IWASAWA λ-INVARIANTS AND Γ-TRANSFORMS OF p-ADIC MEASURES ON $\mathbb{Z}_p^n$

Rupam Barman[1] and Anupam Saikia[2]

Abstract. In this paper we determine a relation between the λ-invariants of a p-adic measure on $\mathbb{Z}_p^n$ and its Γ-transform. Along the way we also determine p-adic properties of certain Mahler coefficients.
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1. **Introduction**

Fix an odd prime p. Let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Q}_p$ with a local parameter π. We write $\mathbb{Z}_p^\times = V \times U$ where V is the group of $(p-1)$st roots of unity in $\mathbb{Z}_p$ and $U = 1 + p\mathbb{Z}_p$. Let u be a topological generator of U. The projections from $\mathbb{Z}_p^\times$ onto V and U are denoted by ω and $<>$ respectively. We have an isomorphism $\phi : \mathbb{Z}_p \to U$ given by $\phi(y) = u^y$.

Let $\Lambda_{(n)}$ denote the $\mathcal{O}$-valued measures on $\mathbb{Z}_p^n$. It is well-known, (see e.g. [1]), that $\Lambda_{(1)}$ is a ring under convolution, and is isomorphic to the formal power series ring $\mathcal{O}[[T-1]]$. Explicitly, for $x \in \mathbb{Z}_p$, let

$$T^x = \sum_{n=0}^{\infty} \binom{x}{n} (T-1)^n \in \mathcal{O}[[T-1]].$$

The power series associated to a measure $\alpha \in \Lambda_{(1)}$ is then defined by

$$\hat{\alpha}(T) = \int_{\mathbb{Z}_p} T^x d\alpha(x) = \sum_{n=0}^{\infty} b_n(\alpha)(T-1)^n$$

where

$$b_n(\alpha) = \int_{\mathbb{Z}_p} \binom{x}{n} d\alpha(x).$$

A classical theorem of Mahler states that any continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ may be written uniquely in the form

$$f(x) = \sum_{n=0}^{\infty} a_n(f) \binom{x}{n},$$

[1]Department of Mathematical Sciences, Tezpur University, Napaam-784028, Assam, INDIA.
[2]Department of Mathematics, Indian Institute of Technology, Guwahati-781039, Assam, INDIA

where $a_n(f) \in \mathbb{Q}_p, a_n(f) \mapsto 0$ as $n \mapsto \infty$. In fact

$$a_n(f) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(j). \tag{1.1}$$

This theorem may be generalized to continuous functions $f : \mathbb{Z}_p \to K$, where K is any finite extension of $\mathbb{Q}_p$. Using this generalization, we obtain the following

$$\int_{\mathbb{Z}_p} f(x) d\alpha(x) = \sum_{n=0}^{\infty} a_n(f) \int_{\mathbb{Z}_p} \binom{x}{n} d\alpha(x) = \sum_{n=0}^{\infty} a_n(f) b_n(\alpha).$$

Note that if $\mathcal{O}$ is the ring of integers of K and $f : \mathbb{Z}_p \to \mathcal{O}$, then $a_n(f) \in \mathcal{O}$.

The natural generalizations of the above results to larger values of n are true. $\mathcal{O}$-valued measures on $\mathbb{Z}_p^n$ correspond to power series in $\mathcal{O}[[T_1 - 1, \cdots, T_n - 1]]$. This correspondence is given by

$$\begin{aligned}\hat{\alpha}(T_1, \cdots, T_n) &= \int_{\mathbb{Z}_p^n} T_1^{x_1} \cdots T_n^{x_n} d\alpha(x_1, \cdots, x_n) \\ &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \left(\int_{\mathbb{Z}_p^n} \binom{x_1}{m_1} \cdots \binom{x_n}{m_n} d\alpha(x_1, \cdots, x_n) \right) \\ &\qquad \times (T_1 - 1)^{m_1} \cdots (T_n - 1)^{m_n}. \end{aligned} \tag{1.2}$$

Furthermore, if $f : \mathbb{Z}_p^n \to \mathcal{O}$ is continuous, we may write (by repeated application of the generalization of Mahler theorem)

$$f(x_1, \cdots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} a_{m_1, \cdots, m_n}(f) \binom{x_1}{m_1} \cdots \binom{x_n}{m_n}, \tag{1.3}$$

where

$$\begin{aligned} a_{m_1, \cdots, m_n}(f) = &\sum_{j_1=0}^{m_1} \cdots \sum_{j_n=0}^{m_n} (-1)^{m_1 - j_1} \cdots (-1)^{m_n - j_n} \binom{m_1}{j_1} \cdots \binom{m_n}{j_n} f(j_1, \cdots, j_n) \\ &\to 0 \text{ in } \mathcal{O}. \end{aligned} \tag{1.4}$$

The constants $a_{m_1, \cdots, m_n}(f)$ are called the Mahler coefficients of the function f.

