Elementary properties of Complex numbers
Let us consider the quadratic equation $x^2 + 1 = 0$.

It has no real root.

Let $i$ (iota) be the solution of the above equation, then

- $i^2 = -1$ i.e. $i = \sqrt{-1}$.
- $i$ is not a real number. So we define it as imaginary number.

A complex number is defined by $z = x + iy$, for any $x, y \in \mathbb{R}$.

Complex analysis is theory of functions of complex numbers.
Why do we need Complex Analysis?

Evaluation of certain integrals which are difficult to workout. Viz.

\[ \int_{-\infty}^{\infty} \frac{e^{x/2}}{1 + e^x} \, dx = \pi. \]

- Fourier Analysis.
- Differential Equations.
- Number Theory.
- All major branches of Mathematics which is applicable in science and engineering.
A complex number denoted by $z$ is an ordered pair $(x, y)$ with $x \in \mathbb{R}$, $y \in \mathbb{R}$.

- $x$ is called real part of $z$ and $y$ is called the imaginary part of $z$. In symbol $x = \text{Re } z$, and $y = \text{Im } z$.

- We denote $i = (0, 1)$ and hence $z = x + iy$ where the element $x$ is identified with $(x, 0)$.

- $\text{Re } z = \text{Im } iz$ and $\text{Im } z = -\text{Re } iz$.

- By $\mathbb{C}$ we denote the set of all complex numbers, that is, $\mathbb{C} = \{z : z = x + iy, x \in \mathbb{R}, y \in \mathbb{R}\}$.
Algebra of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers.

- **Addition and subtraction**: We define
  
  $$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2).$$

- **Multiplication**: We define
  
  $$z_1z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

  Since $i = (0,1)$ it follows from above that $i^2 = -1$.

- **Division**: If $z$ a nonzero complex number then we define
  
  $$\frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}.$$

  From this we get
  
  $$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}.$$
Basic algebraic properties of Complex Numbers

Let $z_1, z_2, z_3 \in \mathbb{C}$.

- **Commutative and associative law for addition**: $z_1 + z_2 = z_2 + z_1$ and $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.
- **Additive identity**: $z + 0 = 0 + z = z \quad \forall z \in \mathbb{C}$
- **Additive inverse**: For every $z \in \mathbb{C}$ there exists $-z \in \mathbb{C}$ such that $z + (-z) = 0 = (-z) + z$.
- **Commutative and associative law for multiplication**: $z_1 z_2 = z_2 z_1$. and $z_1(z_2 z_3) = (z_1 z_2) z_3$.
- **Multiplicative identity**: $z \cdot 1 = z = 1 \cdot z \quad \forall z \in \mathbb{C}$
- **Multiplicative inverse**: For every nonzero $z \in \mathbb{C}$ there exists $w(=\frac{1}{z}) \in \mathbb{C}$ such that $zw = 1 = wz$.
- **Distributive law**: $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

Note: $\mathbb{C}$ is a field.
If \( z = x + iy \) is a complex number then its **conjugate** is defined by \( \bar{z} = x - iy \). Conjugation has the following properties which follows easily from the definition. Let \( z_1, z_2 \in \mathbb{C} \) then,

- \( \text{Re } z = \frac{1}{2}(z + \bar{z}) \) and \( \text{Im } z = \frac{1}{2i}(z - \bar{z}) \).
- \( \bar{z}_1 + \bar{z}_2 = \bar{z}_1 + \bar{z}_2 \).
- \( \bar{z}_1 \bar{z}_2 = \bar{z}_1 \bar{z}_2 \).
- **Note:** If \( \alpha \in \mathbb{R} \) then \( \bar{\alpha z} = \alpha \bar{z} \).
- \( \bar{\bar{z}} = z \).
- \( \text{Re } z = \text{Re } \bar{z} \) and \( \text{Im } z = -\text{Im } \bar{z} \).
The **modulus** or **absolute** value of a complex number \( z = x + iy \) is a non-negative real number denoted by \(|z|\) and defined by

\[
|z| = \sqrt{x^2 + y^2}.
\]

Note that if \( z = x + iy \) then \(|z|\) is the Euclidean distance of the point \((x, y)\) from the origin \((0, 0)\).

**Exercise:** Verify the following properties.

- \( z\bar{z} = |z|^2 \).
- \(|x| = |\text{Re } z| \leq |z|\) and \(|y| = |\text{Im } z| \leq |z|\)
- \(|\bar{z}| = |z|, |z_1 z_2| = |z_1||z_2|\) and \(\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} (z_2 \neq 0)\).
- \(|z_1 + z_2| \leq |z_1| + |z_2|\) (**Triangle inequality**).
- \(||z_1| - |z_2|| \leq |z_1 - z_2|\)
Graphical representation of Complex Numbers

We can represent the complex number $z = x + iy$ by a position vector in the $XY$–plane whose tail is at the origin and head is at the point $(x, y)$.

When $XY$–plane is used for displaying complex numbers, it is called **Argand plane** or **Complex plane** or **z plane**.

