf{F}_{q^n}))t^n/n\right)$$

is a rational function of t of the form,

$$Z(\bar{X}_v, t) = \frac{P_1(t)P_3(t)\cdots P_{2d-1}(t)}{P_0(t)\cdots P_{2d}(t)}$$

where each P_i is a polynomial in t of degree b_i equal to the i-th Betti number of the space $X(\mathbf{C})$.
Here we have an example where the topology of $X(\mathbf{C})$ influences the arithmetic of X. Conversely, we also recover the betti numbers from a knowledge of the arithmetic.

Curvature and arithmetic: Faltings' theorem

Suppose X is a smooth, projective curve over K. A conjecture of Mordell, proved by Faltings is that when the genus of X is at least two, then $X(K)$ is finite.
In this case, by the uniformization theorem, $X(\mathbf{C})$ acquires a metric of constant negative curvature.
On the other hand, when X has genus 0, then either $X(K)$ is empty, or it is Zariski dense in X. Geometrically, $X(\mathbf{C})$ is isomorphic to the Riemann sphere, and it has a Riemannian metric of constant positive curvature.
More generally, if the variety is Fano (anti-canonical bundle is ample), then it is expected that the rational points are Zariski density if non-empty.

Elliptic curves

When the genus of X is one, the geometry of $X(\mathbf{C})$ is flat, and the question of understanding the rank of $X(K)$ is delicate; conjecturally controlled by the L-function (but not by geometric data).

Question

What other geometric information will throw more light on the arithmetic of a variety X?
The answer we would like to propose is that the spectra of Laplace type operators with respect to suitable metrics on $X(\mathbf{C})$ should determine the arithmetic upto some finite error, and conversely.
For this question to make sense, either we work with all metrics on $X(\mathbf{C})$ or with some uniquely defined metrics.

Extremal metrics, K ahler-Einstein metrics

By the work of Calabi, Aubin and Yau, if the first Chern class of a compact, Kaehler manifold is negative or zero, then there exists a unique metric (upto scaling), whose Ricci curvature is proportional to the metric. These are the K ahler-Einstein metrics.

In the case of Fano manifolds, the existence of such metrics and it's relation to the stability of varieties is not completely known; there is current work by Tian, Donaldson and others..

The point is that for a large class of varieties (maybe all varieties upto mild degenerations and modifications), there are uniquely defined metrics generalizing the known class of locally symmetric Riemannian manifolds.

Laplacian and spectrum

Associated to any Riemannian manifold, there exists a natural, second order differential operator acting on the space of smooth functions given by the Laplacian. This is a self-adjoint, elliptic differential operator, and it's spectrum is defined to be the collection of eigenvalues (have finite multiplicity).

Define two Riemannian manifolds to be isospectral if the spectra of the Laplacian acting on the space of smooth functions are equal.

More generally, one can consider the Hodge-deRham Laplacians acting on the space of smooth p-forms, for p ranging from 0 to the real dimension of the manifold. Or the Laplacians acting on the space of sections of natural hermitian vector bundles. The corresponding notion is called strongly isospectral.

Spectrum to arithmetic

One question that can be raised is the following:
Let $X, \ Y$ be smooth, projective varieties defined over a number field K of the same dimension. Suppose that the spectra of the Laplacian acting on p-forms with respect to the (unique) Kahler-Einstein metric are equal. Then are the Hasse-Weil zeta functions of X and Y equal?

Arithmetic to Spectrum

The converse direction does not seem to be clear to state in this generality for an arbitrary variety X. We do not know what collection of natural local systems and their zeta functions to consider to define the arithmetic of a space.

We now try to give some evidence that these questions may have a positive answer. We work in the context of locally Riemannian symmetric spaces. These are spaces whose universal cover is of the form G/M, where G is a semisimple (or reductive Lie group) and M is a maximal compact subgroup of G. First of all, the natural metric is Einstein.
We assume further that G/M is hermitian symmetric. When G is the set of real points of a reductive group defined over a number field, and we take Γ to be a congruent arithmetic lattices, then the various quotient spaces (as Γ varies) $\Gamma\backslash G/M$ can be defined over number fields by the theory of canonical models due to Shimura and Deligne. The point is that these spaces have natural geometric and arithmetical structures, and it makes sense to frame the above questions in this context.

Analogies between spectrum and arithmetic

Various analogies have been known before, starting with the work of Maass and Selberg in the context of Riemannian locally symmetric spaces.

For instance, the Laplace is said to be the Hecke (or Frobenius) operator at infinity.

This analogy can be seen when we compare the Harishchandra isomorphism relating invariant differential operators (like the Casimirs or Laplacians) to the Weyl group invariant polynomials of the Lie algebra of the torus.

The Satake isomorphism, the p-adic analogue of the Harishchandra isomorphism relates the right side above to that of Hecke operators given by the space of compactly supported locally constant functions bi-invariant by a hyperspecial compact open subgroup.

In turn, the Hecke operators are related to the Frobenius by Shimura congruence relations.

In the context of Riemannian locally symmetric spaces, it is better to work with the representation theoretic spectrum: the representation space $L^2(\Gamma\backslash G)$ for G. This can be considered as the 'Fourier' way to look at the spectrum, versus the Laplace approach to spectrum.