Let α be a measure on $\mathbb{Z}_p^n$. For $(a_1, \cdots, a_n) \in (\mathbb{Z}_p^\times)^n$, denote by $\alpha \circ (a_1, \cdots, a_n)$ the measure on $\mathbb{Z}_p^n$ given by

$$\alpha \circ (a_1, \cdots, a_n)(A_1 \times \cdots \times A_n) = \alpha(a_1 A_1, \cdots, a_n A_n),$$

where A_i are compact open subsets of $\mathbb{Z}_p$. Also, if $A = (A_1, \cdots, A_n) \subseteq \mathbb{Z}_p^n$, where all A_i are compact open subsets of $\mathbb{Z}_p$, we let $\alpha|_A$ denote the measure obtained by restricting α to A and extending by 0.

The Γ-transform of a measure $\alpha \in \Lambda_{(n)}$ is defined as a function of the p-adic variables $s_1, \cdots, s_n$ given by

$$\Gamma_\alpha(s_1, \cdots, s_n) = \int_{(\mathbb{Z}_p^\times)^n} < x_1 >^{s_1} \cdots < x_n >^{s_n} d\alpha(x_1, \cdots, x_n).$$

If we put $d\alpha(a_1x_1,\cdots,a_nx_n)$ for $d(\alpha\circ(a_1,\cdots,a_n))(x_1,\cdots,x_n)$, splitting up the integral, we can also write

$$\begin{aligned}\Gamma_\alpha(s_1,\cdots,s_n) &= \sum_{\eta_1\in V}\cdots\sum_{\eta_n\in V}\int_{U^n} <\eta_1x_1>^{s_1}\cdots<\eta_nx_n>^{s_n} d\alpha(\eta_1x_1,\cdots,\eta_nx_n)\\ &= \int_{U^n} x_1^{s_1}\cdots x_n^{s_n}d\beta(x_1,\cdots,x_n),\end{aligned}$$

where

$$\beta=\sum_{\eta_1\in V}\cdots\sum_{\eta_n\in V}(\alpha\circ(\eta_1,\cdots,\eta_n))|_{U^n},$$

a measure on U^n.

Now the measure β may be viewed as a measure on $\mathbb{Z}_p^n$ via the isomorphism ϕ:

$$\tilde{\beta}(A_1,\cdots,A_n)=\beta(\phi(A_1),\cdots,\phi(A_n)).$$

Let us write $d\beta(u^{y_1},\cdots,u^{y_n})$ for $d\tilde{\beta}(y_1,\cdots,y_n)$. Let $G(T_1,\cdots,T_n)$ be the power series associated to $\tilde{\beta}$, that is,

$$G(T_1,\cdots,T_n)=\int_{\mathbb{Z}_p^n} T_1^{y_1}\cdots T_n^{y_n}d\beta(u^{y_1},\cdots,u^{y_n}).$$

Then $\Gamma_\alpha(s_1,\cdots,s_n)=G(u^{s_1},\cdots,u^{s_n})$.

For a more thorough treatment of p-adic measure theory, see [3] and [7].

2. **Iwasawa λ-invariants and Γ- transforms**

The Iwasawa μ and λ- invariants of a power series

$$F(T)=\sum_{n=0}^{\infty}a_n(T-1)^n\in\mathcal{O}[[T-1]]$$

are defined by

$$\begin{aligned}\mu(F(T)) &= \min\{ord(a_n): n\geq 0\}\\ \lambda(F(T)) &= \min\{n: ord(a_n)=\mu(F(T))\}.\end{aligned}$$

Analogously, we define the Iwasawa μ and λ- invariants of a power series

$$F(T_1,\cdots,T_n)=\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}a_{m_1,\cdots,m_n}(T_1-1)^{m_1}\cdots(T_n-1)^{m_n}\in\mathcal{O}[[T_1-1,\cdots,T_n-1]]$$

as follows:

$$\begin{aligned}\mu(F(T_1,\cdots,T_n)) &= \min\{ord(a_{m_1,\cdots,m_n}): m_i\geq 0 \quad \forall i\}\\ \lambda(F(T_1,\cdots,T_n)) &= \min\{m_1+\cdots+m_n: ord(a_{m_1,\cdots,m_n})=\mu(F(T_1,\cdots,T_n))\}.\end{aligned}$$

For a measure $\alpha\in\Lambda_{(n)}$, we understand $\mu(\alpha)$ and $\lambda(\alpha)$ to mean $\mu(\hat{\alpha}(T_1,\cdots,T_n))$ and $\lambda(\hat{\alpha}(T_1,\cdots,T_n))$.

