# $p$-adic measures

A *distribution* on $\mathbb{Z}_p$ is a map

$$\mu : X \to \mathbb{Q}_p$$

defined on compact open subsets $X \subset \mathbb{Q}_p$, so that for all $a + p^N\mathbb{Z}_p \subset X$ one has

$$\mu(a + p^N\mathbb{Z}_p) = \sum_{b=0}^{p-1} \mu(a + bp^n + p^{N+1})\mathbb{Z}_p.$$

An example is $\mu_{\text{Haar}}$ given by

$$\mu_{\text{Haar}}(a + p^N\mathbb{Z}_p) = p^{-N}.$$

A $p$-adic *measure* on $X$ is a distribution such that, for some constant $C$,

$$|\mu(U)|_p \leq C$$

for all compact open $U \subset X$.

Let $\mu$ be a $p$-adic measure on $\mathbb{Z}_p$ and $f : \mathbb{Z}_p \to \mathbb{Q}_p$ a continuous function. For each $N$, choose $x_{a,N} \in \{a + p^N\mathbb{Z}_p\}$ and put

$$S_N := \sum_{0 \leq a \leq p^N - 1} f(x_{a,N})\mu(a + p^N\mathbb{Z}_p).$$

Then there exists a limit

$$\lim_{N\to\infty} S_N =: \int_{\mathbb{Z}_p} f d\mu,$$

and it is independent of the choices made.

Note that there are problems with $\mu_{\text{Haar}}$, even the function $f(x) = x$ is not integrable! Other distributions are given by

$$\mu_{B,k}(a + p^N\mathbb{Z}_p) := p^{N(k-1)} B_k \left(\frac{a}{p^N}\right),$$

where $k \in \mathbb{N}$ and $B_k(x)$ is the $k$-th Bernoulli polynomial.

To regularize these distributions, choose $\alpha \in \mathbb{Q} \cap \mathbb{Z}_p^*$ and put

$$\mu_{k,\alpha}(U) := \mu_{B,k}(U) - \alpha^{-k}\mu_{B,k}(\alpha U).$$

**Theorem 1.** *The distributions $\mu_{k,\alpha}$ are measures. Moreover, for all compact open $U \subset \mathbb{Z}_p$ one has*

$$\mu_{1,\alpha}(U) \leq 1.$$