If two lattices are representation equivalent in G (the above representations are equivalent), then it can be seen that the corresponding locally symmetric spaces are strongly isospectral.

On the other hand, for general G it is not known that strong isospectrality of the compact locally symmetric spaces corresponding to two lattices $\Gamma_1, \ \Gamma_2$ imply representation equivalence of the lattices in G.

Archimedean analogue of Tate's conjectures

Faltings proved that if two arithmetic Riemann surfaces have equal Frobenius actions for almost all primes p, then the Jacobians of the Riemann surfaces are isogenous.
Question: Suppose we know that two Riemann surfaces have the same Laplacian spectrum (are isospectral) with respect to the hyperbolic (or flat) metric. Then do they have the same arithmetic, i.e., will their Jacobians be isogenous?
The answer is trivially false; we can take two complex conjugate lattices in $\mathbf{C}$. They define isospectral elliptic curves but are not isogenous.
However we modify the question, to take care of these counterexamples.

Arithmetic Riemann surfaces

We now look at Shimura varieties attached to indefinite quaternion division algebras D defined over a totally real number field F. When D is split at only one real place, then X_Γ is a compact Riemann surface.
Question: Suppose two such spaces are isospectral. Then do they have the same Hasse-Weil zeta functions?
The first confirmation that these conjectures may be valid is provided by a result of Alan Reid (proved much before our conjectures): if two such Riemann surfaces are isospectral, then the underlying number fields and division algebras are equal.

Gassmann-Sunada method

Sunada constructed examples of isospectral spaces which are not isometric. His method is completely analogous to a related construction in the arithmetic of number fields. Examples were produced by Gassmann of two non-Galois conjugate number fields which have the same Dedekind zeta function.

Let G be a finite group and H_1, H_2 be two non-conjugate subgroups such that every conjugacy class in G meets H_1 and H_2 with the same cardinality.

Example

Take G to be the symmetric group on 6 letters. Let

$$H_1 = \{1, (12)(34), (13)(24), (14)(23)\}.$$

$$H_2 = \{1, (12)(34), (12)(56), (34)(56)\}.$$

Gassmann-Sunada method (contd.)

Suppose G acts freely on a compact Riemannian manifold M. Then the quotient spaces M/H_1 and M/H_2 are covered by M, and we can give them the induced Riemannian metric.
Sunada's theorem is that the spaces M/H_1 and M/H_2 are isospectral.
Theorem (Joint with Dipendra Prasad) The conjectures are true if we are in the situation of the examples constructed by Sunada.
The proof of this is essentially the naturalness of Frobenius reciprocity.

From Arithmetic to Spectrum

Let D be an indefinite quaternion division algebra defined over a totally real number field F. The arithmetic varieties attached to the group $SL_1(D)$ defined by congruent arithmetic lattices, have models defined over an abelian extension of the reflex field F' by Shimura theory.

Theorem Galois conjugation over F' preserves the spectrum (under some technical hypothesis on the nature of the congruent arithmetical lattice).

This gives examples of isospectral spaces which are not isometric, and this generalizes the examples constructed by Vigneras. Vigneras works with maximal lattices given by the units in a maximal order (real dimension two or three). Her method is to use the Selberg trace formula, and compare the length spectrum using formulas of Eichler. The proof of the above theorem instead uses results of Labesse and Langlands on L-indistinguishability, requires only D to be indefinite, and works for a general class of lattices.

Another way of seeing the above theorem is that the connected components in the Deligne canonical model, are all isospectral. Recently, Milne has given examples, where the conjugates (by an automorphism of $\mathbf{C}$)) of the connected compoents of a Shimura variety have different topologies.
Lubotzky, Vishne and Samuels have constructed isospectral pairs of spaces from division algebras, for which the lattices are not commensurable (but the global arithmetic of automorphic forms are isomorphic, related by a Jacquet-Langlands type of correspondence.

Flat tori

We can revisit Milnor's examples of isospectral flat tori. After Faltings theorem, say two lattices in Euclidean n-space to have the same arithmetic if upto an isometry they are commensurable (the lattices are isometric after tensoring with $\mathbf{Q}$).
The following conjecture is attributed to Kitaoka: if two lattices are isospectral (one can weaken the hypothesis to take care of commensurable lattices), then they have the same arithmetic.

We now consider a G_m- analogue. Let $F_1,\ F_2$ be totally real number fields having the same Dedekind zeta function. Then it can be seen that the unit groups with respect to the log embedding are isospectral. On the converse side, we have:

Theorem (with Ted Chinburg) Suppose that the unit groups of two totally real number fields are isometric after tensoring with a number field, Assume further the validity of Schanuel's conjecture on transcendence: if $a_1, \cdots, a_n$ are algebraic real numbers which are multiplicatively independent over the rationals, then $\log a_1, \cdots, \log a_n$ are transcendentally independent. Then the number fields have the same arithmetic, i.e., they have the same Dedekind zeta function.

Remark It also seems that Kitaoka's conjecture as stated above is false: one needs to tensor by a number field. Thus we are assuming that this generalized form of Kitaoka's theorem (apart from Schanuel's conjecture) to relate the spectrum to arithmetic.

The counter-examples come from examples of Smit and Lenstra of arithmetically equivalent totally real number fields with different class groups.