Let $\alpha \in \Lambda_{(n)}$. That is, α is a $\mathcal{O}$-valued measure on $\mathbb{Z}_p^n$. Let u be a fixed topological generator of $U = 1 + p\mathbb{Z}_p$, and let $G(T_1, \cdots, T_n)$ satisfy $G(u^{s_1}, \cdots, u^{s_n}) = \Gamma_\alpha(s_1, \cdots, s_n)$, so that

$$G(T_1, \cdots, T_n) = \int_{\mathbb{Z}_p} T_1^{y_1} \cdots T_n^{y_n} d\beta(u^{y_1}, \cdots, u^{y_n}),$$
$$\text{where } \beta = \sum_{\eta_1 \in V} \cdots \sum_{\eta_n \in V} (\alpha \circ (\eta_1, \cdots, \eta_n))|_{U^n}. \tag{2.1}$$

Note that β is a measure on U^n. We extend β to $\mathbb{Z}_p^n$ by 0 and then we get a power series

$$\hat{\beta}(T_1, \cdots, T_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} b_{m_1, \cdots, m_n} (T_1 - 1)^{m_1} \cdots (T_n - 1)^{m_n}. \tag{2.2}$$

Suppose that

$$G(T_1, \cdots, T_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} g_{m_1, \cdots, m_n} (T_1 - 1)^{m_1} \cdots (T_n - 1)^{m_n}. \tag{2.3}$$

In case of $n = 1$, Sinnott in his paper [6] proved that $\mu(G(T)) = \mu(\alpha^* + \alpha^* \circ (-1))$, if $\hat{\alpha}(T)$ is a rational function of T. Here $\alpha^* = \alpha|_{\mathbb{Z}_p^\times}$. It was Kida who first obtained a relation between the λ-invariant of a measure on $\mathbb{Z}_p$ and its Gamma-Transform with a fixed topological generator [2]. Later, in case of $n = 1$, Childress in her paper [1] proved that $\mu(G(T)) = \mu(\beta)$ and $\lambda(\beta) = p\lambda(G(T))$ if $\lambda(G(T)) \leq p$. Satoh obtained the same result without any condition on $\lambda(G(T))$, but his approach was based on certain properties of Stirling numbers [5]. In our paper [4], exploiting certain combinatorial identities we proved that $\lambda(\beta) = p\lambda(G(T))$ if $\lambda(G(T)) \leq 2p$. In this paper we prove the following main result which gives a relation between $\lambda(G(T_1, \cdots, T_n))$ and $\lambda(\beta)$ for any n.

Theorem 2.1. *Suppose $\lambda(G(T_1, \cdots, T_n)) \leq 2p$, then $\lambda(\beta) = p\lambda(G(T_1, \cdots, T_n))$.*

We will prove this theorem following the approach of Saikia & Barman [4] and Childress [1]. We really do not know whether the method of Satoh based on certain properties of Stirling numbers can be generalized to prove Theorem (2.1). We now state two Lemmas which are easy generalization of Lemma 1 and Lemma 2 proved by Childress in [1]. As before, suppose

$$\hat{\beta}(T_1, \cdots, T_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} b_{m_1, \cdots, m_n} (T_1 - 1)^{m_1} \cdots (T_n - 1)^{m_n}$$

and

$$G(T_1, \cdots, T_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} g_{m_1, \cdots, m_n} (T_1 - 1)^{m_1} \cdots (T_n - 1)^{m_n}.$$

Let us define

$$f_m(x) = \binom{u^x}{m} \quad \text{and} \quad f_{m_1, \cdots, m_n}(x_1, \cdots, x_n) = f_{m_1}(x_1) \cdots f_{m_n}(x_n).$$

Lemma 2.2. *Let α be a measure on $\mathbb{Z}_p^n$. Then $\mu(G(T_1,\cdots,T_n))=\mu(\beta)$.*

Lemma 2.3. *Modulo $p^{n+k_1+\cdots+k_n}$, we have*

$$m_1!\cdots m_n!b_{m_1,\cdots,m_n}\equiv m_1!\cdots m_n!\sum_{j_1=0}^{k_1}\cdots\sum_{j_n=0}^{k_n}g_{j_1,\cdots,j_n}a_{j_1,\cdots,j_n}(f_{m_1,\cdots,m_n}),$$

where $a_{j_1,\cdots,j_n}(f_{m_1,\cdots,m_n})$ are the Mahler coefficients of $f_{m_1,\cdots,m_n}(x_1,\cdots,x_n)$.