The $X$-axis is called as the real axis where as the $Y$-axis is called as the imaginary axis.
Graph the complex numbers:

1. $3 + 4i \quad (3,4)$
2. $2 - 3i \quad (2,-3)$
3. $-4 + 2i \quad (-4,2)$
4. $3$ (which is really $3 + 0i) \quad (3,0)$
5. $4i$ (which is really $0 + 4i) \quad (0,4)$

The complex number is represented by the point or by the vector from the origin to the point.
Add $3 + 4i$ and $-4 + 2i$ graphically.

Graph the two complex numbers $3 + 4i$ and $-4 + 2i$ as vectors.

Create a parallelogram using these two vectors as adjacent sides.

The sum of $3 + 4i$ and $-4 + 2i$ is represented by the diagonal of the parallelogram (read from the origin).

This new (diagonal) vector is called the resultant vector.
Subtract $3 + 4i$ from $-2 + 2i$

Subtraction is the process of adding the additive inverse.

\[
(-2 + 2i) - (3 + 4i) = (-2 + 2i) + (-3 - 4i) = (-5 - 2i)
\]

Graph the two complex numbers as vectors.

Graph the additive inverse of the number being subtracted.

Create a parallelogram using the first number and the additive inverse. The answer is the vector forming the diagonal of the parallelogram.
Consider the unit circle on the complex plane. Any point on the unit circle is represented by \((\cos \varphi, \sin \varphi), \varphi \in [0, 2\pi]\).

Any non zero \(z \in \mathbb{C}\), the point \(\frac{z}{|z|}\) lies on the unit circle and therefore we write \(\frac{z}{|z|} = \cos \varphi + i\sin \varphi\). i.e. \(z = |z|(\cos \varphi + i\sin \varphi)\).

The symbol \(e^{i\varphi}\) is defined by means of \textit{Euler’s formula} as
\[e^{i\varphi} = \cos \varphi + i\sin \varphi.\]
Polar representation of Complex Numbers

Any non-zero \( z = x + iy \) can be uniquely specified by its magnitude (length from origin) and direction (the angle it makes with positive X-axis).

Let \( r = |z| = \sqrt{x^2 + y^2} \) and \( \theta \) be the angle made by the line from origin to the point \((x, y)\) with the positive X-axis.

From the above figure \( x = r \cos \theta \), \( y = r \sin \theta \) and \( \theta = \tan^{-1}(\frac{y}{x}) \).
Polar representation of a Complex Number

- If $z \neq 0$ then $\text{arg}(z) = \{ \theta : z = re^{i\theta} \}$.
- Note that $\text{arg}(z)$ is a multi-valued function.

$$\text{arg}(z) = \{ \theta + 2n\pi : z = re^{i\theta}, n \in \mathbb{Z} \}.$$ 

- $\text{arg } z = \text{Arg } z + 2k\pi$ So, if $\theta$ is argument of $z$ then so is $\theta + 2k\pi$. For example, $\text{arg } i = 2k\pi + \frac{\pi}{2}, k \in \mathbb{Z}$, where as $\text{Arg } i = \frac{\pi}{2}$.
- The principal value of $\text{arg}(z)$, denoted by $\text{Arg}(z)$, is the particular value of $\text{arg}(z)$ chosen in within $(-\pi, \pi]$.
- Let $z_1 = r_1e^{i\theta_1}, z_2 = r_2e^{i\theta_2}$ then $z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$.
- If $z_1 \neq 0$ and $z_2 \neq 0$, $\text{arg}(z_1z_2) = \text{arg}(z_1) + \text{arg}(z_2)$.
- As $|e^{i\theta}| = 1, \forall \theta \in \mathbb{R}$, it follows that $|z_1z_2| = |z_1||z_2|$.
De Moiver’s formula

**De Moiver’s formula:**

\[ z^n = [r(\cos \theta + i \sin \theta)]^n = r^n (\cos n\theta + i \sin n\theta). \]

**Problem:** Given a nonzero complex number \( z_0 \) and a natural number \( n \in \mathbb{N} \). Find all distinct complex numbers \( w \) such that \( z_0 = w^n \).

If \( w \) satisfies the above then \( |w| = |z_0|^{\frac{1}{n}} \). So, if \( z_0 = |z_0|(\cos \theta + i \sin \theta) \) we try to find \( \alpha \) such that

\[ |z_0|(\cos \theta + i \sin \theta) = \left[ |z_0|^{\frac{1}{n}} (\cos \alpha + i \sin \alpha) \right]^n. \]

By De Moiver’s formula \( \cos \theta = \cos n\alpha \) and \( \sin \theta = \sin n\alpha \), that is, \( n\alpha = \theta + 2k\pi \Rightarrow \alpha = \frac{\theta}{n} + \frac{2k\pi}{n} \). The distinct values of \( w \) is given by

\[ |z_0|^{\frac{1}{n}} (\cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n}), \text{ for, } k = 0, 1, 2, \ldots, n - 1. \]