Note that when $\mathrm{ord}_p(m_1!\cdots m_n!)\le k_1+\cdots+k_n$, then

$$b_{m_1,\cdots,m_n}\equiv\sum_{j_1=0}^{k_1}\cdots\sum_{j_n=0}^{k_n}g_{j_1,\cdots,j_n}a_{j_1,\cdots,j_n}(f_{m_1,\cdots,m_n})\pmod{p^n}. \tag{2.4}$$

Also, if $a_m(f_k)$ are the Mahler coefficients of $f_k(x)=\binom{u^x}{k}=\sum_{m=0}^{\infty}a_m(f_k)\binom{x}{m}$, then

$$a_{j_1,\cdots,j_n}(f_{m_1,\cdots,m_n})=a_{j_1}(f_{m_1})\cdots a_{j_n}(f_{m_n}). \tag{2.5}$$

In order to prove the Theorem (2.1), we need to investigate p-adic properties of the Mahler coefficients $a_{j_1,\cdots,j_n}(f_{m_1,\cdots,m_n})$. In the next section we shall study these coefficients.

3. p-adic properties of Mahler coefficients $a_{j_1,\cdots,j_n}(f_{m_1,\cdots,m_n})$

Let us fix a topological generator $u=1+t_1p+t_2p^2+\cdots$ of $1+p\mathbb{Z}_p$. Hence t_1 is a unit. We shall now prove two important binomial expansions in the following lemma.

Lemma 3.1. *For $n\ge 1$, we have*

$$(1+T)^{u^n}\equiv(1+T)(1+T^p)^{nt_1}(1+T^{p^2})^{\frac{n(n-1)}{2}t_1^2+nt_2}+\textit{ higher order terms}\pmod p. \tag{3.1}$$

$$(1+T)^{u^{p+n}}\equiv(1+T)(1+T^p)^{nt_1}(1+T^{p^2})^{t_1+\frac{n(n-1)}{2}t_1^2+nt_2}+\textit{ higher order terms}\pmod p. \tag{3.2}$$

Proof: For any $k\ge 1$, we have

$$(1+T)^{p^k}\equiv(1+T^{p^k})\pmod p.$$

This implies that

$$\begin{aligned}(1+T)^u&=(1+T)^{1+t_1p+t_2p^2+\cdots}\\&\equiv(1+T)(1+T^p)^{t_1}(1+T^{p^2})^{t_2}+\text{ higher order terms}\pmod p.\end{aligned}$$

Hence the statement (3.1) is true for $n=1$. Suppose that it is true for a given n. Then

$$\begin{aligned}(1+T)^{u^{n+1}}&=(1+T)^{u^n(1+t_1p+t_2p^2+\cdots)}\\&\equiv(1+T)^{u^n}(1+T^p)^{t_1u^n}(1+T^{p^2})^{t_2u^n}+\text{ higher order terms}\pmod p.\\&\equiv(1+T)(1+T^p)^{nt_1}(1+T^{p^2})^{\frac{n(n-1)}{2}t_1^2+nt_2}(1+T^p)^{t_1}(1+T^{p^2})^{nt_1{}^2}(1+T^{p^2})^{t_2}\\&\quad+\text{higher order terms}\pmod p.\\&\equiv(1+T)(1+T^p)^{(n+1)t_1}(1+T^{p^2})^{\frac{(n+1)n}{2}t_1^2+(n+1)t_2}\\&\quad+\text{ higher order terms}\pmod p.\end{aligned} \tag{3.3}$$

Hence the result is true for $n+1$. Using the principle of mathematical induction, the statement (3.1) is true for any $n \geq 1$. Again,

$$u^p = 1 + t_1 p^2 + \cdots . \tag{3.4}$$

Using (3.1) and (3.4), we find that

$$\begin{aligned}
(1+T)^{u^{p+n}} &= (1+T)^{u^n(1+t_1p^2+\cdots)} \\
&\equiv (1+T)^{u^n}(1+T^{p^2})^{t_1u^n} + \text{ higher order terms} \pmod p. \\
&\equiv (1+T)(1+T^p)^{nt_1}(1+T^{p^2})^{\frac{n(n-1)}{2}t_1^2+nt_2}(1+T^{p^2})^{t_1} \\
&\quad + \text{higher order terms} \pmod p. \\
&\equiv (1+T)(1+T^p)^{nt_1}(1+T^{p^2})^{t_1+\frac{n(n-1)}{2}t_1^2+nt_2} \\
&\quad + \text{ higher order terms} \pmod p.
\end{aligned} \tag{3.5}$$

This completes the proof of the lemma. □

Using the binomial expansions (3.1) and (3.2), we proved the following results about the Mahler coefficients $a_m(f_n)$ for different m and n in our paper [4].

Result 3.2. *Suppose that* $1 \leq k < p$ *and* $p^2 + (k-1)p \leq m < p^2 + kp$. *Then*

$$a_{p+k}(f_m) \equiv 0 \pmod p.$$

Result 3.3. *Suppose that* $1 \leq k < p$. *Then*

$$a_{p+k}(f_{p^2+kp}) \equiv t_1^{k+1} \pmod p \text{ and } a_{p+k+1}(f_{p^2+kp}) \equiv 0 \pmod p.$$

Result 3.4. *Suppose that* $2p^2 - p \leq m < 2p^2$. *Then* $a_{2p}(f_m) \equiv 0 \pmod p$. *Also,*

$$a_{2p}(f_{2p^2}) \equiv t_1^2 \pmod p,\ a_{2p+1}(f_{2p^2}) \equiv 0 \pmod p, \text{ and } a_{2p+2}(f_{2p^2}) \equiv 0 \pmod p.$$

Let us now state the following result which was proved in [1].

Result 3.5. *Suppose that* $1 \leq k < p$, *then* $a_k(f_{pk}) \equiv {t_1}^k \pmod p$. *Also,*

$$a_p(f_{p^2}) \equiv t_1 \pmod p \text{ and } a_{p+1}(f_{p^2}) \equiv 0 \pmod p.$$

Using the above results and (2.5), one can derive p-adic properties of the Mahler coefficients $a_{j_1,\cdots,j_n}(f_{m_1,\cdots,m_n})$. We now prove certain p-adic properties of the Mahler coefficients in the following lemmas.

Lemma 3.6. *Suppose that* $1 \leq k < p$ *and* $p^2+(k-1)p \leq m < p^2+kp$. *Let* $m_1+\cdots+m_n$ *be a partition of* m *such that* $k_1 + \cdots + k_n = p + k$, *where* $k_i = \mathrm{ord}_p(m_i!), i = 1, \cdots, n$. *Then we have*

$$a_{k_1,\cdots,k_n}(f_{m_1,\cdots,m_n}) \equiv 0 \pmod p.$$

Proof: Clearly, $k_i = \mathrm{ord}_p(m_i!) \neq p$ for all i, because $\mathrm{ord}_p((p^2-p)!) = p-1$ and $\mathrm{ord}_p(p^2!) = p+1$. Hence we have the following two cases only.

Case 1: Suppose that $k_i > p$ for some i. Then $k_i = p + l_i, 0 < l_i \leq k$ and hence $p^2 + (l_i-1)p \leq m_i < p^2 + l_ip$. By Result (3.2), we get

$$a_{k_i}(f_{m_i}) \equiv 0 \pmod p. \tag{3.6}$$

Case 2: Suppose that $k_i < p$ for some i. Then $pk_i \le m_i < p(k_i+1)$. We know that

$$a_{k_i}(f_{m_i}) = \sum_{j=0}^{k_i} (-1)^{k_i - j} \binom{k_i}{j} \binom{u^j}{m_i}. \tag{3.7}$$

But, $\binom{u^j}{m_i}$ is the coefficient of T^{m_i} in the expansion of $(1+T)^{u^j}$. From (3.1), we find that $\binom{u^j}{m_i}$ is congruent to zero modulo p if $m_i \neq pk_i, pk_i + 1$. This implies that

$$a_{k_i}(f_{m_i}) \equiv 0 \ (\mathrm{mod}\, p) \text{ if } m_i \neq pk_i, pk_i + 1. \tag{3.8}$$

Thus, for any partition $m_1 + \cdots + m_n$ of m, where $k_i = ord_p(m_i!)$ are as given in the lemma, (3.6) and (3.8) imply that $a_{k_1,\cdots,k_n}(f_{m_1,\cdots,m_n}) \equiv 0 \ (\mathrm{mod}\, p)$ unless $m_i = pk_i$ or $m_i = pk_i + 1$. With out loss of generality, suppose that $m_i = pk_i$ for $i = 1, \cdots, l$ and $m_i = pk_i + 1$ for $i = l+1, \cdots, n$. Then $m = m_1 + \cdots + m_n = p(p+k) + (n-l)$, which is a contradiction to the fact that $m < p^2 + kp$. This completes the proof of the lemma. □

Lemma 3.7. *Suppose that $2p^2 - p \le m < 2p^2$. Let $m_1 + \cdots + m_n$ be a partition of m such that $k_1 + \cdots + k_n = 2p$, where $k_i = \mathrm{ord}_p(m_i!), i = 1, \cdots, n$. Then we have*

$$a_{k_1,\cdots,k_n}(f_{m_1,\cdots,m_n}) \equiv 0 \ (\mathrm{mod}\, p).$$

Proof: Suppose that $k_i = 2p$ for some i. Then $2p^2 - p \le m_i < 2p^2$ and hence from the Result (3.4), we get

$$a_{k_i}(f_{m_i}) \equiv 0 \ (\mathrm{mod}\, p) \tag{3.9}$$

This implies that $a_{k_1,\cdots,k_n}(f_{m_1,\cdots,m_n}) \equiv 0 \ (\mathrm{mod}\, p)$. The other two cases are $p < k_i < 2p$ and $k_i < p$ as $k_i \neq p$. As shown in the proof of the Lemma (3.6), $a_{k_1,\cdots,k_n}(f_{m_1,\cdots,m_n}) \equiv 0 \ (\mathrm{mod}\, p)$ unless $m_i = pk_i$ or $m_i = pk_i + 1$. With out loss of generality, suppose that $m_i = pk_i$ for $i = 1, \cdots, l$ and $m_i = pk_i + 1$ for $i = l+1, \cdots, n$. Then $m = m_1 + \cdots + m_n = 2p^2 + (n-l)$, which is a contradiction to the fact that $m < 2p^2$. This completes the proof of the lemma. □

4. **Proof of Main Result**

We may assume that $\mu(G(T_1, \cdots, T_n)) = 0$, because $\mu(G(T_1, \cdots, T_n)) = \mu(\beta)$ by Lemma (2.2), and for any power series $F(T_1, \cdots, T_n) \in \mathcal{O}[[T_1 - 1, \cdots, T_n - 1]]$, if $\pi | F(T_1, \cdots, T_n)$ then $\lambda(\pi^{-1} F(T_1, \cdots, T_n)) = \lambda(F(T_1, \cdots, T_n))$.

Case 1: Suppose that $\lambda(G(T_1, \cdots, T_n)) = k < p$. Then there exists a partition $k_1 + \cdots + k_n$ of k such that $g_{k_1,\cdots,k_n}$ is a unit in $\mathcal{O}$ and for every $m_i \ge 0$ satisfying $m_1 + \cdots + m_n < k$, $g_{m_1,\cdots,m_n} \equiv 0 \ (\mathrm{mod}\, \pi)$. If $r < pk$, then for any partition $r_1 + \cdots + r_n$ of r, we find that $\mathrm{ord}_p(r_1! \cdots r_n!) = \mathrm{ord}_p(r_1!) + \cdots + \mathrm{ord}_p(r_n!) \le k-1$. If $l_i = \mathrm{ord}_p(r_i!)$, then from (2.4) we get

$$b_{r_1,\cdots,r_n} \equiv \sum_{j_1=0}^{l_1} \cdots \sum_{j_n=0}^{l_n} g_{j_1,\cdots,j_n} a_{j_1,\cdots,j_n}(f_{r_1,\cdots,r_n}) \equiv 0 \ (\mathrm{mod}\, \pi). \tag{4.1}$$

Now consider the partition $k_1+\cdots+k_n$ of k. Then $pk_1+\cdots+pk_n$ is a partition of pk such that $\mathrm{ord}_p(pk_i)=k_i$. From (2.4), (2.5) and Result (3.5), we get

$$\begin{aligned} b_{pk_1,\cdots,pk_n} &\equiv \sum_{j_1=0}^{k_1}\cdots\sum_{j_n=0}^{k_n} g_{j_1,\cdots,j_n}a_{j_1,\cdots,j_n}(f_{pk_1,\cdots,pk_n}) \\ &\equiv g_{k_1,\cdots,k_n}a_{k_1,\cdots,k_n}(f_{pk_1,\cdots,pk_n}) \\ &\equiv g_{k_1,\cdots,k_n}t_1^k \pmod{\pi}, \end{aligned} \tag{4.2}$$

which is a unit in $\mathcal{O}$. This proves that $\lambda(\beta)=pk_1+\cdots+pk_n=pk=p\lambda(G(T_1,\cdots,T_n))$.

Case 2: Suppose that $\lambda(G(T_1,\cdots,T_n))=p$. Then there exists a partition $k_1+\cdots+k_n$ of p such that $g_{k_1,\cdots,k_n}$ is a unit in $\mathcal{O}$ and for every $m_i\geq 0$ satisfying $m_1+\cdots+m_n<p$, $g_{m_1,\cdots,m_n}\equiv 0 \pmod{\pi}$. Let $m<p^2$. Then for every partition $m_1+\cdots+m_n$ of m, we get $l_1+\cdots+l_n\leq p-1$, where $l_i=\mathrm{ord}_p(m_i!)$. As shown in the previous case, this implies that $b_{m_1,\cdots,m_n}\equiv 0 \pmod{\pi}$. Let us now consider the partition $pk_1+\cdots+pk_n$ of p^2. If $k_i=p$ for some i, then $k_j=0$ for all $j\neq i$. Hence, from (2.4) and Result (3.5), we get

$$\begin{aligned} b_{0,\cdots,0,p^2,0,\cdots,0} &\equiv g_{0,\cdots,0,p,0,\cdots,0}a_p(f_{p^2})+g_{0,\cdots,0,p+1,0,\cdots,0}a_{p+1}(f_{p^2}) \\ &\equiv g_{0,\cdots,0,p,0,\cdots,0}t_1 \pmod{\pi}, \end{aligned} \tag{4.3}$$

which is a unit in $\mathcal{O}$. If all $k_i<p$, then using (4.2), we obtain

$$b_{pk_1,\cdots,pk_n}\equiv g_{k_1,\cdots,k_n}t_1^p \pmod{\pi}, \tag{4.4}$$

which is a unit in $\mathcal{O}$. This proves that $\lambda(\beta)=pk_1+\cdots+pk_n=p^2=p\lambda(G(T_1,\cdots,T_n))$.

Case 3: Suppose that $p<\lambda(G(T_1,\cdots,T_n))<2p$. Let $\lambda(G(T_1,\cdots,T_n))=p+k$, where $1\leq k<p$. Then there exists a partition $k_1+\cdots+k_n$ of $p+k$ such that $g_{k_1,\cdots,k_n}$ is a unit in $\mathcal{O}$ and for every $m_i\geq 0$ satisfying $m_1+\cdots+m_n<p+k$, $g_{m_1,\cdots,m_n}\equiv 0 \pmod{\pi}$. Let $m<p^2+(k-1)p$. Then $\mathrm{ord}_p(m!)<p+k$ and hence for any partition $m_1+\cdots+m_n$ of m, we have $l_1+\cdots+l_n<p+k$, where $l_i=\mathrm{ord}_p(m_i!)$. As shown in the case 1, this implies that $b_{m_1,\cdots,m_n}\equiv 0 \pmod{\pi}$. If $p^2+(k-1)p\leq m<p^2+kp$, then $\mathrm{ord}_p(m!)=p+k$. Therefore, for every partition $m_1+\cdots+m_n$ of m, we get $l_1+\cdots+l_n\leq p+k$, where $l_i=\mathrm{ord}_p(m_i!)$. If $l_1+\cdots+l_n<p+k$, then we have already proved that $b_{m_1,\cdots,m_n}\equiv 0 \pmod{\pi}$. Again if $l_1+\cdots+l_n=p+k$, then

$$b_{m_1,\cdots,m_n}\equiv g_{l_1,\cdots,l_n}a_{l_1,\cdots,l_n}(f_{m_1,\cdots,m_n}) \pmod{\pi}. \tag{4.5}$$

Using Lemma (3.6), we find that $b_{m_1,\cdots,m_n}\equiv 0 \pmod{\pi}$. Let us now consider the partition $k_1+\cdots+k_n$ of $p+k$. Then $pk_1+\cdots+pk_n$ is a partition of p^2+pk. If $k_i<p$, then $\mathrm{ord}_p(pk_i!)=k_i$. Also, $k_i=p$ implies $\mathrm{ord}_p((pk_i)!)=p+1=k_i+1$. If $k_i>p$, then $k_i=p+l_i$, where $1\leq l_i\leq k$ and hence $\mathrm{ord}_p(pk_i)=\mathrm{ord}_p((p^2+pl_i)!)=p+l_i+1=k_i+1$. From Result (3.3) and Result (3.5), we have

$$a_{p+1}(f_{p^2})\equiv 0 \pmod{p} \quad\text{and}\quad a_{p+l_i+1}(f_{p^2+l_ip})\equiv 0 \pmod{p}.$$

Again, if $k_i<p$, then from Result (3.5), we get $a_{k_i}(f_{pk_i})\equiv {t_1}^{k_i} \pmod{p}$. Also, $a_p(f_{p^2})\equiv t_1 \pmod{p}$ and if $1\leq l_i<p$, then $a_{p+l_i}(f_{p^2+l_ip})\equiv t_1^{l_i+1} \pmod{p}$. This implies that, if

$h_i = \mathrm{ord}_p(pk_i!)$ for $i = 1, \cdots, n$, then

$$b_{pk_1,\cdots,pk_n} \equiv \sum_{j_1=0}^{h_1} \cdots \sum_{j_n=0}^{h_n} g_{j_1,\cdots,j_n} a_{j_1,\cdots,j_n}(f_{pk_1,\cdots,pk_n}) \equiv g_{k_1,\cdots,k_n} a_{k_1,\cdots,k_n}(f_{pk_1,\cdots,pk_n}) \pmod{\pi}, \tag{4.6}$$

which is a unit in $\mathcal{O}$. This proves that $\lambda(\beta) = pk_1 + \cdots + pk_n = p^2 + pk = p\lambda(G(T_1, \cdots, T_n))$.

Case 4: Suppose that $\lambda(G(T_1, \cdots, T_n)) = 2p$. Then there exists a partition $k_1 + \cdots + k_n$ of $2p$ such that $g_{k_1,\cdots,k_n}$ is a unit in $\mathcal{O}$ and for every $m_i \geq 0$ satisfying $m_1 + \cdots + m_n < 2p$, $g_{m_1,\cdots,m_n} \equiv 0 \pmod{\pi}$. Let $m < 2p^2 - p$. Then for every partition $m_1 + \cdots + m_n$ of m we have $l_1 + \cdots + l_n < 2p$, where $l_i = \mathrm{ord}_p(m_i!)$. As shown in the case 1, this implies that $b_{m_1,\cdots,m_n} \equiv 0 \pmod{\pi}$. If $2p^2 - p \leq m < 2p^2$, then for every partition $m_1 + \cdots + m_n$ of m we have $l_1 + \cdots + l_n \leq 2p$. If $l_1 + \cdots + l_n < 2p$, then we have already observed that $b_{m_1,\cdots,m_n} \equiv 0 \pmod{\pi}$. Also, if $l_1 + \cdots + l_n = 2p$, then from Lemma (3.7) we get $b_{m_1,\cdots,m_n} \equiv 0 \pmod{\pi}$. Let us now consider the partition $k_1 + \cdots + k_n$ of $2p$. Then $pk_1 + \cdots + pk_n$ is a partition of $2p^2$. If $k_i = 2p$ for some i, then $\mathrm{ord}_p(pk_i) = 2p + 2$. But from Result (3.4), we get

$$a_{2p+1}(f_{2p^2}) \equiv 0 \pmod{p} \quad \text{and} \quad a_{2p+2}(f_{2p^2}) \equiv 0 \pmod{p}.$$

Using this and considering the other possible values of k_i as shown in the previous case, we obtain

$$b_{pk_1,\cdots,pk_n} \equiv \sum_{j_1=0}^{h_1} \cdots \sum_{j_n=0}^{h_n} g_{j_1,\cdots,j_n} a_{j_1,\cdots,j_n}(f_{pk_1,\cdots,pk_n}) \equiv g_{k_1,\cdots,k_n} a_{k_1,\cdots,k_n}(f_{pk_1,\cdots,pk_n}) \pmod{\pi}, \tag{4.7}$$

where $h_i = \mathrm{ord}_p(pk_i!)$. Again, from Result (3.4) we get

$$a_{2p}(f_{2p^2}) \equiv t_1^2 \pmod{p}.$$

Considering the other possibilities as shown in the previous case, (4.7) implies that $b_{pk_1,\cdots,pk_n}$ a unit in $\mathcal{O}$. This proves that $\lambda(\beta) = pk_1 + \cdots + pk_n = 2p^2 = p\lambda(G(T_1, \cdots, T_n))$.

This completes the proof of the main theorem. □

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Department of Mathematical Sciences, Tezpur University, Napaam-784028, Sonitpur, Assam, INDIA

E-mail address: rupamb@tezu.ernet.in

Department of Mathematics, Indian Institute of Technology, Guwahati-781039, Assam, INDIA

E-mail address: a.saikia@iitg.ernet.in